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STUDY OF FRACTIONAL SEMIPOSITONE PROBLEMS ON $\mathbb{R}^N$

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Abstract. Let $s \in (0,1)$ and $N > 2s$. In this paper, we consider the following class of nonlocal semipositone problems:

$$(-\Delta)^s u = g(x) f_a(u) \text{ in } \mathbb{R}^N, \quad u > 0 \text{ in } \mathbb{R}^N,$$

where the weight $g \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is positive, $a > 0$ is a parameter, and $f_a \in \mathcal{C}(\mathbb{R})$ is strictly negative on $(-\infty, 0]$. For f_a having subcritical growth and weaker Ambrosetti–Rabinowitz type nonlinearity, we prove that the above problem admits a mountain pass solution u_a, provided a is near zero. To obtain the positivity of u_a, we establish a Brezis–Kato type uniform estimate of (u_a) in $L^r(\mathbb{R}^N)$ for every $r \in [\frac{2N}{N-2s}, \infty]$.

Keywords: semipositone problems, fractional operator, uniform regularity estimates, positive solutions.

Mathematics Subject Classification: 35R11, 35J50, 35B65, 35B09.

1. INTRODUCTION

In this present paper, we deal with a class of nonlocal semipositone problems on $\mathbb{R}^N$. Precisely, for $s \in (0,1)$ and $N > 2s$, we consider the following nonlocal semilinear equation:

$$(-\Delta)^s u = g(x) f_a(u) \text{ in } \mathbb{R}^N, \tag{SP}$$

where the weight $g \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is positive, $a > 0$ is a parameter, and $f_a \in \mathcal{C}(\mathbb{R})$ has the following form:

$$f_a(t) = \begin{cases} f(t) - a, & \text{if } t \geq 0, \\ -a, & \text{if } t \leq 0, \end{cases} \text{ with } f \in \mathcal{C}(\mathbb{R}^+) \text{ satisfying } f(0) = 0.$$

Moreover, f satisfies the following hypothesis:

$(f1)$ $\lim_{t\to 0} \frac{f(t)}{t} = 0$, $\lim_{t\to\infty} \frac{f(t)}{t} = \infty$, and $\lim_{t\to\infty} \frac{f(t)}{t^{\gamma-1}} \le C(f)$ for some $\gamma \in (2, 2_s^*)$ and $C(f) > 0$,

$(f2)$ there exists $R > 0$ such that $\frac{f(t)}{t}$ is increasing for $t > R$,

where $2_s^* = \frac{2N}{N-2s}$ is the critical fractional exponent. The linear operator $(-\Delta)^s$ is called the fractional Laplacian defined as

$$(-\Delta)^s u(x) := 2 \lim_{\epsilon\to 0^+} \int\limits_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \, \mathrm{d}y, \quad x \in \mathbb{R}^N,$$

where $B_\epsilon(x)$ denotes the ball of radius ϵ and centred at x. Due to the presence of the strictly negative quantity on the R.H.S. of (SP) in the regions where $u \le 0$ and certain portion of $u > 0$, the problem (SP) is called *semipositone* in the literature. Semipositone problems have applications in mathematical physics, biology, engineering etc. More preciously, in the logistic equation, mechanical systems, suspension bridges, population model, etc.; see for example [25, 28].

In the local case, the semipositone problems were first observed by Brown and Shivaji in [7] while studying the perturbed bifurcation problem $-\Delta u = \lambda(u - u^3) - \epsilon$ in Ω, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$, where $\lambda, \epsilon > 0$ and Ω is a bounded domain. In this work, the authors used the sub-super solution method to get positive solutions. Observe that $u = 1$ is a supersolution for this problem since the R.H.S. of the equation is negative at $u = 1$. To obtain an appropriate positive subsolution, the authors used the anti-maximum principle due to Clément and Peletier. Later, many authors studied the following semipositone problem on a bounded domain Ω:

$$-\Delta u = \lambda f(u) \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad \text{and} \quad u = 0 \text{ on } \partial\Omega, \tag{1.1}$$

where $\lambda > 0$, $f : \mathbb{R}^+ \to \mathbb{R}$ is continuous, increasing and $f(0) < 0$. For example, we refer [10–12, 14, 17] where various growth conditions and nonlinearities on the function f are imposed to find the existence of positive solutions for (1.1). In [1], Alves *et al.* considered the following semipositone problem:

$$-\Delta u = g(x) f_a(u) \text{ in } \mathbb{R}^N, \tag{1.2}$$

with $f_a(t) = f(t) - a$ for $t > 0$, $f_a(t) = -a(t+1)$ for $t \in [-1, 0]$, and $f_a(t) = 0$ for $t \le -1$, where the function $f \in \mathcal{C}(\mathbb{R}^+)$ satisfies $f(0) = 0$, is locally Lipschitz, has superlinear growth conditions and Ambrosetti–Rabinowitz (see [2]) type nonlinearities. Meanwhile, the weight function g is assumed to be positive, radial, lies in $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and satisfies the following bound:

$$|x|^{N-2} \int\limits_{\mathbb{R}^N} \frac{g(|y|)}{|x-y|^{N-2}} \, \mathrm{d}y \le C(g), \quad \text{for } x \in \mathbb{R}^N \setminus \{0\}, \text{ where } C(g) > 0. \tag{1.3}$$

The authors used the regularity estimate by Brezis and Kato in [6] and the Reisz potential for the Laplace operator to establish uniform boundedness of mountain pass solutions of (1.2) in $L^\infty(\mathbb{R}^N)$, with respect to the parameter a near zero. The authors then obtained a positive solution of (1.2) using this uniform regularity estimate, the strong maximum principle, and the condition (1.3). For more results related to semipositone problems, we refer [9, 16] and the references therein.

Now, we shift our discussion to the nonlocal case. In the past years, a significant amount of attention has been given to the study of fractional laplacian due to its numerous applications in mathematical physics, engineering and related fields. For example, in linear drift-diffusion equations [15], image denoising [21], quasi-geostrophic flows [3], and bound-state problems [26], to name a few. Several authors recently studied the nonlocal semipositone problems on a bounded domain Ω. In [18], the authors considered a multiparameter fractional semipositone problem $(-\Delta)^s u = \lambda(u^q - 1) + \mu u^r$ in Ω, $u > 0$ in Ω, $u = 0$ in $\mathbb{R}^N \setminus \Omega$, where $\lambda, \mu > 0$ are parameters, $N > 2s$, and $0 < q < 1 < r \leq 2_s^* - 1$. For a certain range of λ and μ, the authors proved the existence of a positive solution for this problem. Their proof relies on the construction of a positive subsolution. Later, in [23], the authors studied the nonlocal nonlinear semipositone problem $(-\Delta)_p^s u = \lambda f(u)$ in Ω, $u = 0$ in $\mathbb{R}^N \setminus \Omega$, where $(-\Delta)_p^s$ is the fractional p-Laplace operator, $\lambda > 0$, $f \in \mathcal{C}(\mathbb{R})$ has superlinear, subcritical growth and $f(s) = 0$ for $s \leq -1$. The authors obtained at least one positive solution provided the parameter λ is sufficiently small. Their proof uses regularity results up to the boundary of Ω and Hopf's Lemma for $(-\Delta)_p^s$. To our knowledge, nonlocal semipositone problems on an unbounded domain have not been studied yet.

In this paper, we consider the nonlocal counterpart of $-\Delta u = g(x)(f(u) - a)$ in $\mathbb{R}^N$, $u > 0$ in $\mathbb{R}^N$ (studied in [1]). On the weight function g, we impose a nonlocal analogue of (1.3). With subcritical, superlinear and without Ambrosetti–Rabinowitz growth conditions (see [2]) for f, our primary concern is to establish the existence of positive solution to (SP), depending on the parameter a. The techniques used in [18, 23] to get positive solution are not adoptable in this context. Our procedure to find non-negative solution for (SP) is motivated by [1], where the uniform regularity estimate (with respect to a) of solutions in $L^\infty(\mathbb{R}^N)$ plays a major role. In [20, Proposition 5.1.1], for $g \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, and $|f(x,t)| \leq C(1 + |t|^p)$; $1 \leq p \leq 2_s^* - 1$, the authors proved that every non-negative solution to the problem $(-\Delta)^s u = g(x) f(x,u)$ in $\mathbb{R}^N$ is bounded. However, this regularity result is not applicable in our situation. Also, the Brezis–Kato type regularity estimate for weak solution to (SP) is unknown.

We consider the homogeneous fractional Sobolev space $\dot{H}^s(\mathbb{R}^N)$ (introduced in [20]) defined as

$$\dot{H}^s(\mathbb{R}^N) := \text{closure of } C_c^1(\mathbb{R}^N) \text{ with respect to } \|\cdot\|_{\dot{H}^s(\mathbb{R}^N)},$$

where $\|\cdot\|_{\dot{H}^s(\mathbb{R}^N)} := [\cdot]_{s,2} + \|\cdot\|_{L^{2_s^*}(\mathbb{R}^N)}$, and

$$[u]_{s,2} := \left(\iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} \,\mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}},$$

is the Gagliardo seminorm. A function $u \in \dot{H}^s(\mathbb{R}^N)$ is a weak solution of (SP) if the following identity holds:

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y)(\phi(x) - \phi(y))}{|x-y|^{N+2s}} \, \mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^N} g(x) f_a(u)\phi(x) \, \mathrm{d}x, \quad \forall \, \phi \in \dot{H}^s(\mathbb{R}^N).$$

We say a weak solution u is a mountain pass solution of (SP) if it is a critical point of $\mathcal{C}^1$ energy functional associated with (SP), which satisfies the mountain pass geometry and a weaker Palais–Smale condition (see Proposition 2.5 and [27, Theorem 2.1]), and moreover, the value of the energy functional at u possesses a min-max characterization (see (3.1)).

Theorem 1.1. *Let $s \in (0,1)$ and $N > 2s$. Assume that f satisfies $(f1)$ and $(f2)$. Let $g \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ be positive. Then there exists $a_1 > 0$ such that for each $a \in (0, a_1)$,* (SP) *admits a mountain pass solution. Moreover, there exist $a_2 \in (0, a_1)$ and $C > 0$, such that $\|u_a\|_{\dot{H}^s(\mathbb{R}^N)} \le C$, for all $a \in (0, a_2)$.*

From the above theorem, observe that the mountain pass solutions of (SP) are uniformly bounded in $L^{2_s^*}(\mathbb{R}^N)$. In the following theorem, we prove a Brezis–Kato type regularity result which says the uniform boundedness of the mountain pass solutions in $L^r(\mathbb{R}^N)$ with $r \ge 2_s^*$, under an additional growth assumption on f near infinity:

$(\tilde{f}1)$ $\lim\limits_{t \to \infty} \dfrac{f(t)}{t^{2_s^*-1}} = 0.$

Theorem 1.2. *Let $s \in (0,1)$ and $N > 2s$. Let f, g, a_2 be as given in Theorem 1.1. In addition, assume that f satisfies $(\tilde{f}1)$. Then for $r \in [2_s^*, \infty]$ and $a \in (0, a_2)$, $u_a \in L^r(\mathbb{R}^N) \cap \mathcal{C}(\mathbb{R}^N)$. Moreover, there exists $C(r, N, s, f, g) > 0$ such that $\|u_a\|_{L^r(\mathbb{R}^N)} \le C$, for all $a \in (0, a_2)$.*

Next, we state the positivity of the solutions to (SP). To state the result, we invoke further hypothesis on f and g:

$(f3)$ f is locally Lipschitz,

$(g1)$ $|x|^{N-2s} \displaystyle\int_{\mathbb{R}^N} \frac{g(y)}{|x-y|^{N-2s}} \, \mathrm{d}y \le C(g)$, where $x \in \mathbb{R}^N \setminus \{0\}$ and $C(g) > 0$.

Theorem 1.3. *Let $s \in (0,1)$ and $N > 2s$. Let f, g, a_2 be as given in Theorem 1.2. Then the following hold:*

(i) *There exists $a_3 \in (0, a_2)$ such that for every $a \in (0, a_3)$, $u_a \ge 0$ on $\mathbb{R}^N$.*
(ii) *In addition, if f satisfies $(f3)$ and g satisfies $(g1)$, then $u_a > 0$ on $\mathbb{R}^N$.*

Let us briefly discuss our approach to prove the above theorems. The existence of mountain pass solution is based on variational methods. To establish the Brezis–Kato type regularity result for (SP), we use the Moser iteration technique and the uniform boundedness of the mountain pass solutions in $\dot{H}^s(\mathbb{R}^N)$ along with the growth assumption $(\tilde{f}1)$. With this regularity result and using the Riesz representation for the operator $(-\Delta)^s$ ([29, Theorem 5]), we show that a sequence of mountain pass solutions uniformly converges to a positive function in $\mathcal{C}(\mathbb{R}^N)$ near $a = 0$. In the end, we conclude the positivity of the solutions by using the properties $(f3)$ and $(g1)$. At this point, it is clear that the range of a, the growth of f near infinity, and the behaviour of the weight function g are essential for the existence of positive solutions. One example of f, g satisfying all these properties is demonstrated in Example 4.1.

We organize the rest of this paper as follows. In Section 2, we obtain some embeddings of $\dot{H}^s(\mathbb{R}^N)$ and set up a variational framework associated with (SP). Section 3 contains the proof of the existence and regularity of the mountain pass solutions to (SP). We establish the positivity of the solution in Section 4.

2. EMBEDDINGS OF $\dot{H}^s(\mathbb{R}^N)$ AND THE VARIATIONAL SETTINGS

In the first part of this section, we discuss compact embeddings of $\dot{H}^s(\mathbb{R}^N)$ into specific Lebesgue spaces and weighted Lebesgue spaces. Using these embeddings, in the second part, we prove qualitative properties of the energy functional associated with (SP). In the rest of this paper, we denote C as a generic positive constant, and denote the norm $\|\cdot\|_{L^p(\mathbb{R}^N)}$ by $\|\cdot\|_p$.

2.1. EMBEDDINGS OF $\dot{H}^s(\mathbb{R}^N)$

Recall that, the homogeneous fractional Sobolev space $\dot{H}^s(\mathbb{R}^N)$ is the closure of $\mathcal{C}_c^1(\mathbb{R}^N)$ with respect to $[\cdot]_{s,2} + \|\cdot\|_{2_s^*}$. In view of [19, Theorem 1.1], $\dot{H}^s(\mathbb{R}^N)$ has the following representation:

$$\dot{H}^s(\mathbb{R}^N) := \left\{ u\colon \mathbb{R}^N \to \mathbb{R} \,:\, u \text{ is measurable}, [u]_{s,2} + \|u\|_{2_s^*} < \infty \right\}.$$

Henceforth, $\dot{H}^s(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$. Moreover, by [8, Theorem 2.2.1] and using the density of $C_c^1(\mathbb{R}^N)$, we get

$$\|u\|_{2_s^*} \le C(N,s)[u]_{s,2}, \quad \forall\, u \in \dot{H}^s(\mathbb{R}^N). \tag{2.1}$$

The above inequality infers that $[\cdot]_{s,2}$ is an equivalent norm in $\dot{H}^s(\mathbb{R}^N)$, i.e., there exists C_1 depending on N, s such that $[u]_{s,2} \le \|u\|_{\dot{H}^s(\mathbb{R}^N)} \le C_1[u]_{s,2}$ holds for all $u \in \dot{H}^s(\mathbb{R}^N)$. In the following proposition, we prove that $\dot{H}^s(\mathbb{R}^N)$ is compactly embedded into spaces of locally integrable functions.

Proposition 2.1. *Let $N > 2s$. Then $\dot{H}^s(\mathbb{R}^N) \hookrightarrow L^q_{loc}(\mathbb{R}^N)$ compactly for every $q \in (1, 2_s^*)$.*

Proof. Combining the embeddings $\dot{H}^s(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$ and $L^{2_s^*}_{loc}(\mathbb{R}^N) \hookrightarrow L^q_{loc}(\mathbb{R}^N)$ with $q \in (1, 2_s^*)$, it is evident that $\dot{H}^s(\mathbb{R}^N) \hookrightarrow L^q_{loc}(\mathbb{R}^N)$ for $q \in (1, 2_s^*)$. First, we show that $\dot{H}^s(\mathbb{R}^N)$ is compactly embedded into $L^2_{loc}(\mathbb{R}^N)$. Let (u_n) be a bounded sequence in $\dot{H}^s(\mathbb{R}^N)$, and $K \subset \mathbb{R}^N$ be a compact set. Then there exists $M_1 > 0$ such that

$\|u_n\|_{L^2(K)} \leq C\|u_n\|_{\dot{H}^s(\mathbb{R}^N)} \leq M_1$ for every $n \in \mathbb{N}$. Consequently, the sequence $(u_n\big|_K)$ is bounded in $L^2(K)$. Further, using [5, Lemma A.1] for every $n \in \mathbb{N}$, we have

$$\sup_{|h|>0} \int_{\mathbb{R}^N} \frac{(u_n(x+h) - u_n(x))^2}{|h|^{2s}} \, \mathrm{d}x \leq C(N)[u_n]^2_{s,2}. \tag{2.2}$$

The boundedness of (u_n) in $\dot{H}^s(\mathbb{R}^N)$ and (2.2) confirm that for all $n \in \mathbb{N}$,

$$\int_{\mathbb{R}^N} (u_n(x+h) - u_n(x))^2 \, dx \to 0$$

as $|h| \to 0$. Now by applying the Riesz–Fréchet–Kolmogorov compactness theorem on $(u_n\big|_K) \subset L^2(K)$, we conclude $(u_n\big|_K)$ is relatively compact. Hence, it has a convergent subsequence in $L^2(K)$. Therefore, $\dot{H}^s(\mathbb{R}^N)$ is compactly embedded into $L^2_{loc}(\mathbb{R}^N)$. For $q \in (1,2)$, using $L^2_{loc}(\mathbb{R}^N) \hookrightarrow L^q_{loc}(\mathbb{R}^N)$ we directly get the compact embeddings of $\dot{H}^s(\mathbb{R}^N)$ into $L^q_{loc}(\mathbb{R}^N)$. Next, we consider $q \in (2, 2^*_s)$. In this case, we express $q = 2t + (1-t)2^*_s$, where $t = \frac{2^*_s - q}{2^*_s - 2} \in (0,1)$. Using $\dot{H}^s(\mathbb{R}^N) \hookrightarrow L^q_{loc}(\mathbb{R}^N)$, the sequence $(u_n\big|_K)$ is bounded in $L^q(K)$. Applying the Hölder inequality with the conjugate pair $(\frac{1}{t}, \frac{1}{1-t})$ and (2.2) we obtain the following estimate for every $|h| > 0$ and $n \in \mathbb{N}$:

$$\begin{aligned} &\int_{\mathbb{R}^N} |u_n(x+h) - u_n(x)|^q \, \mathrm{d}x \\ &\leq \left(\int_{\mathbb{R}^N} (u_n(x+h) - u_n(x))^2 \, \mathrm{d}x \right)^t \left(\int_{\mathbb{R}^N} |u_n(x+h) - u_n(x)|^{2^*_s} \, \mathrm{d}x \right)^{1-t} \\ &\leq C(N) \left(|h|^s [u_n]_{s,2} \right)^{2t} \|u_n\|_{2^*_s}^{2^*_s(1-t)} \leq C(N)|h|^{2st} \|u_n\|^q_{\dot{H}^s(\mathbb{R}^N)}. \end{aligned}$$

Again the boundedness of (u_n) in $\dot{H}^s(\mathbb{R}^N)$, and the Riesz–Fréchet-Kolmogorov compactness theorem confirm a convergent subsequence of $(u_n\big|_K)$ in $L^q(K)$. Therefore, $\dot{H}^s(\mathbb{R}^N)$ is compactly embedded into $L^q_{loc}(\mathbb{R}^N)$ for $q \in (2, 2^*_s)$ as well. This completes the proof. □

We now prove the compact embeddings of $\dot{H}^s(\mathbb{R}^N)$ into weighted Lebesgue spaces.

Proposition 2.2. *Let $N > 2s$ and $q \in [2, 2^*_s)$. Let p be the conjugate exponent of $\frac{2^*_s}{q}$, and $g \in L^p(\mathbb{R}^N)$. Then the embedding $\dot{H}^s(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N, |g|)$ is compact.*

Proof. Let $u_n \rightharpoonup u$ in $\dot{H}^s(\mathbb{R}^N)$. We need to show $u_n \to u$ in $L^q(\mathbb{R}^N, |g|)$. Set

$$L = \sup\{\|u_n - u\|^q_{2^*_s} : n \in \mathbb{N}\}.$$

Clearly, L is finite from the boundedness of (u_n) in $\dot{H}^s(\mathbb{R}^N)$. Let $\epsilon > 0$ be given. Since $\mathcal{C}_c(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$, we choose $g_\epsilon \in \mathcal{C}_c(\mathbb{R}^N)$ such that $\|g - g_\epsilon\|_p < \frac{\epsilon}{2L}$. We estimate

$$\int_{\mathbb{R}^N} |g||u_n - u|^q \leq \int_{\mathbb{R}^N} |g - g_\epsilon||u_n - u|^q + \int_{\mathbb{R}^N} |g_\epsilon||u_n - u|^q. \tag{2.3}$$

Using the Hölder inequality with conjugate pair $(p, \frac{2_s^*}{q})$ we estimate the first integral of the above inequality as

$$\int_{\mathbb{R}^N} |g - g_\epsilon||u_n - u|^q \leq \|g - g_\epsilon\|_p \|u_n - u\|_{2_s^*}^q < \frac{\epsilon}{2}. \tag{2.4}$$

Suppose K is the support of g_ϵ. Using the compact embeddings of $\dot{H}^s(\mathbb{R}^N)$ into $L^q_{loc}(\mathbb{R}^N)$ (Proposition 2.1), there exists $n_1 \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^N} |g_\epsilon||u_n - u|^q = \int_K |g_\epsilon||u_n - u|^q < \frac{\epsilon}{2}, \quad \forall\, n \geq n_1.$$

Therefore, from (2.3) and (2.4) we obtain

$$\int_{\mathbb{R}^N} |g||u_n - u|^q < \epsilon, \quad \forall\, n \geq n_1.$$

Thus $u_n \to u$ in $L^q(\mathbb{R}^N, |g|)$. □

2.2. THE VARIATIONAL SETTINGS

For the existence of a solution of (SP), this subsection sets up a suitable functional framework. In the following remark, we identify some bounds (upper and lower) for f_a and its primitive F_a, defined as $F_a(t) = \int_0^t f_a(\tau)\mathrm{d}\tau$.

Remark 2.3. (i) Let $\epsilon > 0$ and $\gamma \in (2, 2_s^*]$. Using subcritical growth on f and behaviour of f near zero (see $(f1)$), there exists $t_1(\epsilon) > 0$ such that $f(t) < \epsilon t$, for $0 < t < t_1$, and $f(t) \leq C t^{\gamma-1}$ for $t \geq t_1$, where $C = C(f, t_1(\epsilon))$. Hence $f(t) \leq \epsilon t + C t^{\gamma-1}$ for $t \in \mathbb{R}^+$, and

$$|f_a(t)| \leq \epsilon|t| + C|t|^{\gamma-1} - a \quad \text{and} \quad |F_a(t)| \leq \epsilon t^2 + C|t|^\gamma + a|t| \quad \text{for } t \in \mathbb{R}, \tag{2.5}$$

where $C = C(f, t_1(\epsilon))$. Again using the subcritical growth on f, $f(t) \leq C(f) t^{\gamma-1}$ for $t > t_2$. The continuity of f infers that $f(t) \leq C$ on $[0, t_2]$. Hence for $a \in (0, \tilde{a})$, we get

$$|f_a(t)| \leq C(1 + |t|^{\gamma-1}) \quad \text{and} \quad |F_a(t)| \leq C(|t| + |t|^\gamma) \quad \text{for } t \in \mathbb{R}, \tag{2.6}$$

where $C = C(f, t_2, \tilde{a})$. Using $(\tilde{f}1)$, there exists $t_3(\epsilon) > 0$ so that $f(t) \leq \epsilon t^{2_s^*-1}$ for all $t > t_3$. Hence for $a \in (0, \tilde{a})$, we also obtain

$$|f_a(t)| \leq C + \epsilon|t|^{2_s^*-1} \text{ for } t \in \mathbb{R}, \tag{2.7}$$

where $C = C(\epsilon, t_3(\epsilon), \tilde{a})$.

(ii) Let $M > 0$. Since f is superlinear (see $(f1)$), there exists a constant $C = C(M)$ such that $f(t) > Mt - C$, for every $t \in \mathbb{R}^+$. From the superlinearity of f, it also follows that $\lim_{t\to\infty} \frac{F(t)}{t^2} = \infty$, and hence $F(t) > Mt^2 - C(M)$ for every $t \in \mathbb{R}^+$. Accordingly,

$$f_a(t) > Mt - (C + a) \quad \text{and} \quad F_a(t) = F(t) - at > Mt^2 - (at + C) \quad \text{for } t \in \mathbb{R}^+, \tag{2.8}$$

where $C = C(M)$.

For $g \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $a \geq 0$, we consider the following functionals on $\dot{H}^s(\mathbb{R}^N)$:

$$N_a(u) = \int\limits_{\mathbb{R}^N} gF_a(u) \quad \text{and} \quad I_a(u) = \frac{1}{2}[u]_{s,2}^2 - N_a(u).$$

Using the upper bound of F_a (Remark 2.3) it follows that N_a and I_a are well-defined. Moreover, one can also verify that $N_a, I_a \in \mathcal{C}^1(\dot{H}^s(\mathbb{R}^N), \mathbb{R})$ and $N_a'(u), I_a'(u)$ for $u \in \dot{H}^s(\mathbb{R}^N)$ are given by

$$\begin{aligned} N_a'(u)(v) &= \int\limits_{\mathbb{R}^N} gf_a(u)v, \\ I_a'(u)(v) &= \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x) - u(y)(v(x) - v(y))}{|x-y|^{N+2s}} \,\mathrm{d}x\mathrm{d}y - N_a'(u)(v), \end{aligned} \tag{2.9}$$

where $v \in \dot{H}^s(\mathbb{R}^N)$. Every critical point of I_a corresponds to a solution of (SP). In the following proposition, we prove the compactness of N_a and N_a'.

Proposition 2.4. *Let $N > 2s$ and $\gamma \in (2, 2_s^*)$. Let $g \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Assume that f satisfies $(f1)$. Then the following hold for $a \geq 0$:*

(i) *The functional N_a is compact on $\dot{H}^s(\mathbb{R}^N)$. Moreover, if $u_n \rightharpoonup u$ in $\dot{H}^s(\mathbb{R}^N)$ and $a_n \to a$ in $\mathbb{R}^+$, then $N_{a_n}(u_n) \to N_a(u)$.*
(ii) *The map $N_a' : \dot{H}^s(\mathbb{R}^N) \to (\dot{H}^s(\mathbb{R}^N))'$ is compact. Moreover, if $u_n \rightharpoonup u$ in $\dot{H}^s(\mathbb{R}^N)$ and $a_n \to a$ in $\mathbb{R}^+$, then $N_{a_n}'(u_n)(v) \to N_a'(u)(v)$ for every $v \in \dot{H}^s(\mathbb{R}^N)$.*

Proof. (i) Let $u_n \rightharpoonup u$ in $\dot{H}^s(\mathbb{R}^N)$. We show that $N_a(u_n) \to N_a(u)$. The idea of the proof is similar to Proposition 2.2. Set

$$L = \sup\left\{ \|u_n\|_{2_s^*}^\gamma + \|u\|_{2_s^*}^\gamma + \|u_n\|_{2_s^*} + \|u\|_{2_s^*} : n \in \mathbb{N} \right\},$$

which is finite since (u_n) is bounded in $\dot{H}^s(\mathbb{R}^N)$. Let $\epsilon > 0$ be given. Let p be the conjugate exponent of $\frac{2_s^*}{\gamma}$. Using the density of $\mathcal{C}_c(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ and $L^{\frac{N}{2s}}(\mathbb{R}^N)$, we take $g_\epsilon \in C_c(\mathbb{R}^N)$ satisfying

$$|g_\epsilon| < |g| \quad \text{and} \quad \|g - g_\epsilon\|_p + \|g - g_\epsilon\|_{\frac{N}{2s}}^{\frac{1}{2}} < \frac{\epsilon}{L}.$$

For $a \geq 0$, we write

$$|N_a(u_n) - N_a(u)| \leq \int\limits_{\mathbb{R}^N} |g - g_\epsilon||F_a(u_n) - F_a(u)| + \int\limits_{\mathbb{R}^N} |g_\epsilon||F_a(u_n) - F_a(u)|. \tag{2.10}$$

Using (2.6) and (2.4), the first integral of (2.10) can be estimated as

$$\begin{aligned}
&\int\limits_{\mathbb{R}^N} |g - g_\epsilon||F_a(u_n) - F_a(u)| \\
&\leq \int\limits_{\mathbb{R}^N} |g - g_\epsilon| \left(|F_a(u_n)| + |F_a(u)|\right) \\
&\leq C \int\limits_{\mathbb{R}^N} |g - g_\epsilon|\big(|u_n|^\gamma + |u|^\gamma + |u_n| + |u|\big) \\
&\leq C\Big(\|g - g_\epsilon\|_p \left(\|u_n\|_{2_s^*}^\gamma + \|u\|_{2_s^*}^\gamma \right) \\
&\qquad + \left(\|g - g_\epsilon\|_{\frac{N}{2s}} \|g - g_\epsilon\|_1 \right)^{\frac{1}{2}} \left(\|u_n\|_{2_s^*} + \|u\|_{2_s^*} \right) \Big) \\
&\leq C \frac{\epsilon}{L} \left(\|u_n\|_{2_s^*}^\gamma + \|u\|_{2_s^*}^\gamma + \|u_n\|_{2_s^*} + \|u\|_{2_s^*} \right) \\
&< C\epsilon,
\end{aligned} \tag{2.11}$$

where $C = C(f, a)$. Next, we show that the second integral of (2.10) converges to zero as $n \to \infty$. Let K be the support of g_ϵ. Since $\dot{H}^s(\mathbb{R}^N)$ is compactly embedded into $L^\gamma_{loc}(\mathbb{R}^N)$ (Proposition 2.1), $u_n \to u$ in $L^\gamma(K)$. In particular, up to a subsequence, $u_n(x) \to u(x)$ a.e. in K. From the continuity of F_a, $F_a(u_n(x)) \to F_a(u(x))$ a.e. in K. Further, since $|F_a(u_n)| \leq C(|u_n|^\gamma + |u_n|)$, and $\int_K |u_n|^\gamma \to \int_K |u|^\gamma, \int_K |u_n| \to \int_K |u|$, using the generalized dominated convergence theorem, $F_a(u_n) \to F_a(u)$ in $L^1(K)$. Thus

$$\int\limits_{\mathbb{R}^N} |g_\epsilon||F_a(u_n) - F_a(u)| \leq \|g_\epsilon\|_\infty \int\limits_K |F_a(u_n) - F_a(u)| \to 0, \text{ as } n \to \infty.$$

Therefore, from (2.10), $N_a(u_n) \to N_a(u)$, as $n \to \infty$. Now for a sequence (a_n), we write

$$|N_{a_n}(u_n) - N_a(u)| \leq \int\limits_{\mathbb{R}^N} |g||F_{a_n}(u_n) - F_a(u_n)| + \int\limits_{\mathbb{R}^N} |g||F_a(u_n) - F_a(u)|. \tag{2.12}$$

By the compactness of N_a, the second integral of (2.12) converges to zero. Further, by noting that $F_{a_n}(u_n) - F_a(u_n) = (a - a_n)u_n$, the first integral of (2.12) can be estimated as

$$\int\limits_{\mathbb{R}^N} |g||F_{a_n}(u_n) - F_a(u_n)| \leq |a_n - a| \int\limits_{\mathbb{R}^N} |g||u_n| \leq |a_n - a| \|g\|_1^{\frac{1}{2}} \left(\int\limits_{\mathbb{R}^N} |g| u_n^2 \right)^{\frac{1}{2}}. \tag{2.13}$$

Thus, combining (2.12) and (2.13), and using Proposition 2.2 we get $N_{a_n}(u_n) \to N_a(u)$, as $n \to \infty$.

(ii) The compactness of N_a' similarly follows using the splitting arguments shown above. The proof of the convergence of $(N_{a_n}'(u_n)(v))$ holds similarly. □

Now we prove that the energy functional I_a satisfies all the conditions of the mountain pass theorem. It is worth mentioning that I_a may not satisfy the Palais–Smale condition due to the weaker Ambrosetti–Rabinowitz type nonlinearities for f in $(f1)$. Nevertheless, in the first two parts of the following proposition, I_a satisfies a weaker Palais–Smale condition, called the Cerami condition introduced in [13].

Proposition 2.5. *Let $N > 2s$ and $\gamma \in (2, 2_s^*)$. Let f satisfies $(f1)$ and $(f2)$. Let $g \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ be positive. Then the following hold:*

(i) *Let (u_n) be a bounded sequence in $\dot{H}^s(\mathbb{R}^N)$ such that $I_a(u_n) \to c$ in $\mathbb{R}$ and $I_a'(u_n) \to 0$ in $(\dot{H}^s(\mathbb{R}^N))'$. Then (u_n) has a convergent subsequence in $\dot{H}^s(\mathbb{R}^N)$.*
(ii) *For any $c \in \mathbb{R}$, there exist $\eta, \beta, \rho > 0$ such that $\|I_a'(u)\|[u]_{s,2} \geq \beta$ for $u \in I_a^{-1}([c-\eta, c+\eta])$ with $[u]_{s,2} \geq \rho$.*
(iii) *There exist $\rho > 0, \beta = \beta(\rho) > 0$, and $a_1 = a_1(\rho) > 0$ such that if $a \in (0, a_1)$, then $I_a(u) \geq \beta$ for every $u \in \dot{H}^s(\mathbb{R}^N)$ satisfying $[u]_{s,2} = \rho$.*
(iv) *There exists $\tilde{u} \in \dot{H}^s(\mathbb{R}^N)$ with $[\tilde{u}]_{s,2} > \rho$ such that $I_a(\tilde{u}) < 0$.*

Proof. (i) By the reflexivity, up to a subsequence $u_n \rightharpoonup u$ in $\dot{H}^s(\mathbb{R}^N)$. We consider the functional $J(v) = [v]_{s,2}$, for $v \in \dot{H}^s(\mathbb{R}^N)$. From (2.9),

$$J'(u_n)(u_n - u) = I_a'(u_n)(u_n - u) + N_a'(u_n)(u_n - u).$$

By the hypothesis,

$$|\langle I_a'(u_n), u_n - u\rangle| \leq \|I_a'(u_n)\|[u_n - u]_{s,2} \to 0, \text{ as } n \to \infty.$$

Moreover, since N_a' is compact (Proposition 2.4),

$$N_a'(u_n)(u_n - u) \to 0, \text{ as } n \to \infty$$

Therefore,

$$J'(u_n)(u_n - u) \to 0, \text{ as } n \to \infty.$$

Further, since $u_n \rightharpoonup u$ in $\dot{H}^s(\mathbb{R}^N)$ and $J \in \mathcal{C}^1(\dot{H}^s(\mathbb{R}^N), \mathbb{R})$, we also have

$$J'(u)(u_n - u) \to 0.$$

Therefore,

$$[u_n - u]_{s,2}^2 = J'(u_n)(u_n - u) - J'(u)(u_n - u) \to 0,$$

as required.

(ii) Our proof adapts the arguments given in [4, Proposition 3.6]. On the contrary, assume that (u_n) is a sequence in $\dot{H}^s(\mathbb{R}^N)$ satisfying

$$I_a(u_n) \to c, \quad [u_n]_{s,2} \to \infty, \quad \text{and} \quad \|I_a'(u_n)\|[u_n]_{s,2} \to 0, \quad \text{as } n \to \infty. \tag{2.14}$$

Set $w_n = \frac{u_n}{[u_n]_{s,2}}$. Then $[w_n]_{s,2} = 1$, and by the reflexivity, up to a subsequence, $w_n \rightharpoonup w$ in $\dot{H}^s(\mathbb{R}^N)$.

Step 1. This step shows that $w^+ = 0$ a.e. in $\mathbb{R}^N$. We consider the set

$$\Omega = \left\{x \in \mathbb{R}^N : w(x) > 0\right\}.$$

On the contrary, we assume $|\Omega| > 0$. Using (2.14),

$$[u_n]_{s,2}^2 - N_a'(u_n)(u_n) = I_a'(u_n)(u_n) \to 0.$$

Hence for each $n \in \mathbb{N}$, we have

$$1 = \frac{1}{[u_n]_{s,2}^2}\left(\int_{\mathbb{R}^N} g f_a(u_n) u_n + \epsilon_n\right), \tag{2.15}$$

where $\epsilon_n \to 0$ as $n \to \infty$. From the compactness of $\dot{H}^s(\mathbb{R}^N) \hookrightarrow L^2_{loc}(\mathbb{R}^N)$ and by the Egorov theorem, there exists $\Omega_1 \subset \Omega$ with $|\Omega_1| > 0$ such that (w_n) converges to w uniformly on Ω_1. Thus there exists $n_1 \in \mathbb{N}$ such that for $n \geq n_1$, $w_n \geq 0$ and hence $u_n \geq 0$ a.e. on Ω_1. This implies that, for every $n \geq n_1$, $\Omega_1 \subset \Omega_n^+$, where

$$\Omega_n^+ := \left\{x \in \mathbb{R}^N : u_n(x) \geq 0\right\}.$$

From the definition of f_a, $f_a(u_n)u_n = -au_n \geq 0$ on $\mathbb{R}^N \setminus \Omega_n^+$. Therefore, using (2.15) and using the lower bound of f_a in (2.8), for all $n \geq n_1$ we obtain

$$\begin{aligned} 1 &\geq \frac{1}{[u_n]_{s,2}^2}\left\{\int_{\Omega_n^+} g f_a(u_n) u_n + \epsilon_n\right\} \\ &\geq M \int_{\Omega_n^+} g \frac{u_n^2}{[u_n]_{s,2}^2} - \frac{(C_M + a)}{[u_n]_{s,2}} \int_{\Omega_n^+} g \frac{u_n}{[u_n]_{s,2}} + \frac{\epsilon_n}{[u_n]_{s,2}^2} \\ &\geq M \int_{\Omega_1} g w_n^2 - \frac{(C_M + a)}{[u_n]_{s,2}} \int_{\Omega_n^+} g w_n + \frac{\epsilon_n}{[u_n]_{s,2}^2}. \end{aligned} \tag{2.16}$$

Further,

$$\int_{\Omega_1} g w_n^2 \to \int_{\Omega_1} g w^2$$

(by Proposition 2.2) and

$$\int_{\Omega_n^+} g w_n \leq \int_{\mathbb{R}^N} g w_n \leq C(N, s)\left(\|g\|_1 \|g\|_{\frac{N}{2s}}\right)^{\frac{1}{2}}.$$

We take the limit as $n \to \infty$ in (2.16) and using $[u_n]_{s,2} \to \infty$, to obtain

$$1 \geq M \int_{\Omega_1} g w^2, \text{ for arbitrarily large } M > 0,$$

which is a contradiction. Thus, $|\Omega| = 0$.

Step 2. For a fixed $n \in \mathbb{N}$, we set

$$m_n := \sup \left\{ I_a(t w_n) : 0 \leq t \leq [u_n]_{s,2} \right\}.$$

Since the map $t \mapsto I_a(t w_n)$ is continuous on $[0, [u_n]_{s,2}]$, there exists $t_n \in [0, [u_n]_{s,2}]$ such that $m_n = I_a(t_n w_n)$. First, we claim that $m_n \to \infty$, as $n \to \infty$. Since the sequence (u_n) is unbounded, there exists $n_2 \in \mathbb{N}$ so that for $n \geq n_2$, $[u_n]_{s,2} \geq M$. Hence by definition, $m_n \geq I_a(M w_n)$. Using the compactness of N_a (Proposition 2.4) and $|\Omega| = 0$ (Step 1), we get

$$\begin{aligned}
\lim_{n\to\infty} I_a(M w_n) &= \frac{M^2}{2} - \lim_{n\to\infty} \int_{\mathbb{R}^N} g F_a(M w_n) \\
&= \frac{M^2}{2} - \int_{\mathbb{R}^N} g F_a(M w) = \frac{M^2}{2} + aM \int_{\Omega^c} g w.
\end{aligned}$$

Since the quantity $\frac{M^2}{2} + aM \int_{\Omega^c} g w$ is sufficiently large, we conclude that $I_a(M w_n) \to \infty$, as $n \to \infty$, and hence the claim holds. Next, for each $n \in \mathbb{N}$,

$$I_a(t_n w_n) - I_a(u_n) = \frac{t_n^2 - [u_n]_{s,2}^2}{2} + \int_{\mathbb{R}^N} g \left(F_a(u_n) - F_a(t_n w_n) \right). \tag{2.17}$$

Set $s_n = \frac{t_n}{[u_n]_{s,2}}$. Then $s_n \in [0,1]$ and $s_n u_n = t_n w_n$. Clearly,

$$F_a(u_n(x)) - F_a(s_n u_n(x)) = 0$$

whenever $u_n(x) = 0$. If $u_n(x) \neq 0$, then for $R > 0$ given in $(f2)$ we apply [4, Proposition 3.3] to get

$$F_a(u_n(x)) - F_a(s_n u_n(x)) \leq \frac{1 - s_n^2}{2} u_n(x) f_a(u_n(x)) + C(R).$$

Therefore, (2.17) yields

$$\begin{aligned}
I_a(t_n w_n) - I_a(u_n) &\leq \frac{1 - s_n^2}{2} \left(-[u_n]_{s,2}^2 + \int_{\mathbb{R}^N} g u_n f_a(u_n) \right) + C(R) \|g\|_1 \\
&= \frac{s_n^2 - 1}{2} I_a'(u_n)(u_n) + C(R) \|g\|_1.
\end{aligned}$$

Hence, in view of (2.14), the sequence $(I_a(t_n w_n) - I_a(u_n))$ is bounded. On the other hand, individually $(I_a(u_n))$ is bounded (see (2.14)) and $(I_a(t_n w_n))$ is unbounded, resulting in a contradiction. Therefore, such an unbounded sequence (u_n) in (2.14) does not exist.

(iii) Let $\epsilon > 0$ be such that $\epsilon B_g \|g\|_{\frac{N}{2s}} < \frac{1}{2}$, where B_g is the best constant of (2.1). Then using (2.5) and the embeddings $\dot{H}^s(\mathbb{R}^N) \hookrightarrow L^\gamma(\mathbb{R}^N, |g|)$ (Proposition 2.2) we get

$$\begin{aligned} I_a(u) &\geq \frac{[u]_{s,2}^2}{2} - \epsilon \int_{\mathbb{R}^N} gu^2 - C \int_{\mathbb{R}^N} g|u|^\gamma - a \int_{\mathbb{R}^N} g|u| \\ &\geq [u]_{s,2}^2 \left(\frac{1}{2} - \epsilon B_g \|g\|_{\frac{N}{2s}} - C[u]_{s,2}^{\gamma-2} \right) - aC[u]_{s,2}, \end{aligned} \tag{2.18}$$

where $C = C(f, g, N, s)$. Taking $[u]_{s,2} = \rho$, $I_a(u) \geq A(\rho) - aC\rho$, where $A(\rho) = C\rho^2(1 - C_1\rho^{\gamma-2})$ for some constants C, C_1 independent of a. Let ρ_1 be the first nontrivial zero of A. For $\rho < \rho_1$, fix $a_1 \in (0, \frac{A(\rho)}{C\rho})$ and $\beta = A(\rho) - a_1 C\rho$. Therefore, using (2.18), $I_a(u) \geq \beta$ for every $a \in (0, a_1)$.

(iv) We consider $\phi \in \mathcal{C}^2(\mathbb{R}^N)$, $\phi \geq 0$, and $[\phi]_{s,2} = 1$. For $M, t > 0$, using (2.8) we get

$$F_a(t\phi) > M(t\phi)^2 - (at\phi + C(M)).$$

Hence

$$I_a(t\phi) \leq t^2 \left(\frac{1}{2} - M \int_{\mathbb{R}^N} g\phi^2 \right) + at \int_{\mathbb{R}^N} g\phi + C(M)\|g\|_1.$$

Choose $M > \left(2 \int_{\mathbb{R}^N} g\phi^2\right)^{-1}$. Then $I_a(t\phi) \to -\infty$, as $t \to \infty$, i.e., there exists $t_1 > \rho$ so that $I_a(t\phi) < 0$ for $t > t_1$. Thus $\tilde{u} = t\phi$ with $t > t_1$ is the required function. □

3. EXISTENCE, UNIFORM BOUNDEDNESS, AND REGULARITY OF THE SOLUTIONS

In this section, we study the existence of solutions to (SP) and their various properties. This section contains the proof of Theorem 1.1–1.2.

Proof of Theorem 1.1. Recall $a_1, \beta, \tilde{u}$ as given in (iii) and (iv) of Proposition 2.5. For $a \in (0, a_1)$ using Proposition 2.5 and the fact that $[\cdot]_{s,2}$ is an equivalent norm in $\dot{H}^s(\mathbb{R}^N)$ (from (2.1)), we observe that all the properties of the mountain pass theorem in [27, Theorem 2.1] are verified. Therefore, applying [27, Theorem 2.1] there exists $u_a \in \dot{H}^s(\mathbb{R}^N)$ satisfying

$$I_a(u_a) = \inf_{\gamma \in \Gamma_{\tilde{u}}} \max_{s \in [0,1]} I_a(\gamma(s)) \geq \beta \quad \text{and} \quad I'_a(u_a) = 0, \tag{3.1}$$

where

$$\Gamma_{\tilde{u}} := \left\{ \gamma \in \mathcal{C}([0,1], \dot{H}^s(\mathbb{R}^N)) : \gamma(0) = 0 \text{ and } \gamma(1) = \tilde{u} \right\}.$$

Thus, u_a is a nontrivial solution of (SP). First, we show that the set $\{I_a(u_a) : a \in (0, a_1)\}$ is uniformly bounded. Define $\tilde{\gamma} : [0,1] \to \dot{H}^s(\mathbb{R}^N)$ by $\tilde{\gamma}(s) = s\tilde{u}$, where $\tilde{u} = t\phi$ for some $t > t_1$. Clearly, $\tilde{\gamma} \in \Gamma_{\tilde{u}}$ and hence using (3.1) for $a \in (0, a_1)$,

$$I_a(u_a) \le \max_{s\in[0,1]} I_a(\tilde{\gamma}(s)) = \max_{s\in[0,1]} I_a(st\phi). \tag{3.2}$$

Further, since $F_a(st\phi) \ge M(st\phi)^2 - a_1 st\phi - C(M)$ (see (2.8)), where $M \int_{\mathbb{R}^N} g\phi^2 > \frac{1}{2}$, we get

$$\begin{aligned}\max_{s\in[0,1]} I_a(st\phi) &\le \max_{s\in[0,1]} \left(s^2t^2 \left(\frac{1}{2} - M \int_{\mathbb{R}^N} g\phi^2 \right) + sta_1 \int_{\mathbb{R}^N} g\phi + C(M)\|g\|_1 \right) \\ &\le ta_1 \int_{\mathbb{R}^N} g\phi + C(M)\|g\|_1.\end{aligned}$$

Thus from (3.2), it is evident that $I_a(u_a) \le C$ for all $a \in (0, a_1)$. Next, we prove the existence of $a_2 \in (0, a_1)$ such that the set $\{[u_a]_{s,2} : a \in (0, a_2)\}$ is uniformly bounded, i.e., $[u_a]_{s,2} \le C$ for all $a \in (0, a_2)$ and for some C. On the contrary, assume that no such a_2 and C exist. Then there exists a sequence (a_n) in $(0, a_1)$, such that $a_n \to 0$, and $[u_{a_n}]_{s,2} \to \infty$, as $n \to \infty$. Observe that $I'_{a_n}(u_{a_n}) = 0$ for each $n \in \mathbb{N}$, and up to a subsequence, $I_{a_n}(u_{a_n}) \to c$ in $\mathbb{R}$. Set $w_{a_n} = u_{a_n}[u_{a_n}]_{s,2}^{-1}$. Suppose $w_{a_n} \rightharpoonup w$ in $\dot{H}^s(\mathbb{R}^N)$. Now using the convergence $N_{a_n}(w_{a_n}) \to N_0(w)$ (by Proposition 2.4(i)), we can proceed with the same arguments as given in the proof of Proposition 2.5(ii) (with a replaced by a_n) to get the following contradiction:

$$I_{a_n}(t_{a_n} w_{a_n}) \le C(R)\|g\|_1 + I_{a_n}(u_{a_n}), \quad \forall\, n \in \mathbb{N},$$

and

$$I_{a_n}(t_{a_n} w_{a_n}) \to \infty, \text{ as } n \to \infty.$$

Thus there exists C such that $[u_a]_{s,2} \le C$ for all $a \in (0, a_2)$. Therefore, (u_a) is uniformly bounded in $\dot{H}^s(\mathbb{R}^N)$. □

Now we discuss the regularity of the mountain pass solution u_a. Before proceeding to the proof of Theorem 1.2, we recall a result in [24, Theorem 1.1], where the author provided a sufficient condition for Hölder regularity of weak solutions to a class of nonlocal equations. To state the result, we define the following spaces:

$$L^1_{2s}(\mathbb{R}^N) := \left\{ u \in L^1_{loc}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \frac{|u(x)|}{1+|x|^{N+2s}}\, \mathrm{d}x < \infty \right\},$$

$$W^{s,2}_{loc}(\mathbb{R}^N) := \left\{ u \in L^2_{loc}(\mathbb{R}^N) : \left(\iint_{K\times K} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\, \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}} < \infty \right\},$$

where $K \subset \mathbb{R}^N$ is any relatively compact open set.

Proposition 3.1. *Let* $s \in (0,1)$ *and* $N > 2s$. *Let* $h \in L^q_{loc}(\mathbb{R}^N)$ *for* $q > \frac{N}{2s}$. *Assume that* $u \in W^{s,2}_{loc}(\mathbb{R}^N) \cap L^1_{2s}(\mathbb{R}^N)$ *is a weak solution of the equation* $(-\Delta)^s u = h$ *in* $\mathbb{R}^N$. *Then* $u \in C^{0,\alpha}_{loc}(\mathbb{R}^N)$ *for* $\alpha \in (0, \min\{2s - \frac{N}{q}, 1\})$.

Proof of Theorem 1.2. Let a_2 be as given in Theorem 1.1 and $a \in (0, a_2)$. For $\tau > 0$, we consider the truncation function $u_\tau \in L^\infty(\mathbb{R}^N)$ associated with u_a, defined as

$$u_\tau = \max\{-\tau, \min\{u_a, \tau\}\}.$$

For $r \geq 2$, set $\phi = u_a|u_\tau|^{r-2}$. Clearly, $\phi \in L^\infty(\mathbb{R}^N) \cap \dot{H}^s(\mathbb{R}^N)$. Taking ϕ as a test function in the weak formulation of u_a, we have

$$\begin{aligned} &\iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u_a(x) - u_a(y)(u_a(x)|u_\tau(x)|^{r-2} - u(y)|u_\tau(y)|^{r-2})}{|x-y|^{N+2s}} \,\mathrm{d}x\mathrm{d}y \\ &= \int\limits_{\mathbb{R}^N} g(x) f_a(u_a) u_a(x) |u_\tau(x)|^{r-2} \,\mathrm{d}x. \end{aligned} \tag{3.3}$$

We use [22, Lemma 3.1] and the embedding (2.1) to estimate the L.H.S of (3.3) as

$$\begin{aligned} &\iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u_a(x) - u_a(y)(u_a(x)|u_\tau(x)|^{r-2} - u_a(y)|u_\tau(y)|^{r-2})}{|x-y|^{N+2s}} \,\mathrm{d}x\mathrm{d}y \\ &\geq \frac{4(r-1)}{r^2} \iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u_a(x)|u_\tau(x)|^{\frac{r}{2}-1} - u_a(y)|u_\tau(y)|^{\frac{r}{2}-1})}{|x-y|^{N+2s}} \,\mathrm{d}x\mathrm{d}y \\ &\geq \frac{4(r-1)}{r^2} C(N,s) \left(\int\limits_{\mathbb{R}^N} \left| u_a(x) |u_\tau(x)|^{\frac{r}{2}-1} \right|^{2^*_s} \mathrm{d}x \right)^{\frac{2}{2^*_s}}. \end{aligned}$$

Hence from (3.3) we get for every $\tau > 0$ that

$$\left(\int\limits_{\mathbb{R}^N} \left| u_a(x) |u_\tau(x)|^{\frac{r}{2}-1} \right|^{2^*_s} \mathrm{d}x \right)^{\frac{2}{2^*_s}} \leq \frac{r^2}{4(r-1)} C(N,s) \int\limits_{\mathbb{R}^N} g(x) |f_a(u_a)| |u_a(x)|^{r-1} \,\mathrm{d}x.$$

Letting $\tau \to \infty$ the monotone convergence theorem yields

$$\left(\int\limits_{\mathbb{R}^N} |u_a(x)|^{\frac{r}{2} 2^*_s} \,\mathrm{d}x \right)^{\frac{2}{2^*_s}} \leq \frac{r^2}{4(r-1)} C(N,s) \int\limits_{\mathbb{R}^N} g(x) |f_a(u_a)| |u_a(x)|^{r-1} \,\mathrm{d}x. \tag{3.4}$$

Step 1. In this step, for $r_1 = 2^*_s + 1$, we show that $|u_a|^{r_1} \in L^{\frac{2^*_s}{2}}(\mathbb{R}^N)$ and there exists C such that $\| |u_a|^{r_1} \|_{\frac{2^*_s}{2}} \leq C$ for all $a \in (0, a_2)$. Let $\epsilon > 0$. Using the growth condition $(\tilde{f}1)$, for every $a < a_2$, we have $|f_a(u_a)| \leq C + \epsilon |u_a|^{2^*_s - 1}$, where $C = C(\epsilon, a_2)$

(see (2.7)). Applying Hölder's inequality with the conjugate pair $(\frac{2_s^*}{2}, \frac{2_s^*}{2_s^*-2})$ and uniform boundedness of (u_a) in $\dot{H}^s(\mathbb{R}^N)$ (from Theorem 1.1), we have the following estimates for all $a \in (0, a_2)$:

$$\int_{\mathbb{R}^N} g(x)|u_a(x)|^{2_s^*}\,\mathrm{d}x \le \|g\|_\infty \|u_a\|^{2_s^*}_{\dot{H}^s(\mathbb{R}^N)} \le C\|g\|_\infty,$$

$$\begin{aligned}\int_{\mathbb{R}^N} g(x)|u_a(x)|^{2_s^*-2}|u_a(x)|^{2_s^*+1}\,\mathrm{d}x &\le \|g\|_\infty \|u_a\|^{2_s^*-2}_{\dot{H}^s(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} |u_a(x)|^{\frac{2_s^*}{2}(2_s^*+1)}\,\mathrm{d}x\right)^{\frac{2}{2_s^*}} \\ &\le C\|g\|_\infty \left(\int_{\mathbb{R}^N} |u_a(x)|^{\frac{2_s^*}{2}(2_s^*+1)}\,\mathrm{d}x\right)^{\frac{2}{2_s^*}},\end{aligned}$$

where C does not depend on a and ϵ. Hence (3.4) yields

$$\begin{aligned}&\left(\int_{\mathbb{R}^N} |u_a(x)|^{\frac{2_s^*}{2}r_1}\,\mathrm{d}x\right)^{\frac{2}{2_s^*}} \\ &\le \frac{r_1^2}{4(r_1-1)} C\|g\|_\infty \left(C(\epsilon, a_2) + \epsilon \left(\int_{\mathbb{R}^N} |u_a(x)|^{\frac{2_s^*}{2}r_1}\,\mathrm{d}x\right)^{\frac{2}{2_s^*}}\right).\end{aligned} \tag{3.5}$$

Now we choose ϵ such that

$$\left(\frac{r_1^2}{4(r_1-1)} C\|g\|_\infty\right)\epsilon < \frac{1}{2}.$$

Therefore, from (3.5), there exists C such that

$$\frac{1}{2}\left(\int_{\mathbb{R}^N} |u_a(x)|^{\frac{2_s^*}{2}r_1}\,\mathrm{d}x\right)^{\frac{2}{2_s^*}} \le \frac{r_1^2}{4(r_1-1)} C\|g\|_\infty, \quad \forall\, a \in (0, a_2). \tag{3.6}$$

Thus the set $\{|u_a|^{r_1} : a \in (0, a_2)\}$ is uniformly bounded in $L^{\frac{2_s^*}{2}}(\mathbb{R}^N)$.

Step 2. In this step, we obtain the uniform L^∞ bound of (u_a). Using (2.5) and (3.4), for $r > r_1$ we have

$$\begin{aligned}&\left(\int_{\mathbb{R}^N} |u_a(x)|^{\frac{r}{2}2_s^*}\,\mathrm{d}x\right)^{\frac{2}{2_s^*}} \\ &\le \frac{r^2}{4(r-1)} C(N, s, f, a_2) \int_{\mathbb{R}^N} g(x)(1 + |u_a(x)|^{2_s^*-1})|u_a(x)|^{r-1}\,\mathrm{d}x.\end{aligned} \tag{3.7}$$

Set $m_1 = \frac{2_s^*(2_s^*-1)}{r-2}$ and $m_2 := r-1-m_1$. Observe that $m_1 < 2_s^*$ whenever $r > r_1$. Applying Young's inequality with $(\frac{2_s^*}{m_1}, \frac{2_s^*}{2_s^*-m_1})$ we have the following estimate:

$$|u|^{r-1} = |u|^{m_1}|u|^{m_2} \leq \frac{m_1}{2_s^*}|u|^{2_s^*} + \frac{2_s^*-m_1}{2_s^*}|u|^{\frac{2_s^* m_2}{2_s^*-m_1}} \leq |u|^{2_s^*} + |u|^{\frac{2_s^* m_2}{2_s^*-m_1}},$$

where we can verify $\frac{2_s^* m_2}{2_s^*-m_1} = 2_s^* - 2 + r$. Hence, by Theorem 1.1,

$$\begin{aligned}\int_{\mathbb{R}^N} |u_a(x)|^{r-1}\,\mathrm{d}x &\leq \int_{\mathbb{R}^N} |u_a(x)|^{2_s^*}\,\mathrm{d}x + \int_{\mathbb{R}^N} |u_a(x)|^{2_s^*-2+r}\,\mathrm{d}x \\ &\leq \|u_a\|^{2_s^*}_{\dot{H}^s(\mathbb{R}^N)} + \int_{\mathbb{R}^N} |u_a(x)|^{2_s^*-2+r}\,\mathrm{d}x \\ &\leq C\left(1 + \int_{\mathbb{R}^N} |u_a(x)|^{2_s^*-2+r}\,\mathrm{d}x\right),\end{aligned}$$

where C does not depend on a. Therefore, from (3.7) we obtain the following estimate:

$$\begin{aligned}&\left(1 + \int_{\mathbb{R}^N} |u_a(x)|^{\frac{r}{2}2_s^*}\,\mathrm{d}x\right)^{\frac{2}{2_s^*(r-2)}} \\ &\leq r^{\frac{1}{r-2}}\left(C\|g\|_\infty\right)^{\frac{1}{r-2}}\left(1 + \int_{\mathbb{R}^N} |u_a(x)|^{2_s^*-2+r}\,\mathrm{d}x\right)^{\frac{1}{r-2}},\end{aligned} \tag{3.8}$$

where $C = C(N, s, f, a_2)$. We consider the sequence (r_j) defined as follows:

$$r_1 = 2_s^* + 1, r_2 = 2 + \frac{2_s^*}{2}(r_1 - 2), \ldots, r_{j+1} = 2 + \frac{2_s^*}{2}(r_j - 2).$$

Notice that $2_s^* - 2 + r_{j+1} = \frac{2_s^*}{2} r_j$ and $r_{j+1} - 2 = \left(\frac{2_s^*}{2}\right)^j (r_1 - 2)$. Then (3.8) yields

$$\begin{aligned}&\left(1 + \int_{\mathbb{R}^N} |u_a(x)|^{\frac{r_{j+1}}{2}2_s^*}\,\mathrm{d}x\right)^{\frac{2}{2_s^*(r_{j+1}-2)}} \\ &\leq \left(r_{j+1}C\|g\|_\infty\right)^{\frac{1}{r_{j+1}-2}}\left(1 + \int_{\mathbb{R}^N} |u_a(x)|^{\frac{2_s^*}{2}r_j}\,\mathrm{d}x\right)^{\frac{2}{2_s^*(r_j-2)}}.\end{aligned}$$

Set

$$D_j = \left(1 + \int_{\mathbb{R}^N} |u_a(x)|^{\frac{2_s^*}{2}r_j}\,\mathrm{d}x\right)^{\frac{2}{2_s^*(r_j-2)}}.$$

We iterate the above inequality to get

$$D_{j+1} \leq C^{\sum_{k=2}^{j+1} \frac{1}{r_k-2}} \left(\prod_{k=2}^{j+1} r_k^{\frac{1}{r_k-2}} \right) D_1, \tag{3.9}$$

where $C = C(N, s, f, g, a_2)$. Using (3.6) of Step 1, $D_1 \leq C$ for some C independent of a. Moreover,

$$D_{j+1} \geq \left(\left(\int_{\mathbb{R}^N} u_a(x)^{\frac{2_s^* r_{j+1}}{2}} \, \mathrm{d}x \right)^{\frac{2}{2_s^* r_{j+1}}} \right)^{\frac{r_{j+1}}{r_{j+1}-2}} = \|u_a\|_{L^{\frac{2_s^* r_{j+1}}{2}}(\mathbb{R}^N)}^{\frac{r_{j+1}}{r_{j+1}-2}}.$$

Therefore, from (3.9) it is evident that

$$\|u_a\|_{L^{\frac{2_s^* r_{j+1}}{2}}(\mathbb{R}^N)}^{\frac{r_{j+1}}{r_{j+1}-2}} \leq C^{\sum_{k=2}^{j+1} \frac{1}{r_k-2}} \left(\prod_{k=2}^{j+1} r_k^{\frac{1}{r_k-2}} \right) C, \quad \forall\, a \in (0, a_2). \tag{3.10}$$

By noting that $r_j \to \infty$ as $j \to \infty$, we use the above estimate and the interpolation arguments to get $u_a \in L^r(\mathbb{R}^N)$ for any $r \in [2_s^*, \infty)$, and $\|u_a\|_r \leq C(r, N, s, f, g, a_2)$ for all $a \in (0, a_2)$. Furthermore,

$$\sum_{k=2}^{\infty} \frac{1}{r_k - 2} = \frac{N}{2s(2_s^* - 2)}$$

and

$$\prod_{k=2}^{\infty} r_k^{\frac{1}{r_k-2}} = \exp\left(\frac{2}{(2_s^* - 2)^2} \log\left(2 \left(\frac{2_s^*(2_s^* - 2)}{2} \right)^{2_s^*} \right) \right).$$

Thus taking the limit as $j \to \infty$ in (3.10), we conclude that $\|u_a\|_\infty \leq C(N, s, f, g, a_2)$ for all $a \in (0, a_2)$.

Step 3. This step verifies the continuity of u_a. Now, $u_a \in L^\infty(\mathbb{R}^N) \subset L^1_{2s}(\mathbb{R}^N)$. For $q > \frac{N}{2s}$,

$$\int_{\mathbb{R}^N} (g f_a(u_a))^q \leq C \int_{\mathbb{R}^N} g^q (1 + |u_a|^{(2_s^*-1)q}) \leq C \|g\|_q^q (1 + \|u_a\|_\infty^{(2_s^*-1)q}) \leq C, \quad \forall\, a \in (0, a_2).$$

Further, using Proposition 2.1, $\dot{H}^s(\mathbb{R}^N) \hookrightarrow L^2_{loc}(\mathbb{R}^N)$, and hence from the characterization of $\dot{H}^s(\mathbb{R}^N)$, $\dot{H}^s(\mathbb{R}^N) \hookrightarrow W^{s,2}_{loc}(\mathbb{R}^N)$. Therefore, applying Proposition 3.1 we conclude that $u_a \in \mathcal{C}^{0,\alpha}_{loc}(\mathbb{R}^N)$ for $\alpha \in (0, \min\{2s - \frac{N}{q}, 1\})$. In particular, $u_a \in \mathcal{C}(\mathbb{R}^N)$ for all $a \in (0, a_2)$. This completes the proof. □

Next, we prove a uniform lower bound for (u_a) in $L^\infty(\mathbb{R}^N)$.

Proposition 3.2. *Let f, g, a_2, u_a be as given in Theorem 1.2. Then there exist $\tilde{a}_2 \in (0, a_2)$ and $\beta_1 > 0$ such that $\|u_a\|_\infty \geq \beta_1$, for all $a \in (0, \tilde{a}_2)$.*

Proof. By the definition, $F_a(t) \geq -a|t|$, for all $t \in \mathbb{R}$. For β as given in Proposition 2.5 we see that $I_a(u_a) \geq \beta$, for all $a \in (0, a_2)$. Hence using the uniform boundedness of (u_a) in $\dot{H}^s(\mathbb{R}^N)$ (Theorem 1.1), we get for all $a < a_2$,

$$\frac{[u_a]_{s,2}^2}{2} = I_a(u_a) + \int_{\mathbb{R}^N} gF_a(u_a) \geq \beta - a \int_{\mathbb{R}^N} g|u_a| \geq \beta - aC \left(\|g\|_1 \|g\|_{\frac{N}{2s}} \right)^{\frac{1}{2}}.$$

Choose $0 < \tilde{a}_2 < \min \left\{ \beta C^{-1} \left(\|g\|_1 \|g\|_{\frac{N}{2s}} \right)^{-\frac{1}{2}}, a_2 \right\}$. Then

$$\frac{[u_a]_{s,2}^2}{2} \geq \beta_0 := \beta - \tilde{a}_2 C \left(\|g\|_1 \|g\|_{\frac{N}{2s}} \right)^{\frac{1}{2}} > 0, \quad \forall\, a \in (0, \tilde{a}_2).$$

Hence using $|f_a(u_a)| \leq C(1 + |u_a|^{2_s^*-1})$ and the fact that $u_a \in L^\infty(\mathbb{R}^N)$ (Theorem 1.2), we get

$$\beta_0 \leq \frac{1}{2} \int_{\mathbb{R}^N} g|f_a(u_a)u_a| \leq C\|g\|_1 \left(\|u_a\|_\infty + \|u_a\|_\infty^{2_s^*} \right),$$

where C does not depend on a. Therefore, there exists $\beta_1 > 0$ such that $\|u_a\|_\infty \geq \beta_1$, for all $a \in (0, \tilde{a}_2)$. □

4. POSITIVITY OF THE SOLUTIONS

This section contains the proof of Theorem 1.3. Afterwards, we give an example of a function satisfying all the hypotheses in this paper. The idea of our proof for the positivity of solutions is motivated by [1, Theorem 1.1] (also, see [4, Theorem 4.14]).

Proof of Theorem 1.3. (i) For $\tilde{a_2}$ as in Proposition 3.2, take $a_3 \in (0, \tilde{a_2})$. Let (a_n) be a sequence in $(0, a_3)$ such that $a_n \to 0$ as $n \to \infty$. We aim to show that u_{a_n} is nonnegative on $\mathbb{R}^N$ for large n. By Theorem 1.1, $u_{a_n} \in \dot{H}^s(\mathbb{R}^N)$ is a mountain pass solution of (SP), such that the following hold (up to a subsequence):

$$I'_{a_n}(u_{a_n}) = 0, \text{ for each } n \in \mathbb{N}, \; I_{a_n}(u_{a_n}) \to c, \text{ as } n \to \infty, \text{ and } \|u_{a_n}\|_{s,2} \leq C. \quad (4.1)$$

Therefore, (u_{a_n}) is a bounded Palais–Smale sequence in $\dot{H}^s(\mathbb{R}^N)$. For brevity, we denote the sequence (u_{a_n}) by (u_n). Since I_a satisfies the Cerami condition, using the same arguments as in Proposition 2.5(i) (replacing a by a_n), we obtain that (up to a subsequence) $u_n \to \tilde{u}$ in $\dot{H}^s(\mathbb{R}^N)$, and $u_n(x) \to \tilde{u}(x)$ a.e. in $\mathbb{R}^N$. We split the rest of our proof into two steps. In the first step, we prove that $\tilde{u}$ is nonnegative and (u_n) converges uniformly to $\tilde{u}$ on $\mathbb{R}^N$. In the second step, we obtain the non-negativity of u_n.

Step 1. We consider the following function:

$$f_0(t) = \begin{cases} f(t), & \text{if } t \geq 0, \\ 0, & \text{if } t \leq 0. \end{cases}$$

Since $a_n \to 0$ in $\mathbb{R}^+$ and $u_n \to \tilde{u}$ in $\dot{H}^s(\mathbb{R}^N)$, using (ii) of Proposition 2.4 it follows that

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^N} g(x) f_{a_n}(u_n)\phi(x)\,\mathrm{d}x = \int\limits_{\mathbb{R}^N} g(x) f_0(\tilde{u})\phi(x)\,\mathrm{d}x, \quad \forall\,\phi \in \dot{H}^s(\mathbb{R}^N).$$

So we have the following identity for every $\phi \in \dot{H}^s(\mathbb{R}^N)$:

$$\begin{aligned}
&\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(\tilde{u}(x) - \tilde{u}(y)(\phi(x) - \phi(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y \\
&= \lim_{n\to\infty} \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u_n(x) - u_n(y)(\phi(x) - \phi(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y \\
&= \lim_{n\to\infty} \int\limits_{\mathbb{R}^N} g(x) f_{a_n}(u_n)\phi(x)\,\mathrm{d}x = \int\limits_{\mathbb{R}^N} g(x) f_0(\tilde{u})\phi(x)\,\mathrm{d}x.
\end{aligned}$$

Therefore, $\tilde{u}$ satisfies the following equation weakly:

$$(-\Delta)^s u = g(x) f_0(u) \text{ in } \mathbb{R}^N. \tag{4.2}$$

Since $|x|^{2s-N}$ is a fundamental solution of $(-\Delta)^s$ (see [29, Theorem 5]), we get

$$\tilde{u}(x) = C(N,s) \int\limits_{\mathbb{R}^N} \frac{g(y) f_0(\tilde{u}(y))}{|x-y|^{N-2s}}\,\mathrm{d}y \geq 0 \ \text{ a.e. in } \mathbb{R}^N. \tag{4.3}$$

Further, using the similar set of arguments as given in Theorem 1.2, we see that $\tilde{u} \in L^\infty(\mathbb{R}^N) \cap \mathcal{C}(\mathbb{R}^N)$. Moreover, since u_n is a solution of (SP), we also have

$$u_n(x) = C(N,s) \int\limits_{\mathbb{R}^N} \frac{g(y) f_{a_n}(u_n(y))}{|x-y|^{N-2s}}\,\mathrm{d}y \ \text{ a.e. in } \mathbb{R}^N. \tag{4.4}$$

Using (4.3) and (4.4) we estimate $|u_n - \tilde{u}|$ as follows:

$$\begin{aligned}
|u_n(x) - \tilde{u}(x)| \leq C(N,s)\Bigg(&\int\limits_{B_1(x)} g(y) \frac{|f_{a_n}(u_n(y)) - f_0(\tilde{u}(y))|}{|x-y|^{N-2s}}\,\mathrm{d}y \\
&+ \int\limits_{\mathbb{R}^N\setminus B_1(x)} g(y) \frac{|f_{a_n}(u_n(y)) - f_0(\tilde{u}(y))|}{|x-y|^{N-2s}}\,\mathrm{d}y\Bigg).
\end{aligned} \tag{4.5}$$

Take $1 < \delta < \frac{N}{N-2s}$. Applying Hölder's inequality with the conjugate pair (δ, δ') we estimate the first integral of (4.5) as

$$\int\limits_{B_1(x)} g(y)\frac{|f_{a_n}(u_n(y)) - f_0(\tilde{u}(y))|}{|x-y|^{N-2s}}\,\mathrm{d}y$$

$$\leq \left(\int\limits_{B_1(x)} \frac{1}{|x-y|^{(N-2s)\delta}}\,\mathrm{d}y\right)^{\frac{1}{\delta}} \left(\int\limits_{B_1(x)} g(y)^{\delta'}|f_{a_n}(u_n(y)) - f_0(\tilde{u}(y))|^{\delta'}\,\mathrm{d}y\right)^{\frac{1}{\delta'}}.$$

We calculate

$$\int\limits_{B_1(x)} \frac{1}{|x-y|^{(N-2s)\delta}}\,\mathrm{d}y = \omega_N \int\limits_0^1 \frac{r^{N-1}}{r^{(N-2s)\delta}}\,\mathrm{d}r \leq C(N).$$

We show that the second integral of the above inequality converges to zero. Observe that

$$|f_{a_n}(u_n(y)) - f_0(\tilde{u}(y))|^{\delta'} \leq 2^{\delta'-1}\left(a_n^{\delta'} + |f_0(u_n(y)) - f_0(\tilde{u}(y))|^{\delta'}\right), \tag{4.6}$$

where using (2.6) and the uniform boundedness of the mountain pass solution in $L^\infty(\mathbb{R}^N)$ (Theorem 1.2), we get

$$|f_0(u_n(y)) - f_0(\tilde{u}(y))|^{\delta'} \leq C2^{\delta'-1}(1 + |u_n(y)|^{(2_s^*-1)\delta'} + |\tilde{u}(y)|^{(2_s^*-1)\delta'}) \leq C.$$

Moreover, $f_0(u_n(y)) \to f_0(\tilde{u}(y)$ and $a_n \to 0$ as $n \to \infty$. Therefore, by the dominated convergence theorem,

$$\int\limits_{B_1(x)} g(y)^{\delta'}(a_n^{\delta'} + |f_0(u_n(y)) - f_0(\tilde{u}(y))|^{\delta'})\,\mathrm{d}y \to 0.$$

Hence using (4.6) and the generalized dominated convergence theorem, we conclude that

$$\int\limits_{B_1(x)} g(y)^{\delta'}|f_{a_n}(u_n(y)) - f_0(\tilde{u}(y))|^{\delta'}\,\mathrm{d}y \to 0, \text{ as } n \to \infty.$$

Next, the second integral of (4.5) has the following bound:

$$\int\limits_{\mathbb{R}^N\setminus B_1(x)} g(y)\frac{|f_{a_n}(u_n(y)) - f_0(\tilde{u}(y))|}{|x-y|^{N-2s}}\,\mathrm{d}y \leq \int\limits_{\mathbb{R}^N\setminus B_1(x)} g(y)|f_{a_n}(u_n(y)) - f_0(\tilde{u}(y))|.$$

Again by the generalized dominated convergence theorem,

$$\int\limits_{\mathbb{R}^N\setminus B_1(x)} g(y)|f_{a_n}(u_n(y)) - f_0(\tilde{u}(y))| \to 0.$$

Therefore, (4.5) yields $u_n \to \tilde{u}$ in $L^\infty(\mathbb{R}^N)$ as $n \to \infty$. Thus (u_n) converges uniformly to $\tilde{u}$ on $\mathbb{R}^N$.

Step 2. Now $\tilde{u} \in \mathcal{C}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is a non-negative function and satisfies $(-\Delta)^s \tilde{u} \geq 0$ in the weak sense in $\mathbb{R}^N$ (from (4.2)). Suppose $\tilde{u}(x_0) = 0$ for some $x_0 \in \mathbb{R}^N$. Since $\tilde{u}$ satisfies all the properties of the strong maximum principle [20, Proposition 5.2.1], we conclude that $\tilde{u}$ vanishes identically on $\mathbb{R}^N$. Further, from the uniform lower bound of (u_n) in Proposition 3.2 and the uniform convergence of $u_n \to \tilde{u}$ (Step 1), there exists $\beta_2 > 0$ such that $\|\tilde{u}\|_\infty \geq \beta_2$, a contradiction. Thus $\tilde{u} \neq 0$ on $\mathbb{R}^N$ and [20, Proposition 5.2.1] yields $\tilde{u} > 0$ on $\mathbb{R}^N$. Therefore, again from the uniform convergence of (u_n), there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$, $u_n \geq 0$ on $\mathbb{R}^N$.

(ii) For a sequence (a_n) given in (i), we show u_{a_n} (denoted by u_n) is positive on $\mathbb{R}^N$ for large n. For each $n \in \mathbb{N}$, since f_{a_n} is locally Lipschitz (from $(f3)$) and $0 \leq u_n, \tilde{u} \leq C$, we have $|f_{a_n}(u_n(y)) - f_{a_n}(\tilde{u}(y))| \leq M|u_n(y) - \tilde{u}(y)|$ for some $M > 0$. For $x \in \mathbb{R}^N \setminus \{0\}$, using (4.3) and (4.4) we write

$$|u_n(x) - \tilde{u}(x)| \leq C(N,s) \left(M \int_{\mathbb{R}^N} \frac{g(y)|u_n(y) - \tilde{u}(y)|}{|x-y|^{N-2s}} \, \mathrm{d}y + a_n \int_{\mathbb{R}^N} \frac{g(y)}{|x-y|^{N-2s}} \, \mathrm{d}y \right).$$

Since g satisfies $(g1)$, from the above inequality and Step 1 we get

$$|u_n(x) - \tilde{u}(x)| \leq C(N,s) \left(M \|u_n - \tilde{u}\|_\infty + a_n \right) \frac{C(g)}{|x|^{N-2s}}.$$

Hence

$$\sup_{x \in \mathbb{R}^N \setminus \{0\}} \left\{ |x|^{N-2s} |u_n(x) - \tilde{u}(x)| \right\} \to 0, \text{ as } n \to \infty. \tag{4.7}$$

Now we show that $\lim\limits_{|x| \to \infty} |x|^{N-2s} \tilde{u}(x) > 0$. Using (4.3) we get

$$\begin{aligned} \lim_{|x| \to \infty} |x|^{N-2s} \tilde{u}(x) &= C(N,s) \lim_{|x| \to \infty} \int_{\mathbb{R}^N} \frac{g(y) f_0(\tilde{u}(y)) |x|^{N-2s}}{|x-y|^{N-2s}} \, \mathrm{d}y \\ &\geq C(N,s) \lim_{|x| \to \infty} \int_{B_R} \frac{g(y) f_0(\tilde{u}(y)) |x|^{N-2s}}{|x-y|^{N-2s}} \, \mathrm{d}y, \end{aligned} \tag{4.8}$$

for any $R > 0$. Choose $R > 0$ arbitrarily. Then there exists $x \in \mathbb{R}^N$ such that $|x| > 2R + 1$. Hence

$$|x-y|^{N-2s} \geq ||x| - |y||^{N-2s} \geq ||x| - R|^{N-2s} \geq 2^{2s-N} (1 + |x|)^{N-2s}, \text{ for } y \in B_R.$$

Using the above estimate, for $y \in B_R$ we get

$$\frac{g(y) f_0(\tilde{u}(y)) |x|^{N-2s}}{|x-y|^{N-2s}} \leq 2^{N-2s} g(y) f_0(\tilde{u}(y)).$$

Further,

$$\frac{g(y)f_0(\tilde{u})|x|^{N-2s}}{|x-y|^{N-2s}} \to g(y)f_0(\tilde{u}) \text{ a.e. in } B_R,$$

as $|x| \to \infty$. Therefore, the dominated convergence theorem yields

$$\lim_{|x|\to\infty} \int_{B_R} \frac{g(y)f_0(\tilde{u}(y))|x|^{N-2s}}{|x-y|^{N-2s}} \, \mathrm{d}y = \int_{B_R} g(y)f_0(\tilde{u}(y)) \, \mathrm{d}y.$$

Hence from (4.8) we conclude that

$$\lim_{|x|\to\infty} |x|^{N-2s}\tilde{u}(x) \geq C(N,s) \int_{B_R} g(y)f_0(\tilde{u}(y)) \, \mathrm{d}y.$$

Letting $R \to \infty$ and applying the Fatou's lemma,

$$\lim_{|x|\to\infty} |x|^{N-2s}\tilde{u}(x) \geq C(N,s) \int_{\mathbb{R}^N} g(y)f_0(\tilde{u}(y)) \, \mathrm{d}y.$$

Further, using (4.3) and $\tilde{u} > 0$ on $\mathbb{R}^N$, it follows that $gf_0(\tilde{u}) \gneqq 0$ on $\mathbb{R}^N$. Hence,

$$\lim_{|x|\to\infty} |x|^{N-2s}\tilde{u}(x) > 0.$$

Therefore, from (4.7) there exists $n_2 \in \mathbb{N}$ and $R >> 1$ such that for $n \geq n_2$, $u_n > 0$ on B_R^c. Moreover, since $\tilde{u} \in \mathcal{C}(\mathbb{R}^N)$, there exists $\eta > 0$ such that $\tilde{u} > \eta$ on $\overline{B_R}$. Therefore, from the uniform convergence of (u_n) (in Step 1), there exists $n_3 \in \mathbb{N}$ such that for $n \geq n_3$, $u_n > 0$ on $\overline{B_R}$. Thus, by choosing $n_4 = \max\{n_2, n_3\}$, we see that for $n \geq n_4$, $u_n > 0$ on $\mathbb{R}^N$. This completes the proof. □

Example 4.1. Let $s \in (0,1)$ and $N > 2s$. For $R > 0$, we consider the following functions:

$$f(t) = 2t\ln(1+|t|), \text{ for } t \in \mathbb{R}^+, \quad g(y) = \frac{\chi_{B_R(0)}(y)}{(1+|y|)^{2(N-2s)}}, \text{ for } y \in \mathbb{R}^N.$$

(i) We can verify that f satisfies $(f1)$–$(f3)$ and $(\tilde{f}1)$.
(ii) Clearly $g \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. We show that g satisfies $(g1)$. For $x \in \mathbb{R}^N \setminus \{0\}$, split

$$\int_{\mathbb{R}^N} \frac{g(y)}{|x-y|^{N-2s}} \, \mathrm{d}y = \int_{|x-y|\geq \frac{|x|}{2}} \frac{g(y)}{|x-y|^{N-2s}} \, \mathrm{d}y + \int_{|x-y|\leq \frac{|x|}{2}} \frac{g(y)}{|x-y|^{N-2s}} \, \mathrm{d}y.$$

The first integral has the following bound:

$$\int_{|x-y|\geq \frac{|x|}{2}} \frac{g(y)}{|x-y|^{N-2s}} \, \mathrm{d}y \leq \left(\frac{2}{|x|}\right)^{N-2s} \|g\|_1.$$

Now consider the case $|x-y| \leq \frac{|x|}{2}$. Set $z = x - y$. Then $|x-z| \geq ||x| - |z|| \geq \frac{|x|}{2}$ and hence $|x-z| \geq |z|$. Using the fact that $g(y) \leq g(\frac{x}{2})$ and $g(y) \leq g(z)$ we obtain

$$\begin{aligned}\int\limits_{|x-y|\leq \frac{|x|}{2}} \frac{g(y)}{|x-y|^{N-2s}}\,\mathrm{d}y &\leq \int\limits_{|z|\leq \frac{|x|}{2}} \frac{(g(\frac{x}{2})g(z))^{\frac{1}{2}}}{|z|^{N-2s}}\,\mathrm{d}z\\ &\leq \frac{2^{N-2s}\chi_{B_R(0)}(\frac{x}{2})}{(2+|x|)^{N-2s}} \int\limits_{B_R(0)} \frac{\mathrm{d}z}{(1+|z|)^{N-2s}|z|^{N-2s}}\\ &\leq \left(\frac{2}{|x|}\right)^{N-2s} \omega(N) \int\limits_0^R \frac{r^{2s-1}}{(1+r)^{N-2s}}\,\mathrm{d}r\\ &\leq \left(\frac{2}{|x|}\right)^{N-2s} C(N),\end{aligned}$$

for some constant $C(N)$. Here $\omega(N)$ is the measure of $B_1(0)$ in $\mathbb{R}^N$. Therefore,

$$|x|^{N-2s} \int\limits_{\mathbb{R}^N} \frac{g(y)}{|x-y|^{N-2s}}\,\mathrm{d}y \leq C(g,N)$$

for $x \in \mathbb{R}^N \setminus \{0\}$.

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