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Dedicated to Professor Jan Stochel
on the occasion of his 70th birthday

ON THE MÖBIUS INVARIANT PRINCIPAL FUNCTIONS OF PINCUS

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Abstract. In this semi-expository short note, we prove that the only homogeneous pure hyponormal operator T with $\operatorname{rank}(T^*T - TT^*) = 1$, modulo unitary equivalence, is the unilateral shift.

Keywords: hyponormal operator, multiplicity, trace formula, homogeneous operators, principal function.

Mathematics Subject Classification: 47B20.

1. INTRODUCTION

In this paper, a Hilbert space $\mathcal{H}$ is assumed to be complex and separable and an operator T on $\mathcal{H}$ is assumed to be linear and bounded. The algebra of bounded linear operators on a $\mathcal{H}$ is denoted by $\mathcal{L}(\mathcal{H})$. An operator $A \in \mathcal{L}(\mathcal{H})$ is said to be *hyponormal* if $[A^*, A] := A^*A - AA^*$ is *non-negative*, that is, $\langle [A^*, A]f, f\rangle \geqslant 0$ for all $f \in \mathcal{H}$.

Let $\mathcal{H}$ be a Hilbert space and $\{e_n\}_{n\geqslant 0}$ be an orthonormal basis in $\mathcal{H}$. For any bounded non-negative operator B acting on $\mathcal{H}$, define its trace by setting

$$\operatorname{tr}(B) = \sum_n \langle Be_n, e_n\rangle.$$

This definition of $\operatorname{tr}(B)$ does not depend on the choice of the orthonormal basis that was chosen to define it.

An operator $A \in \mathcal{L}(\mathcal{H})$ is said to be in the *trace class* $\mathcal{S}_1(\mathcal{H})$ if $\sum_{n=0}^{\infty}\langle |A|e_n, e_n\rangle$ is finite. As usual, here $|A|$ is the unique positive square root of the self-adjoint operator A^*A.

The s-numbers $\{s_j(T)\}_{j=1}^{\infty}$ of a compact operator T are the eigenvalues of $(T^*T)^{\frac{1}{2}}$, counted with multiplicity and arranged in decreasing order. The trace norm is also given by the formula

$$\|T\|_1 = \sum_{j=1}^{\infty} s_j(T).$$

Let T be a trace class operator. Set

$$\Lambda_T := \{\lambda_j(T) : j = 1, 2, \ldots, \nu(T)\}$$

be an enumeration of the non-zero eigenvalues of T counting multiplicities. The determinant of the operator $I + T$ is defined as follows:

$$\det(I+T) = \begin{cases} \prod_{j=1}^{\nu(T)} (1 + \lambda_j(T)), & \Lambda_T \neq \emptyset, \\ 1 & \Lambda_T = \emptyset. \end{cases}$$

In case $\nu(T)$ is infinite, the convergence of the product defining the determinant follows from the inequality $\sum_{j=1}^{\infty} |\lambda_j(T)| \leq \|T\|_1$, see [12, Chapter II]. We need the following crucial relationship between the trace and the determinant.

Recall the Jacobi formula for matrix exponential, namely, $\det \exp(B) = \exp(\mathrm{tr}(B))$. Now, suppose that T is a trace class operator with $\|T\|_1 < 1$. Then we define $\det(I + T) = \exp \mathrm{tr}(\log(I + T))$. Here $\log(I + T)$ is the logarithm of $I + T$ given by the series (convergent in the norm $\|\cdot\|_1$)

$$\log(I+T) = -\sum_{n=1}^{\infty} (-1)^n \frac{T^n}{n}, \tag{1.1}$$

see [9, p. 81].

Definition 1.1. A natural number m is said to be the (rational) multiplicity of an operator $T \in \mathcal{L}(\mathcal{H})$ if there exist vectors $\{x_i\}_{i \in I}$, for some indexing set I with $|I| = m$, such that

$$H = \bigvee \big\{ f(T)x_i, i \in I, f \in \mathrm{Rat}(\sigma(T)) \big\},$$

where $\mathrm{Rat}(\sigma(T))$ is the set of all rational functions r of the form $\frac{p}{q}$ for a pair of polynomials p and q with q not zero on $\sigma(T)$.

In this short note we study the class of hyponormal operators T with finite multiplicity. The remarkable inequality [5] of Berger and Shaw

$$\mathrm{tr}\,[\mathrm{T}^*,\ \mathrm{T}] \leqslant \frac{m}{\pi}\,\mathrm{Area}(\sigma(\mathrm{T})) \tag{1.2}$$

ensures that the self-commutator $[T^*, T]$ of such an operator is in the trace class. An immediate corollary is an inequality due to Putnam [23]: If $T \in \mathcal{L}(\mathcal{H})$ is hyponormal, then

$$\|[T^*, T]\| \leq \frac{1}{\pi}\,\mathrm{Area}(\sigma(T)). \tag{1.3}$$

The verification below of Putnam's inequality is taken from [16, Chapter VI, Theorem 2.1]. Pick a non-zero vector $x \in \mathcal{H}$ and set

$$\mathcal{H}_x := \bigvee \big\{ f(T)x : f \in \mathrm{Rat}(\sigma(T)) \big\}.$$

Let $T_x : \mathcal{H}_x \to \mathcal{H}_x$ be the restriction of the operator T to $\mathcal{H}_x$. The operator T_x is evidently hyponormal and it is rationally cyclic of multiplicity 1. We have

$$\begin{aligned}\langle [T^*, T]x, x]\rangle &= \|Tx\|^2 - \|T^*x\|^2\\ &\leqslant \|T_x x\|^2 - \|T_x^* x\|^2\\ &= \langle [T_x^*, T_x]x, x\rangle\\ &\leqslant \operatorname{tr}([T_x^*, T_x])\\ &\leqslant \frac{1}{\pi}\operatorname{Area}(\sigma(T_x))\\ &\leqslant \frac{1}{\pi}\operatorname{Area}(\sigma(T)),\end{aligned}$$

where the penultimate inequality follows from Berger-Shaw inequality (1.2) and the last inequality is a consequence of the spectral inclusion $\sigma(T_x) \subseteq \sigma(T)$.

Remark 1.2. Among many consequences of Putnam's inequality, we single out one that we will need in what follows, namely, if T is a *pure* hyponormal operator, then $\operatorname{Area}(\sigma(T)) > 0$.

Moreover, we note that the *determinantal formula* due to Carey and Pincus, discussed below, connects the *principal function* g_T of the operator T with the trace of $[T^*, T]$ using the Helton–Howe *trace formula*. For a recent account, one may consult the book [13].

Definition 1.3. The bi-holomorphic automorphism group Möb of the unit disc consists of rational functions φ of the form:

$$\varphi(z) = \beta\frac{z-a}{1-\bar{a}z}, \quad \beta \in \mathbb{T}, a \in \mathbb{D},$$

where $\mathbb{T}$ and $\mathbb{D}$ denote the unit circle and the open unit disc, respectively.

For an opeartor T with the spectrum $\sigma(T)$ contained in the closed unit disc $\overline{\mathbb{D}}$, by the spectral mapping theorem, $0 \notin \sigma(I - \bar{a}T)$ for any $a \in \mathbb{D}$. Hence, the operator $I - \bar{a}T$, $a \in \mathbb{D}$, is invertible.

Definition 1.4. An operator T with $\sigma(T) \subseteq \overline{\mathbb{D}}$ is said to be *homogeneous* if the operator

$$\varphi(T) := \beta(T-a)(I-\bar{a}T)^{-1}, \quad \beta \in \mathbb{T},\, a \in \mathbb{D},$$

is unitarily equivalent to T for all $\varphi \in$ Möb.

The problem of determining all the homogeneous normal operators, homogeneous contractions and homogeneous operators in the Cowen-Douglas class has been addressed in a series of papers [2, 4, 15] previously. One of the goals of this paper is to determine modulo unitary equivalence, all hyponormal operators T such that $[T^*, T]$ is in trace class that are homogeneous. This involves, among other things, finding a transformation rule for the principal function of an operator under the Möbius transformations.

2. PRELIMINARIES

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *hyponormal* if the self-commutator $[T^*, T] = T^*T - TT^*$ is non-negative definite. A hyponormal operator T is said to be a *pure* if there is no nontrivial reducing subspace for T on which it is normal. Every hyponormal operator T, modulo unitary equivalence, is of the form $T_p \oplus T_n$, where $T_p = T|_{\mathcal{H}_p}$, $T_n = T|_{\mathcal{H}_n}$ and $\mathcal{H} = \mathcal{H}_p \oplus \mathcal{H}_n$ such that T_p is pure and T_n is normal, see [16, Chapter II, Theorem 1.3].

Any operator $T \in \mathcal{L}(\mathcal{H})$ can be written in the form $T = A + iB$, where $A = \frac{T+T^*}{2}$ and $B := \frac{T-T^*}{2i}$ are self-adjoint. It follows that $[T^*, T] = 2i[A, B]$.

2.1. PRINCIPAL FUNCTION

The principal function of an operator T is defined by means of an auxiliary operator valued function E of two complex variables, called the determining function of T. The principal function g_T of T then appears by expressing the multiplicative determinant of the self-commutator, or the trace of the self-commutator $D := [T^*, T]$ as an integral. We recall that the determining function E is given by the formula

$$E(z, w) = I - 2iD^{\frac{1}{2}}(A - z)^{-1}(B - w)^{-1}D^{\frac{1}{2}}, \quad z, w \in \mathbb{C} \setminus \sigma(A) \times \sigma(B).$$

Pincus in [18, 20] proved the existence of a function $g(v, u) \geqslant 0$ such that

$$\det E(z, w) = \exp\left(\frac{1}{2\pi i}\iint_{\mathbb{C}} g(u, v)\frac{du}{u - z}\frac{dv}{v - w}\right), \quad z, w \in \mathbb{C} \setminus \sigma(A) \times \sigma(B). \quad (2.1)$$

The support of any almost everywhere determined version of $g(u, v)$ is said to be the "determining set" of the pair A, B, or equivalently that of the operator T. The essential closure of the determining set is denoted by $D(A, B)$. It is proved in [19] that $\sigma(T) = D(A, B)$. Thus, $\mathrm{Supp}(g) \subseteq \sigma(T)$ and if T is pure, then $\mathrm{Supp}(g) = \sigma(T)$, see also [9, p. 105].

Remark 2.1. For every integrable, compactly supported function g on $\mathbb{C}$, with $0 \leqslant g \leqslant 1$, there exists a pure semi-normal operator T, with rank $[T^*,\ \mathrm{T}] = 1$ such that $[g] = [g_T]$ in $\mathrm{L}^1(\,\mathrm{d}\mu)$. The proof is in [7, Theorem 1], see also [21].

2.2. THE TRACIAL BI-LINEAR FORM

Let $\mathbb{C}[x, y]$ denote the algebra polynomials over the complex field in the two indeterminates x, y. Thus, any $p \in \mathbb{C}[x, y]$ is of the form

$$p(x, y) = \sum_{j,k=1}^{m} a_{i,j}x^j y^k, \quad a_{i,j} \in \mathbb{C}.$$

Let A, B be a pair of self adjoint operators in $\mathcal{L}(\mathcal{H})$ such that $\|[A, B]\|_1 < \infty$. Also, let $\mathbb{C}[A, B]$ be the algebra of operators generated by substituting A, B in place of the commuting variables x, y of the polynomial $p \in \mathbb{C}[x, y]$. Thus, if X, Y is any pair of operators in $\mathbb{C}[A, B]$, then the operator

$$p(X,Y) = \sum_{j,k=1}^{m} a_{i,j} X^j Y^k$$

is well defined modulo operators of trace class. The tracial bi-linear form associated with the pair X, Y is

$$(p, q) = \operatorname{tr} i\,[p(X,Y), q(X,Y)], \quad p, q \in \mathbb{C}[x, y].$$

An amazing formula discovered by Helton and Howe [14] for the tracial bilinear form is given below.

Theorem 2.2 (Helton–Howe)**.** *Suppose that X, Y are a pair of operators such that $[X, Y]$ is in trace class. Then there exists a regular signed Borel measure μ with compact support in $\mathbb{C}$ such that for $p, q \in \mathbb{C}[x, y]$,*

$$(p, q) = \operatorname{tr} i\,[p(X,Y), q(X,Y)] = \iint_{\mathbb{C}} J(p,q) d\mu,$$

where

$$J(p,q) = \frac{\partial p}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial p}{\partial y}\frac{\partial q}{\partial x}.$$

Soon after the discovery of the Helton–Howe formula, Pincus established that the measure μ in the Helton–Howe formula is mutually absolutely continuous with respect to the area measure $dxdy$, that is, $d\mu = g_T(x,y)dxdy$, where g_T is the principal function of the operator $T = X + iY$.

2.3. UNITARY INVARIANTS

For z, w in a neighbourhood of infinity, the operator valued determining function $E(z, w)$ of an irreducible pure hyponormal operator T of trace class is a complete unitary invariant of T. The principal function g_T, on the other hand, is a unitary invariant in general but it is a complete invariant when the rank of $[T^*, T]$ is 1, see [18]. Indeed, we describe a large class of operators showing that the principal function need not be a complete invariant if we drop the requirement: rank of $[T^*, T]$ is 1. A different unitary invariant is in [10]. In what follows, we assume that the operator T is a irreducible hyponormal (hence, pure), and that rank of $[T^*, T] = 1$. Thus, we assume without loss of generality that $[T^*, T] = x \otimes x$ for some $x \in \mathcal{H}$, where $x \otimes x$ denotes the *non-negative definite* rank one operator $h \mapsto \langle h, x \rangle x$, $h \in \mathcal{H}$. In this case the multiplicative commutator and therefore, the determining function E of the operator T can be calculated explicitly as follows.

For any pair of complex numbers z, w not in the spectrum of T, the operators $(T^* - \bar{w})^{-1}$ and $(T - z)^{-1}$ exist and the multiplicative commutator

$$(T - z)(T^* - \bar{w})(T - z)^{-1}(T^* - \bar{w})^{-1},$$

is in the *determinant class*, that is, it is of the form $I + K$, where K is in trace class:

$$\begin{aligned}
&(T - z)(T^* - \bar{w})(T - z)^{-1}(T^* - \bar{w})^{-1} \\
&= ((T^*T - x \otimes x - zT^* - \bar{w}T + z\bar{w})(T - z)^{-1}(T^* - \bar{w})^{-1}) \\
&= ((T^* - \bar{w})(T - z)(T - z)^{-1}(T^* - \bar{w})^{-1} - (x \otimes x)(T - z)^{-1}(T^* - \bar{w})^{-1}) \\
&= I - (x \otimes x)(T - z)^{-1}(T^* - \bar{w})^{-1} \\
&= I + K,
\end{aligned}$$

where

$$K = -(x \otimes x)(T - z)^{-1}(T^* - \bar{w})^{-1}$$

is in trace class, and moreover,

$$\operatorname{tr} K = -\langle (T^* - \bar{w})^{-1}x, (T^* - \bar{z})^{-1}x\rangle.$$

Therefore,

$$\begin{aligned}
\det(I - K) &= \exp\big(\operatorname{tr}\log(I - K)\big) \\
&= \exp\Big(\operatorname{tr}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j}(-K)^j\Big) \\
&= \exp\Big(\sum_{j=1}^{\infty}\frac{(-1)^{2j+1}}{j}\operatorname{tr}(K^j)\Big) \\
&= \exp\Big(\sum_{j=1}^{\infty}\frac{(-1)^{2j+1}}{j}(\operatorname{tr}(K))^j\Big) \\
&= \exp\Big(\log(1 - \operatorname{tr}(K))\Big) \\
&= 1 - \operatorname{tr}(K) \\
&= 1 - \langle (T^* - \bar{w})^{-1}x, (T^* - \bar{z})^{-1}x\rangle.
\end{aligned}$$

Consequently, combining with the formula (2.1), we have the equality

$$1 - \langle (T^* - \bar{w})^{-1}x, (T^* - \bar{z})^{-1}x\rangle = \exp\left(-\frac{1}{\pi}\int\limits_{\mathbb{C}}\frac{g_T(\zeta)}{(\zeta - z)(\bar{\zeta} - \bar{w})}dA(\zeta)\right). \qquad (2.2)$$

For a different approach to establishing this formula, see [8, Theorem 4.3].

2.4. AN EXAMPLE

Let S be the unilateral shift operator acting on the Hilbert space ℓ_2 of square summable complex sequences by the rule: $Se_k = e_{k+1}$, where $\{e_0, e_1, e_2, \ldots\}$ is the standard basis of ℓ_2. The self-commutator $[S^*, S] = e_0 \otimes e_0$. Since $(S^* - \bar{w}I)^{-1}e_0 = -\frac{1}{\bar{w}}e_0$, we have that

$$1 - \langle (S^* - \bar{w}I)^{-1}e_0, (S^* - \bar{z}I)^{-1}e_0 \rangle = 1 - \left\langle -\frac{1}{\bar{w}}e_0, -\frac{1}{\bar{z}}e_0 \right\rangle = 1 - \frac{1}{z\bar{w}}.$$

We claim that

$$\exp\left(-\frac{1}{\pi} \int\limits_{\bar{\mathbb{D}}} \frac{1}{(\zeta - z)(\bar{\zeta} - \bar{w})} dA(\zeta) \right) = 1 - \frac{1}{z\bar{w}}.$$

Taking $|\zeta| \le 1$, and $|z|, |w| > 1$, and expanding $\frac{1}{\zeta - z}$ as well as $\frac{1}{\bar{\zeta} - \bar{w}}$ in a power series of $\frac{\zeta}{z}$ and $\frac{\bar{\zeta}}{\bar{w}}$, respectively, the claim is verified by integrating the product term by term. Thus the principal function of the unilateral shift S is the characteristic function $\mathbb{1}_{\bar{\mathbb{D}}}$ of the closed unit disc $\bar{\mathbb{D}}$.

Remark 2.3. Let $\sigma_{\text{ess}}(T)$ be the essential spectrum of an operator T. For $\lambda \in \mathbb{C} \backslash \sigma_{\text{ess}}(T)$, the principal function $g(\lambda) = -\operatorname{ind}(T - \lambda)$, see [9, 5^0, p. 105]. Consequently, the principal function g_S of the unilateral shift S is $\mathbb{1}_{\bar{\mathbb{D}}}$.

3. THE ACTION OF THE MÖBIUS GROUP

The hyponormal operators share an important property with normal operators, namely, the spectral radius $\rho(T)$ of a hyponormal operator equals its norm $\|T\|$. However, unlike normal operators, if T is a *pure* hyponormal operators, then by Putnam's inequality, the area measure of spectrum $\sigma(T)$ must be positive.

3.1. INVARIANCE

It is not hard to verify that if T is hyponormal, then $\varphi(T)$ is also hyponormal for any φ in Möb, the biholomorphic automorphism group of the unit disc $\mathbb{D}$. We reproduce the proof below from [24, Lemma 1].

Proposition 3.1 (Stampfli)**.** *If T is hyponormal, then $\varphi(T)$, φ in Möb, is also hyponormal.*

Proof. Any Möbius transformation is a composition of an affine transformation and an inversion of some other affine transformation. We have

$$[(aT + b)^*, aT + b)] = |a|^2[T^*, T] \ge 0.$$

Therefore, to complete the proof, it is enough to verify that $[(T^*)^{-1}, T^{-1}]$ is hyponormal for an invertible operator T. By hypothesis, we have that

$$0 \leqslant T^{-1}(T^*T - TT^*)(T^*)^{-1} = T^{-1}T^*T(T^*)^{-1} - I.$$

If A is invertible and $A \geqslant I$, then $A^{-1} \leqslant I$. Therefore,

$$I - T^*T^{-1}(T^*)^{-1}T \geqslant 0.$$

Hence,

$$\begin{aligned}[(T^*)^{-1}, T^{-1}] &= ((T^*)^{-1}T^{-1} - T^{-1}(T^*)^{-1}) \\ &= (T^*)^{-1}(I - T^*T^{-1}(T^*)^{-1}T)T^{-1} \geqslant 0\end{aligned}$$

completing the proof of the proposition. □

We now re-write the formula for the tracial bi-linear form in complex co-ordinates and in slightly greater generality, see [16, Chapter X, Theorem 2.4, and Equation (12), p. 242].

Theorem 3.2 (Carey–Helton–Howe–Pincus). *Suppose that $T \in \mathcal{L}(\mathcal{H})$ is a hyponormal operator with $[T^*, T]$ is in the trace class $\mathcal{S}_1(\mathcal{H})$. Then for any pair of functions p, q in the Frechet Space $C^\infty(\sigma(T))$ of all smooth functions on $\sigma(T)$, we have the equality*

$$\operatorname{tr}[p(T, T^*), q(T, T^*)] = \frac{1}{\pi} \int\limits_{\sigma(T)} J(p, q) g_T d\mu,$$

where

$$J(p, q) := \frac{\partial p}{\partial \bar{z}} \frac{\partial q}{\partial z} - \frac{\partial p}{\partial z} \frac{\partial q}{\partial \bar{z}}.$$

The exact relationship of this formula to that of the Helton–Howe formula given earlier is obtained by choosing $X = \operatorname{Re} T$, $Y = \operatorname{Im} T$; and converting the complex derivatives in $J(p, q)$ to real ones by using transformation from complex to real coordinates:

$$\frac{\partial f}{\partial z} = \frac{1}{2}\Big(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\Big), \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big)$$

and vice-versa.

The proof of the lemma below follows directly from the Carey–Helton–Howe–Pincus formula.

Lemma 3.3. *Suppose that $T \in \mathcal{L}(\mathcal{H})$ is a hyponormal operator and $[T^*, T]$ is in the trace class $\mathcal{S}_1(\mathcal{H})$ and that $\sigma(T) \subseteq \bar{\mathbb{D}}$. Then $[(T^* - \bar{\lambda})^{-1}, (T - \lambda)^{-1}]$ is also in $\mathcal{S}_1(\mathcal{H})$ for $\lambda \notin \sigma(T)$. In particular, $[\varphi(T)^*, \varphi(T)]$ is in $\mathcal{S}_1(\mathcal{H})$ for any $\varphi \in$ Möb.*

Proof. Pick $p(\zeta,\overline{\zeta}) = \frac{1}{\overline{\zeta}-\overline{\lambda}}$ and $q(\zeta,\overline{\zeta}) = \frac{1}{\zeta-\lambda}$. Then $J(p,q) = \frac{1}{|\zeta-\lambda|^4} < k$ for some $k > 0$ since $\lambda \notin \overline{\mathbb{D}}$. Therefore,

$$\begin{aligned}\operatorname{tr}[(T^* - \overline{\lambda})^{-1}, (T-\lambda)^{-1}] &= \operatorname{tr}[p(T,T^*), q(T,T^*)] \\ &= \frac{1}{\pi}\int\limits_{\sigma(T)} \frac{1}{|\zeta-\lambda|^4} g_T d\mu \\ &\leqslant \frac{k}{\pi}\|g_T\|_{L^1(\sigma(T))} < \infty.\end{aligned}$$

Since affine transform of a trace class operator is again in trace class, the proof is complete. ☐

We now compute the self commutator of the operator $\varphi(T)$. For this, we note that $\varphi(z) = \frac{z-a}{1-\bar{a}z} = -(\bar{a})^{-1} + c(z-\bar{a}^{-1})^{-1}$, where $c = \frac{a-\bar{a}^{-1}}{\bar{a}}$. Then

$$\begin{aligned}[\varphi(T)^*, \varphi(T)] &= [-a^{-1}I + \overline{c}(T^* - a^{-1})^{-1}, -\overline{a}^{-1}I + c(T-\overline{a}^{-1})^{-1}] \\ &= (-a^{-1}I + \overline{c}(T^* - a^{-1})^{-1})(-\overline{a}^{-1}I + c(T-\overline{a}^{-1})^{-1}) \\ &\quad - (-\overline{a}^{-1}I + c(T-\overline{a}^{-1})^{-1})(-a^{-1}I + \overline{c}(T^* - a^{-1})^{-1}) \\ &= |c|^2((T^*-a^{-1})^{-1}(T-\overline{a}^{-1})^{-1} - (T-\overline{a}^{-1})^{-1}(T^*-a^{-1})^{-1}) \\ &= |c|^2\big((T-\overline{a}^{-1})(T^*-a^{-1})\big)^{-1}[(T^*-a^{-1}), (T-\overline{a}^{-1})] \\ &\quad \big((T^*-a^{-1})(T-\overline{a}^{-1})\big)^{-1} \\ &= |c|^2\big((T-\overline{a}^{-1})(T^*-a^{-1})\big)^{-1}[T^*, T]\big((T^*-a^{-1})(T-\overline{a}^{-1})\big)^{-1}.\end{aligned}$$

The computation of $[\varphi(T)^*, \varphi(T)]$ facilitates the proof of the lemma below.

Lemma 3.4. *Suppose that $T \in \mathcal{L}(\mathcal{H})$ is a hyponormal operator and the rank of $[T^*, T]$ is* 1 *and that $\sigma(T) \subseteq \overline{\mathbb{D}}$. Then the rank of the self-commutator $[\varphi(T)^*, \varphi(T)]$ is also* 1.

Proof. For the proof, in view of the preceding discussion, it is enough to verify that whenever T is an invertible operator with $\operatorname{rank}[T^*, T] = 1$, the rank of $[T^{*-1}, T^{-1}]$ is also 1. By hypothesis, $[T^*, T] = x \otimes x$ for some vector x in $\mathcal{H}$. Hence,

$$\begin{aligned}T^{-1}T^*T(T^*)^{-1} &= I + T^{-1}(T^*T - TT^*)(T^*)^{-1} \\ &= I + T^{-1}(x\otimes x)(T^*)^{-1}.\end{aligned}$$

Taking inverses on both sides, we have

$$\begin{aligned}T^{-1}(T^*)^{-1} &= (T^*)^{-1}[I + T^{-1}(x\otimes x)(T^*)^{-1}]^{-1}T^{-1} \\ &= (T\{I + T^{-1}(x\otimes x)(T^*)^{-1}\}T^*)^{-1} \\ &= (TT^* + x\otimes x)^{-1}.\end{aligned}$$

Similarly,

$$(T^*)^{-1}T^{-1} = (T^*T - x\otimes x)^{-1}.$$

Therefore,

$$\begin{aligned}[(T^*)^{-1}, T^{-1}] &= (T^*)^{-1}T^{-1} - T^{-1}(T^*)^{-1}\\ &= (T^*T - x\otimes x)^{-1} - (TT^* + x\otimes x)^{-1}\\ &= (T^*T - x\otimes x)^{-1}(\{TT^* + x\otimes x\} - \{T^*T - x\otimes x\})\\ &\quad (TT^* + x\otimes x)^{-1}\\ &= (T^*T - x\otimes x)^{-1}[x\otimes x](TT^* + x\otimes x)^{-1}\\ &= (TT^*)^{-1}(x\otimes x)(T^*T)^{-1}.\end{aligned}$$

It follows that the self-commutator of T^{-1} is also of rank one completing the proof. □

Remark 3.5. Combining Proposition 3.1 and Lemma 3.4, we conclude that the set of *pure* hypnormal operators with rank 1 self-commutator is left invariant under the action of the Möbius group. Similarly, combining Proposition 3.1, this time with Lemma 3.3, we see that the set of *pure* hypnormal operators T with $\|[T^*, T]\|_1$ finite is also left invariant under the action of Möb.

3.2. A CHANGE OF VARIABLE FORMULA FOR THE PRINCIPAL FUNCTION

A change of variable formula for the principal function appears in [9, pp. 106–107] and also in [16, p. 245]. However, for our purposes, we need a change of variable formula for the principal function in the form given below.

Proposition 3.6. *Let T be a pure hyponormal operator with trace class self-commutator and set $W := \varphi(T)$, φ in Möb. Assume that the spectrum of T is contained in the closed unit disc. Then the relationship between the two principal functions g_T and g_W is given by the change of variable formula*

$$g_W(\zeta) = g_T(\varphi^{-1}(\zeta)), \quad \zeta \in \sigma(W).$$

Proof. We have proved that W is a hyponormal operator with $\|[W^*, W]\|_1 < \infty$. We note that $\varphi(T)^* = \varphi^*(T^*)$, where $\varphi^*(z) = \overline{\varphi(\bar{z})}$. Setting

$$\tilde{p}(z, \bar{z}) := p(\varphi(z), \overline{\varphi(z)}) \quad \text{and} \quad \tilde{q}(z, \bar{z}) := q(\varphi(z), \overline{\varphi(z)}),$$

we have that

$$\operatorname{tr}[p(\varphi(T), \varphi(T)^*), q(\varphi(T), \varphi(T)^*)] = \frac{1}{\pi}\int_{\sigma(\varphi(T))} J(p, q) g_{\varphi(T)}(\zeta) dA(\zeta).$$

On the other hand,

$$\begin{aligned}\operatorname{tr}[p(\varphi(T), \varphi(T)^*), q(\varphi(T), \varphi(T)^*)] &= \operatorname{tr}[\tilde{p}(T, T^*), \tilde{q}(T, T^*)]\\ &= \frac{1}{\pi}\int_{\sigma(T)} J_\zeta(\tilde{p}, \tilde{q}) g_T(\zeta) dA(\zeta)\\ &= \frac{1}{\pi}\int_{\sigma(\varphi(T))} J_\eta(p, q) g_T(\varphi^{-1}(\eta)) dA(\eta),\end{aligned}$$

where $\eta = \varphi(\zeta)$. By the chain rule, we have $\frac{\partial \tilde{p}}{\partial \zeta} = \frac{\partial \tilde{p}}{\partial \bar{\eta}} \frac{\partial \bar{\varphi}}{\partial \zeta}$, and similarly $\frac{\partial \tilde{p}}{\partial \zeta} = \frac{\partial \tilde{p}}{\partial \eta} \frac{\partial \varphi}{\partial \zeta}$. Thus, we have the equality

$$J_\zeta(\tilde{p}, \tilde{q}) = J_\eta(p, q)\Big(\frac{\partial(\overline{\varphi(\zeta)})}{\partial \bar{\zeta}} \frac{\partial(\varphi(\zeta))}{\partial \zeta}\Big).$$

Consequently,

$$\begin{aligned} dA(\eta) &= -\frac{1}{2i} d\eta \wedge d\overline{\eta} \\ &= -\frac{1}{2i}\Big(\frac{\partial(\overline{\varphi(\zeta)})}{\partial \bar{\zeta}} \frac{\partial(\varphi(\zeta))}{\partial \zeta}\Big) d\zeta \wedge d\bar{\zeta} \\ &= \Big(\frac{\partial(\overline{\varphi(\zeta)})}{\partial \bar{\zeta}} \frac{\partial(\varphi(\zeta))}{\partial \zeta}\Big) dA(\zeta). \end{aligned}$$

Hence,

$$J_\zeta(\tilde{p}, \tilde{q}) dA(\zeta) = J_\eta(p, q) dA(\eta).$$

Since p and q are arbitrary C^∞ functions on $\sigma(T)$, we conclude that

$$g_{\varphi(T)}(\zeta) = g_T(\varphi^{-1}(\zeta))$$

completing the proof. □

4. HOMOGENEOUS HYPONORMAL OPERATORS T WITH $\operatorname{rank}[T^*, T] = 1$

We have already remarked that the principal function of a pure hyponornmal operator in the trace class $\mathcal{S}_1(\mathcal{H})$ is not a complete unitary invariant for the operator T in general. However, it is not hard to see that it is a unitary invariant.

Proposition 4.1. *Let T be a pure hyponormal operator in $\mathcal{S}_1(\mathcal{H})$. If W is an operator unitarily equivalent to T, then the principal functions of W and T coincide.*

Proof. Let $W = UTU^*$ for some unitary operator U. The operator W is hyponormal and is in $\mathcal{S}_1(\mathcal{H})$. For any polynomial $p \in \mathbb{C}[z, \bar{z}]$, we have $p(W, W^*) = Up(T, T^*)U^*$. Hence, by the Helton–Howe formula, we find that

$$\begin{aligned} \frac{1}{\pi} \int\limits_{\sigma(W)} J(p, q) g_W(\zeta) dA(\zeta) &= \operatorname{tr}[p(W, W^*), q(W, W^*)] \\ &= \operatorname{tr}[Up(T, T^*)U^*, Uq(T, T^*)U^*] \\ &= \operatorname{tr}(U[p(T, T^*), q(T, T^*)]U^*) \\ &= \operatorname{tr}[p(T, T^*), q(T, T^*)] \\ &= \frac{1}{\pi} \int\limits_{\sigma(T)} J(p, q) g_T(\zeta) dA(\zeta) \end{aligned}$$

Since $\sigma(T) = \sigma(W)$, we have that

$$\frac{1}{\pi} \int\limits_{\sigma(T)} J(p,q)(g_T - g_W)(\zeta) dA(\zeta) = 0$$

for p, q in $\mathbb{C}[x,y]$, and in consequence $g_W = g_T$. □

Imposing the condition of homogeneity on a pure hyponormal opeartor T in $\mathcal{S}_1(\mathcal{H})$, we investigate what happens to the principal function g_T.

We begin with the simple observation that if T is a homogeneous operator, then by the spectral mapping theorem, the spectrum $\sigma(T)$ must be invariant under the action of the Möbius group. Consequently, $\sigma(T)$ has to be either the closed unit disc $\overline{\mathbb{D}}$, or the unit circle $\mathbb{T}$. However, if T is also a pure hyponormal operator, then as we have noted earlier, $\sigma(T)$ cannot be $\mathbb{T}$.

Proposition 4.2. *Suppose that T is a pure hyponormal homogeneous operator such that $[T^*, T]$ is in $\mathcal{S}_1(\mathcal{H})$. Then the principal function g_T is constant on the spectrum $\sigma(T)$.*

Proof. Since $\varphi(T)$ is unitarily equivalent to T, $\varphi \in$ Möb, it follows that $g_T(z) = g_{\phi(T)}(z)$. By the change of variable formula for the principal function, we have $g_{\phi(T)}(z) = g_T(\phi^{-1}(z))$. Combining these two equalities, we conclude that

$$g_T(z) = g_{\varphi(T)}(z) = g_T(\varphi^{-1}(z)), \tag{4.1}$$

for all $\varphi \in$ Möb. For a fixed but arbitrary $z \in \mathbb{D}$, pick a Möbius transformation φ_z with the property: $\varphi_z(0) = z$. Using this φ_z in Equation (4.1), we have

$$g_T(z) = g_{\phi_z(T)}(z) = g_T({\phi_z}^{-1}(z)) = g_T(0).$$

We therefore conclude that g_T must be a constant on $\sigma(T)$, $0 < g_T(0) \le 1$. □

We have now all the tools to prove the only new result of this short note. Let us recapitulate what we have proved so far. Assume that T is a pure homogeneous hyponormal operator with rank of $[T^*, T] = 1$. Then for such an operator T we must have that:

(i) the spectrum $\sigma(T) = \overline{\mathbb{D}}$,
(ii) the principal function g_T must be a constant, moreover, this constant value is in $(0, 1]$,
(iii) if $T = S$ is the unilateral shift, then $g_S = \mathbb{1}_{\overline{\mathbb{D}}}$, see Example 2.4.

Finally, note that:

(a) the unilateral shift S is a homogeneous, see [3, List 4.1(2)], pure hyponormal operator and $\operatorname{rank}([S^*, S]) = 1$,
(b) the principal function of a pure hyponormal operator T with rank-one self-commutator $x \otimes x$ is a complete unitary invariant of T.

The remarkable assertion of (b) is due to Pincus and is in [18].

Moreover, in the case of a homogeneous pure hyponormal operator T with $\operatorname{rank}[T^*,T] = 1$, the spectrum $\sigma(T) = \overline{\mathbb{D}}$. Since the support of g_T equals $\sigma(T)$, therefore to find homogeneous pure hyponormal operators T with $\operatorname{rank}[T^*,T] = 1$, we have to check if the operator with $0 < g_T(0) \leqslant 1$ is homogeneous. From what is said so far, it follows that the unilateral shift S is the unique (modulo unitary equivalence) pure hyponormal operator with $\operatorname{rank}([S^*,S]) = 1$ such that $g_S = \mathbb{1}_{\overline{\mathbb{D}}}$.

Theorem 4.3. *The only homogeneous pure hyponormal operator T with* rank *of* $[T^*,T] = 1$*, modulo unitary equivalence, is the unilateral shift.*

Proof. In view of the discussion preceding the theorem, we have to show that there is no homogeneous *pure* hyponormal operator T with rank of $[T^*,T] = 1$ such that $g_T = c < 1$. Let us suppose to the contrary that there exists such an operator T with $g_T = c < 1$. In the determinant expansion formula (2.2), setting $g_T = c$, we have (as in Example 2.4):

$$\begin{aligned} 1 - \langle (T^* - \bar{w})^{-1}x, (T^* - \bar{z})^{-1}x\rangle &= \exp\Big(-\frac{1}{\pi} \int\limits_{\sigma(T)} \frac{g_T(\zeta)dA(\zeta)}{(\zeta - z)(\bar{z} - \bar{w})}\Big) \\ &= \Big(1 - \frac{1}{z\bar{w}}\Big)^c. \end{aligned} \tag{4.2}$$

Putting $z = w$ in Equation (4.2) we have the equality

$$1 - \|(T^* - \overline{w})^{-1}x\|^2 = \Big(1 - \frac{1}{|w|^2}\Big)^c. \tag{4.3}$$

Since T is homogeneous and hyponormal, the spectrum $\sigma(T)$ can only be $\overline{\mathbb{D}}$, the possibility of $\sigma(T) = \mathbb{T}$ is ruled out by Putnam's inequality. For a hyponormal operator, the spectral radius $\rho(T) = \|T\|$ and we conclude that that

$$\|(T - wI)^{-1}\| = \rho((T - wI)^{-1}) \leqslant \frac{1}{|w|}.$$

Since $[T^*,T] = x \otimes x$ for some $x \in \mathcal{H}$ by hypothesis, taking $p(z,\bar{z}) = \bar{z}$ and $q(z,\bar{z}) = z$ in the Helton–Howe formula we conclude that $\|x\| = \sqrt{c}$. Therefore,

$$\|(T^* - \overline{w})^{-1}x\| \leqslant \sqrt{c}\|(T^* - \overline{w})^{-1}\|,$$

and we conclude that

$$\begin{aligned} 1 - \|(T^* - \overline{w})^{-1}x\|^2 &\geq 1 - \|(T^* - \overline{w})^{-1}\|^2\|x\|^2 \\ &= 1 - c\|(T^* - \overline{w})^{-1}\|^2 \\ &\geqslant 1 - \frac{c}{|w|^2}. \end{aligned} \tag{4.4}$$

Combining the equality (4.3) with the inequality (4.4), we have

$$\Big(1 - \frac{c}{|w|^2}\Big) \leqslant \Big(1 - \frac{1}{|w|^2}\Big)^c, \quad |w| > 1. \tag{4.5}$$

It is easy to verify that the inequality (4.5) is false unless $c = 1$ completing the proof. □

We give an example of a class $\mathcal{T}$ consisting of unitarily inequivalent homogeneous hyponormal operators such that $[T^*, T] \in \mathcal{S}_1$, rank $[T^*, T] = \infty$ such that $g_{T_1} = g_{T_2}$ for every pair of operators $T_1, T_2 \in \mathcal{T}$. Also, see remark below Lemma 1 in [22, p. 252].

Let $\mathcal{T}$ be the set of weighted shift operator $\{T_\lambda : \lambda > 1\}$ with weight sequences $\{w_n(\lambda)\}_{n\geqslant 0}$, $w_n(\lambda) = \sqrt{\frac{n+1}{n+\lambda}}$. For $\lambda > 1$, the weight sequence $\{w_n(\lambda)\}$ is strictly increasing and hence T_λ is hyponormal. The operator T_λ is also pure and cyclic. Clearly, $[T_\lambda^*, T_\lambda]$ is a diagonal operator D_λ with

$$D_\lambda(0,0) = w_0 \quad \text{and} \quad D_\lambda(i,i) = w_{i+1}^2(\lambda) - w_i^2(\lambda)) \neq 0.$$

Thus, $\operatorname{rank}[T^*, T] = \infty$. Moreover,

$$\operatorname{tr}[T_\lambda^*, T_\lambda] = \sum_{i=0}^{\infty}(w_{i+1}^2(\lambda) - w_i^2(\lambda)) + w_0^2(\lambda) = 1.$$

For $\lambda_1 \neq \lambda_2$, the two operators T_{λ_1} and T_{λ_2} are unitarily inequivalent. But all these operators are homogeneous, see [3]. Therefore, the principal function g_{T_λ} is constant, say c, on $\bar{\mathbb{D}}$. But then

$$1 = \operatorname{tr}[T_\lambda^*, T_\lambda] = \frac{1}{\pi}\int\limits_{\bar{\mathbb{D}}} c\, dA(\zeta).$$

Thus, $c = 1$ and it follows that g_{T_λ} is identically 1 on $\bar{\mathbb{D}}$ for all $\lambda > 1$.

4.1. OPEN PROBLEM

Find all the pure hyponormal operators T such that $[T^*, T]$ is in $\mathcal{S}_1(\mathcal{H})$ and that g_T is constant on $\sigma(T)$ modulo unitary equivalence.

Remark 4.4. In studying homogeneous contractions T assuming that both the defect indices of T are equal to 1, it was shown that the Sz.-Nagy–Foias characteristic function of T must be constant. This observation leads to a class of homogeneous bi-lateral shifts (all of them inequivalent among themselves), parametrized by $c > 0$, possessing a constant characteristic function, see [1, 11].

Similarly, homogeneous operators T in the Cowen-Douglas class $B_1(\mathbb{D})$ are determined by specifying the curvature $\lambda = -\mathcal{K}_T(0) > 0$ just at one point. From this, one infers that an operator T in $B_1(\mathbb{D})$ is homogeneous if and only if T is of the form T_λ^*, $\lambda > 0$, discussed above (see [17]).

The situation involving the hyponormal operators T with $\operatorname{rank}[T^*, T] = 1$, appears to be very different. Here again, the unitary invariant g_T, under the assumption of homogeneity, is a constant function, say c, with $0 < c \leqslant 1$. But there is only one *homogeneous* hyponormal operator T with $[T^*, T] = x \otimes x$, namely, the unilateral shift corresponding to $c = 1$.

4.2. POSTSCRIPT

In a conversation with the second author, in the year 1983, Kevin Clancey had remarked that the only homogeneous *pure* hyponormal operator with rank 1 self-commutator might be the unilateral shift. We have verified this statement to be correct in this short note.

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