ZESZYTY NAUKOWE POLITECHNIKI ŚLĄSKIEJ 2011
Seria: MATEMATYKA STOSOWANA z. 1 Nr kol. 1854

Beata BAJORSKA
Institute of Mathematics
Silesian University of Technology

ON JORDAN-CHEVALLEY DECOMPOSITION

Summary. Expressing a linear operator f on a finite-dimensional vector space over any field K as a sum of two commuting operators – semisimple and nilpotent – is called the Jordan-Chevalley decomposition of f. It is known that this decomposition exists for an arbitrary f if only K is perfect. In this paper we give some methods for determining the decomposition.

O ROZKŁADZIE JORDANA-CHEVALLEYA

Streszczenie. Zapis operatora liniowego działającego na skończenie wymiarowej przestrzeni wektorowej nad dowolnym ciałem K w postaci sumy dwóch przemiennych operatorów – półprostego i nilpotentnego – nazywamy rozkładem Jordana-Chevalleya tego operatora. Wiadomo, że jeśli K jest ciałem doskonałym, to taki rozkład istnieje dla dowolnego operatora. Celem artykułu jest omówienie metod wyznaczenia postulowanego rozkładu.

1. Introduction

If a matrix is upper-triangular, it is easy to decompose it into a "diagonal part" and "strictly upper-triangular part" (which is in particular nilpotent). For instance

2010 Mathematics Subject Classification: 15A04, 15A21, 12F10.
Wpłynęło do Redakcji (received): 14.7.2011

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}.$$

It is obvious that the matrices on the right commute. But this is not always so, for instance

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix},$$

but the matrices on the right do not commute. Since each linear operator on a vector space of dimension n can be represented by a square matrix of order n, the above decompositions can be also viewed as decompositions of linear operators. Diagonality (or, in general – semisimplicity) and nilpotency of summands can provide some profits in examining the properties of a matrix itself. However, to derive really nice and interesting properties, one usually needs commutativity of the summands, like in the binomial theorem (for finding powers of a sum).

Expressing a linear operator f on a finite-dimensional vector space over any field K as a sum of two commuting operators – semisimple and nilpotent – is called the Jordan-Chevalley decomposition of f (or Jordan decomposition). This decomposition can be easily described if f has so-called Jordan normal form (which explains the first part of the name), but it may exist even if the Jordan normal form does not. The most general result (known as the Jordan-Chevalley decomposition theorem) guarantees the existence of the unique Jordan-Chevalley decomposition if only K is perfect. Thus it plays a very important role in examining Lie algebras, leading eventually to the corresponding decompositon of every element in finite-dimensional Lie algebras in two main cases – the semisimple Lie algebras over algebraically closed fields of characteristic 0 (like $\mathbb{C}$, examined by Claude Chevalley (which explains the second part of the name), see e.g. [2] for further details) and the restricted Lie algebras over perfect fields of prime characteristic (usually called the Jordan-Chevalley-Seligman decomposition then). The Jordan-Chevalley decomposition is also called Dunford decomposition (after Nelson Dunford, who generalized it to Banach spaces) or SN decomposition (Semisimple & Nilpotent).

In this paper we focus on determining the decomposition. For the convenience of the Reader we include Section 2, which is a collection of notions and facts crucial for further considerations. However, we still assume the Reader is familiar with the very basic concepts of linear algebra, fields and polynomial rings. In Section 3 we formulate and prove the Jordan-Chevalley decomposition theorem (our proof is inspired by those in [2] and [3]) and point out some observations. Also, we give an example of a matrix over a non-perfect field having no Jordan-Chevalley decomposition, confirming that the perfectness of the field is crucial. Finally, in

Section 4 we discuss a few methods (based on the results from Section 3) for determining the Jordan-Chevalley decomposition and give some examples.

2. Preliminaries

Note that this section plays an informative role only – considering the topics in details would exceed the scope of this paper. We refer the Reader to any book on linear algebra, fields and polynomial rings (e.g. [1]) for further details.

Notions: From now on K will denote a field, V – a finite-dimensional vector space over K, $End(V)$ – the ring of linear operators (endomorphisms) on V, $K[x]$ – the ring of polynomials (in x) over K and $M_n(K)$ – the ring of square matrices of order n over K. To refer to the elements of a given matrix A we write $A = [a_{ij}]$. The image of $f \in End(V)$ will be denoted by $f(V)$, the identity operator by Id, the identity matrix (of any order) by I and the zero operator or matrix by 0.

Definition 1. (i) *A field L is called an* ***extension field of*** *K if K is a subfield of L. An extension field L of K is called* ***finite-dimensional*** *if the dimension of L (viewed as a vector space over K) is finite.*

(ii) *Let L be an extension field of K. $a \in L$ is called* ***algebraic over*** *K if it is a root of some polynomial over K, otherwise it is called* ***transcendental over*** *K. L is called an* ***algebraic extension field of*** *K if every element in L is algebraic over K. The smallest extension field of K containing all roots of a given polynomial $P(x) \in K[x]$ is called a* ***root*** *(or splitting)* ***field of*** *$P(x)$* ***over*** *K.*

(iii) *Let L be an algebraic extension field of K and G – a group of automorphisms σ of L that fix each $k \in K$ (that is $\sigma(k) = k$). A set of all elements in L fixed by each automorphism in G forms a field called the* ***fixed field for*** *G and it contains K. L is called a* ***Galois extension of*** *K if K itself is the fixed field of G. Then G is called the* ***Galois group of*** *L* ***over*** *K and is denoted by $G(L/K)$.*

Examples: (i) $\mathbb{R}$ is a subfield of $\mathbb{C}$ and the geometrical representation implies that $\mathbb{C} \cong \mathbb{R}^2$ (as vector spaces). Hence $\mathbb{C}$ is a finite-dimensional extension of $\mathbb{R}$. (ii) $\mathbb{C}$ is also an algebraic extension of $\mathbb{R}$, since $a+bi$ is a root of $(x-a)^2+b^2$. In particular, $\mathbb{C}$ is a root field of x^2+1 over $\mathbb{R}$. π is transcendental over $\mathbb{Q}$. Note that transcendental over any field K always exists and it it frequently used – e.g. it is x we use to define $K[x]$. (iii) $\mathbb{C}$ is a Galois extension of $\mathbb{R}$, because one of the automorphisms of $\mathbb{C}$ fixing $\mathbb{R}$ elementwise is conjugation and the set of elements it fixes is exactly $\mathbb{R}$. Note that (iii) is not the standard definition of a Galois extension – usually it is a theorem (see e.g. [1, Chapter VIII, Theorem 1]).

Definition 2. *K is called **perfect** if every irreducible polynomial over K decomposes into distinct linear factors (over some extension field of K).*

Examples: Every field of characteristic 0 is perfect. If a field has a prime characteristic p, then it is perfect if and only if every element is a p-th power of some element. In particular, every finite or algebraically closed field is perfect.

Definition 3. (i) *Let $f \in End(V)$. Then $\varphi_f : K[x] \to End(V)$ defined as*

$$\varphi_f(a_0 + a_1x + \cdots + a_nx^n) = a_0Id + a_1f + \cdots + a_nf^n$$

*is a homomorphism of rings. Its every value, for a given polynomial $P(x)$ denoted by $P(f)$, is called a **polynomial of** f. $\rho_f(x) \in K[x]$ is called the **minimal polynomial of** f if it is monic of the least possible degree such that $\rho_f(f) = 0$.*

(ii) *Let $A \in M_m(K)$. Then $\varphi_A : K[x] \to M_m(K)$ defined as*

$$\varphi_A(a_0 + a_1x + \cdots + a_nx^n) = a_0I + a_1A + \cdots + a_nA^n$$

*is a homomorphism of rings. Its every value, for a given polynomial $P(x)$ denoted by $P(A)$, is called a **polynomial of** A. $\rho_A(x) \in K[x]$ is called the **minimal polynomial of** A if it is monic of the least possible degree such that $\rho_A(A) = 0$.*

Examples: For any K, V, f, A we have $\varphi_f(x) = f$, $\varphi_A(x) = A$, so f (resp. A) is a polynomial of f (resp. A). Moreover, $\rho_{Id}(x) = x - 1$, since $Id \neq 0$ and $\rho_{Id}(Id) = Id - Id = 0$. Similarly, $\rho_I(x) = x - 1$. Note that if A is a matrix representation of f, then $\rho_A(x) = \rho_f(x)$. Also, if $f \neq 0$ ($A \neq 0$) then the minimal polynomial has a positive degree.

Definition 4. (i) *Let $A \in M_n(K)$. Then $\chi_A(x) := \det(xI - A)$ is a polynomial over K called the **characteristic polynomial of** A. Roots of $\chi_A(x)$ are called **eigenvalues of** A.*

(ii) *Let $f \in End(V)$. The characteristic polynomial of a matrix representation of f with respect to every basis of V is the same and is called the **characteristric polynomial of** f. It is denoted by $\chi_f(x)$. Its roots are called **eigenvalues of** f.*

Examples: The characteristic polynomial of a zero matrix of order n is x^n, the characteristic polynomial of the identity matrix of order n is $(x - 1)^n$. Both have only one eigenvalue, 0 and 1, respectively. Note that $\deg(\chi_A(x))$ is equal to the order of A, whence $\deg(\chi_f(x)) = \dim V$. Sometimes the characteristic polynomial is defined as $\det(A - xI)$.

Definition 5. *Let V be a vector space over K and let $V_1, V_2, \dots, V_r$ be its subspaces. We say that V is a* ***direct sum of subspaces*** *$V_1, V_2, \dots, V_r$ and write $V = \bigoplus_{i=1}^{r} V_i$ if $V = \sum_{i=1}^{r} V_i$ and $V_i \cap \left(\sum_{k \neq i} V_k \right) = \{0_V\}$ for each $i = 1, 2, \dots, r$.*

Example: Let $V = \mathbb{R}^2,\ V_1 = \{(a,0) |\ a \in \mathbb{R}\},\ V_2 = \{(0,b) |\ b \in \mathbb{R}\}$. Then $V = \bigoplus_{i=1}^{2} V_i$ since $V_1 + V_2 = \{v_1 + v_2 |\ v_1 \in V_1,\ v_2 \in V_2\} = V$ and $V_1 \cap V_2 = \{(0,0)\}$.

Definition 6. *(i) Let $A = [a_{ij}] \in M_m(K),\ B = [b_{ij}] \in M_n(K)$.*
Then $C = [c_{ij}] \in M_{m+n}(K)$ such that $c_{ij} = \begin{cases} a_{ij} & 1 \leqslant i,j \leqslant m \\ b_{i-m,j-m} & m < i,j \leqslant m+n \\ 0 & \text{otherwise} \end{cases}$
is called a ***direct sum of*** *A, B and denoted by $A \oplus B$.*

(ii) A direct sum of any finite number of square matrices, $A_1, .., A_n$ say, is defined in the standard inductive way and denoted by $\bigoplus_{i=1}^{n} A_i$ for short.

Examples: $K = \mathbb{R}$ (i) $A = [-3]$ $B = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$. Then $A \oplus B = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 4 & 5 \end{bmatrix}$.
(ii) $A_1 = [7],\ A_2 = [8],\ A_3 = [1]$. Then $\bigoplus_{i=1}^{3} A_i = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Definition 7. *(i) For $a \in K$ let* $m_{ij}^{a} := \begin{cases} a & i = j \\ 1 & j = i+1 \\ 0 & \text{otherwise} \end{cases}$.
A square matrix $J_a := [m_{ij}^{a}]$ of any order is called a ***Jordan block*** *(corresponding to a). A direct sum of a finite number of Jordan blocks is called a* ***Jordan matrix****.*

(ii) Let $A \in M_n(K)$. A Jordan matrix $J \in M_n(K)$ is called a ***Jordan normal form of*** *A if there exists an invertible matrix $C \in M_n(K)$ such that $A = CJC^{-1}$.*

(iii) We say that $f \in End(V)$ ***has a Jordan normal form*** *if there exists a basis of V with respect to which a matrix representation of f is a Jordan matrix.*

Examples: (i) The identity matrix of any order n is a Jordan matrix since $I = \bigoplus_{i=1}^{n} [1]$. (ii) Each Jordan matrix is a Jordan normal form of itself (for $C = I$). (iii) Id has a Jordan normal form (since its matrix representation with respect to the standard basis is I). Note that elements lying on the diagonal of J are eigenvalues of A. Also note, that if A is a matrix representation of f (with respect to the standard basis), J, C as in (ii), then columns of C are in fact vectors in the basis with respect to which a matrix representation of f is J.

Definition 8. *(i) $A \in M_n(K)$ is called **diagonalizable** if there exists a diagonal matrix $D \in M_n(K)$ and an invertible matrix $C \in M_n(K)$ such that $A = CDC^{-1}$. $A \in M_n(K)$ is called **semisimple** if it is diagonalizable over some extension field of K. A is called **nilpotent** if there exists a natural number n such that $A^n = 0$.*

*(ii) $f \in End(V)$ is called **diagonalizable** (respectively: **semisimple**) if any matrix representation of f is diagonalizable (respectively: semisimple). f is called **nilpotent** if there exists a natural number n such that $f^n = 0$.*

Examples: Any diagonal matrix (operator) is diagonalizable. Any diagonalizable matrix (operator) is semisimple. A zero matrix (operator) is obviously nilpotent. Also, every upper-triangular (or lower-triangular) matrix A of order n with only zeros on the diagonal is nilpotent, since its characteristic polynomial is x^n whence $A^n = 0$ (see Fact 4 below).

Definition 9. *Let $P_1(x),\ P_2(x), \ldots, P_r(x) \in K[x]$. The **greatest common divisor (gcd)** of $P_1(x), \ldots, P_r(x)$ is a monic polynomial of the greatest possible degree that divides each $P_i(x)$. $P_1(x), \ldots, P_r(x)$ are called **coprime** if their gcd equals 1.*

Examples: $P_1(x) = 2x - 4, P_2(x) = (2x - 4)^2, P_3(x) = x - 3 \in \mathbb{R}[x]$. Then gcd of $P_1(x),\ P_2(x)$ is $x - 2$ and $P_1(x), P_2(x), P_3(x)$ are coprime. Note that gcd is unique and can be found e.g. by the Euclidean Algorithm (see e.g. [1, Chapter V, Theorem 2]).

The following facts will be given exactly in the form needed for further considerations, however some of them are more general.

Fact 1. *If $f, g \in End(V)$, $fg = gf$ and $P(x) \in K[x]$, then $gP(f) = P(f)g$. In particular, $Q(f)P(f) = P(f)Q(f)$ for every polynomial $Q(x) \in K[x]$.*

Remark: The proof is straightforward.

Fact 2. *If $P_1(x), P_2(x), \ldots, P_r(x) \in K[x]$ are coprime polynomials, then there exist polynomials $Q_1(x), Q_2(x), \ldots, Q_r(x) \in K[x]$ such that $\sum_{i=1}^{r} P_i(x)Q_i(x) = 1$.*

Remark: *gcd* of two polynomials can be expressed as their "linear" combination by the Inverse Euclidean Algorithm (the proof is similar to that for integers). Since $gcd(P_1(x), .., P_r(x)) = gcd(gcd(P_1(x), \ldots, P_{r-1}(x)), P_r(x))$, the result for every finite number of polynomials follows by induction. Fact 2 is a special case of that.

Fact 3. *Let K be a perfect field, $P(x) \in K[x]$ and let L be a root field for $P(x)$ over K. Then L is a Galois extension of K.*

Remark: Follows from [1, Chapter VII, Theorem 4] and [1, Corollary, p.190].

Fact 4 (cf. Cayley-Hamilton theorem)**.** *If $f \in End(V)$ then $\chi_f(f) = 0$.*

Remark: For the proof see e.g. [1, Chapter XV, Theorem 8].

Fact 5. *Let $A, B \in M_n(K)$. If $A = \bigoplus_{i=1}^{n} A_i,\ B = \bigoplus_{i=1}^{n} B_i$ and for each i A_i, B_i have the same order, then $A + B = \bigoplus_{i=1}^{n} (A_i + B_i)$ and $AB = \bigoplus_{i=1}^{n} A_i B_i$.*

Remark: Obtanaible by straightforward calculations.

Fact 6 (cf. Jordan normal form)**.** *If $A \in M_n(K)$ and K contains all eigenvalues of A, then A has a Jordan normal form, which is unique up to the order of Jordan blocks.*

Remark: For the proof see e.g. [1, Chapter XV, §3].

Fact 7. *A difference of commuting nilpotent (respectively: semisimple) operators is nilpotent (respectively: semisimple).*

Proof. Let $f, g \in End(V),\ f^n = 0,\ g^m = 0$ for some natural n, m and $fg = gf$. Then, by binomial theorem, we obtain

$$(f - g)^{n+m} = \sum_{i=0}^{n+m} \binom{n+m}{i} (-1)^i g^i f^{m+n-i} =$$

$$= f^n \sum_{i=0}^{m} \binom{n+m}{i} (-1)^i g^i f^{m-i} + g^m \sum_{i=m+1}^{n+m} \binom{n+m}{i} (-1)^i g^{i-m} f^{m+n-i} = 0$$

so $f - g$ is nilpotent. For the semisimple case see e.g. [3, Lemma 6.1.3]. □

Fact 8. *If $f \in End(V)$ is nilpotent, then $\chi_f(x) = x^{\dim V}$. Consequently, the only operator that is both nilpotent and semisimple is the zero operator.*

Proof. If f is nilpotent, then $f^n = 0$ for some n. Therefore the minimal polynomial $\rho_f(x)$ divides x^n (see e.g. [1, Chapter XV, §3]) and hence $\rho_f(x) = x^m,\ m \leqslant n$ (since factorization in $K[x]$ is unique, see e.g. [1, Corollary, p. 121]). The irreducible factors in the characteristic and minimal polynomials of f are the same (see e.g. [1, Corollary, p.402]) whence the only eigenvalue of f is 0 and its multiplicity equals $\dim V$, that is $\chi_f(x) = x^{\dim V}$. Hence f has a Jordan normal form, a matrix representation of which, J say, has only zeros on the diagonal. If f is also semisimple, J has to be diagonal, whence $J = 0$ and consequently $f = 0$. □

3. Jordan-Chevalley decomposition – facts & proofs

We shall now discuss the existence of the Jordan-Chevally decomposition.

Definition 10. *Let A be a linear operator on a finite-dimensional vector space (respectively: a square matrix) over a field K and suppose that there exist a pair of commuting operators (respectively: square matrices) – semisimple A_s and nilpotent A_n – such that $A = A_s + A_n$. Then A_s is called the* ***semisimple part of*** *A, A_n is called the* ***nilpotent part of*** *A and $A = A_s + A_n$ is called the* ***Jordan-Chevalley decomposition of*** *A.*

Theorem (Jordan-Chevalley decomposition theorem). *Let f be a linear operator on a finite-dimensional vector space V over a field K.*

If K is perfect, then there exists exactly one pair of commuting operators – semisimple f_s and nilpotent f_n – such that $f = f_s + f_n$.

Proof. The proof organized as follows: we first prove the statement under assumption, that K contains all roots of $\chi_f(x)$, which is done in a few steps. Namely, we start with constructing an operator f_s and showing it is semisimple, then we show that $f - f_s$ is nilpotent and commutes with f_s, and the last step is to prove the uniqueness. Finally, we prove the statement in general.

1. Construction of f_s. Suppose that K contains all roots of the characteristic polynomial $\chi_f(x)$ of f. Thus there exist pairwise distint elements $\lambda_1, \lambda_2, \ldots, \lambda_r \in K$ and $n_1, n_2, \ldots, n_r \in \mathbb{N}$ such that $\sum_{i=1}^{r} n_i = \dim V$ and

$$\chi_f(x) = \prod_{i=1}^{r} (x - \lambda_i)^{n_i}. \tag{1}$$

For each i consider the product of all but i-th factors in $\chi_f(x)$, that is $\prod_{k \neq i} (x - \lambda_k)^{n_k}$. These polynomials have no common factor of degree > 0, which means that they are coprime. Thus (Fact 2) there exist polynomials $Q_i(x),\ i = 1, 2, \ldots, r$ such that

$$\sum_{i=1}^{r} \left(Q_i(x) \prod_{k \neq i} (x - \lambda_k)^{n_k} \right) = 1. \tag{2}$$

To simplify the notation let

$$P_i := Q_i(f) \prod_{k \neq i} (f - \lambda_k Id)^{n_k},\ i = 1, 2, \ldots, r \tag{3}$$

and we define

$$f_s := \sum_{i=1}^{r} \lambda_i P_i. \tag{4}$$

2. Semisimplicity of f_s. To prove that f_s is semisimple we first show that

$$V = \bigoplus_{i=1}^{r} V_i = \bigoplus_{i=1}^{r} P_i(V) \quad \text{where} \quad V_i := \{v \in V | \ (f - \lambda_i Id)^{n_i}(v) = 0\}. \tag{5}$$

We shall do it by proving that $V = \bigoplus_{i=1}^{r} P_i(V)$ and $P_i(V) = V_i$ for each i.

Indeed, the Cayley-Hamilton theorem (Fact 4) states that

$$\chi_f(f) = 0 \tag{6}$$

so from (3) $(f - \lambda_i Id)^{n_i} P_i = Q_i(f)\chi_f(f) = 0$ for $i = 1, 2, \dots, r$, which means that

$$P_i(V) \subseteq V_i, \ i = 1, 2, \dots, r. \tag{7}$$

Moreover, from (2) and (3) we have

$$Id = \sum_{i=1}^{r} P_i \tag{8}$$

so $V = Id(V) = \sum_{i=1}^{r} P_i(V) \subseteq \sum_{i=1}^{r} V_i \subseteq V$, whence

$$V = \sum_{i=1}^{r} P_i(V) = \sum_{i=1}^{r} V_i. \tag{9}$$

Now we show that $\sum P_i(V)$ is direct. From (3) it follows that if $i \neq j$ then $P_j P_i = Q_j(f)Q_i(f) \prod_{k \neq i,j} (f - \lambda_k Id)^{n_k} \chi_f(f)$ so (6) yields

$$P_j P_i = 0, \ i \neq j, \ i, j = 1, 2, \dots, r \tag{10}$$

and if we multiply both sides in (8) by P_j we obtain

$$P_j = \sum_{i=1}^{r} P_j P_i = P_j^2, \ j = 1, 2, \dots, r. \tag{11}$$

Now, fix j and suppose that $v \in P_j(V) \cap \Big(\sum_{i \neq j} P_i(V) \Big)$ that is

$$v = P_j(v_j) = \sum_{i \neq j} P_i(v_i) \quad \text{for some} \quad v_1, v_2, \dots, v_r \in V. \tag{12}$$

Multiplying both sides of the latter equality by P_j for each j and applying (10) gives $P_j^2(v_j) = \sum_{i \neq j} P_j P_i(v_i) = 0$, thus (11) and (12) yields $v = P_j(v_j) = P_j^2(v_j) = 0$. So $P_j(V) \cap \left(\sum_{i \neq j} P_i(V) \right) = \{0_V\}$ for every $j = 1, 2, \ldots, r$, which means (by Definition 5) that $V = \bigoplus_{i=1}^{r} P_i(V)$.

To show the directness of $\sum V_i$ it is enough to show that $V_j = P_j(V)$ for each $j = 1, 2, \ldots, r$. So fix j and suppose $v \in V_j$, that is $(f - \lambda_j Id)^{n_j}(v) = 0$. Then (3) implies that $P_i(v) = 0$ for $i \neq j$ so from (8) we have $v = Id(v) = \sum_{i=1}^{r} P_i(v) = P_j(v)$ which means that $V_j \subseteq P_j(V)$. Together with (7) it gives

$$P_j(V) = V_j = \{v \in V | \ (f - \lambda_j Id)^{n_j}(v) = 0\}, \ j = 1, 2, \ldots, r \tag{13}$$

and finishes the proof of (5).

Now, take any basis in each V_i, say

$$B_i = (b_{i1}, b_{i2}, \ldots, b_{is_i}), \ s_i := \dim(V_i), \ i = 1, 2, \ldots, r. \tag{14}$$

According to (5), $B = (B_1, B_2, \ldots, B_r)$ is a basis in V. Since (10) and (11) gives

$$P_i(v) = 0 \ \text{ if } v \in P_j(V), \ j \neq i \qquad P_i(v) = v \ \text{ if } v \in P_i(V), \tag{15}$$

from (4) and (13) we get

$$f_s(v_i) = \sum_{i=1}^{r} \lambda_i P_i(v_i) = \lambda_i v_i, \quad v_i \in V_i, \ i = 1, 2, \ldots, r, \tag{16}$$

whence in particular

$$f_s(b_{ik}) = \lambda_i b_{ik}, \qquad i = 1, 2, \ldots, r, \ k = 1, 2, \ldots, s_i \tag{17}$$

so f_s is diagonal with respect to B (thus semisimple).

3. Commutativity of f_s and $f - f_s$. If we take

$$f_n := f - f_s \tag{18}$$

then by (4) f_s, f_n, f are polynomials of f and (Fact 1) we get

$$f_s f_n = f_n f_s, \qquad f f_s = f_s f \tag{19}$$

so in particular f_s, f_n commute.

4. Nilpotency of $f - f_s$. On account of (16), (18) and (19), from binomial theorem we obtain

$$v_i \in V_i,\ N \geqslant n_i \implies (f_n)^N(v_i) = (f - f_s)^N(v_i) = \sum_{k=1}^{N} \binom{N}{k} f^{N-k}(-f_s)^k(v_i) =$$
$$= \sum_{k=1}^{N} \binom{N}{k} f^{N-k}(-\lambda_i Id)^k(v_i) = (f - \lambda_i Id)^N(v_i) = 0. \tag{20}$$

Moreover, by (9) every element $v \in V$ can be written as $v = \sum_{i=1}^{r} v_i$ for some $v_i \in V_i$ so if only $N \geqslant max(n_i) \geqslant n_i,\ i = 1, 2, .., r$, then

$$\forall\ v \in V \qquad (f_n)^N(v) = (f_n)^N\Big(\sum_{i=1}^{r} v_i\Big) = \sum_{i=1}^{r}(f_n)^N(v_i) = 0 \tag{21}$$

which means that $(f_n)^N = 0$, so f_n is nilpotent as required.

5. Uniqueness. Thus there always exists a decomposition of f given by (4) and (18). To show the uniqueness assume that there exists another pair of commuting operators – semisimple g_s and nilpotent g_n – such that $f_s + f_n = f = g_s + g_n$. Let

$$m := f_s - g_s = g_n - f_n. \tag{22}$$

By definition of g_s we have $g_s f = g_s(g_s + g_n) = g_s^2 + g_s g_n = g_s^2 + g_n g_s = f g_s$ and similarly, for g_n, we get $g_n f = g_n(g_s + g_n) = (g_s + g_n)g_n = f g_n$. Therefore g_s and g_n commutes with every polynomial of f (Fact 2), in particular with f_s and f_n. Thus $f_s - g_s$ is semisimple and $g_n - f_n$ is nilpotent (Fact 7), so by (22) m is both nilpotent and semisimple, whence it has to be a zero operator (Fact 8) and consequently $f_s = g_s,\ f_n = g_n$, which provides the uniqeness.

6. The general case. To simplify the notation we shall carry out this part of the proof using matrices instead of operators. Note that if we take A to be a matrix representation of f, then all the below can be rewritten for f and all the above can be rewritten for A.

So, if $A \in M_n(K)$ ($n = \dim V$) and K contains all roots of $\chi_A(x)$, then there exists a unique pair of commuting matrices – semisimple A_s and nilpotent A_n – such that $A = A_s + A_n$. Moreover, they are polynomials of A, that is $A_s = S(A)$, $A_n = N(A)$ for some $S(x), N(x) \in K[x]$. Also, we may assume these polynomials have degree smaller than the degree of the minimal polynomial $\rho_A(x)$. Indeed, otherwise we can divide both $S(x)$ and $N(x)$ by $\rho_A(x)$ obtaining

$$S(x) = R(x)\rho_A(x) + \overline{S(x)}, \quad N(x) = T(x)\rho_A(x) + \overline{N(x)},$$

where both $\overline{S(x)}$ and $\overline{N(x)}$, being remainders, have degree smaller than $\rho_A(x)$. By definition $\rho_A(A) = 0$ whence

$$S(A) = R(A)\rho_A(A) + \overline{S(A)} = \overline{S(A)}, \quad N(A) = T(A)\rho_A(A) + \overline{N(A)} = \overline{N(A)}$$

so although $S(x), \overline{S(x)}$ (respectively $N(x), \overline{N(x)}$) may differ, they define the same operator A_s (respectively A_n).

Now, let $A \in M_n(K)$ and let L be a root field of $\chi_A(x)$ over K. In particular $K \subseteq L$, whence $A \in M_n(L)$ and according to all the above there exist $S(x), N(x) \in L[x]$ such that $\deg(S(x)), \deg(N(x)) < \deg(\rho_A(x))$, $S(A)$ is semisimple, $N(A)$ is nilpotent, $S(A)N(A)=N(A)S(A)$ and $A=S(A)+N(A)$. To show that this is the required decompositon of A we only have to prove that $S(x), N(x) \in K[x]$.

Since K is perfect, L is a Galois extension of K (Fact 3). It means that K is a fixed field of a Galois group $G(L/K)$. Let σ be any element of $G(L/K)$, that is σ is an automorphism on L such that $\sigma(k)=k$ for every $k \in K$. Then σ induces an operator τ on $M_n(L)$ in the natural way, namely

$$\tau([x_{ij}]) := [\sigma(x_{ij})]. \tag{23}$$

Observe first that τ is an endomorphism. Indeed, since σ is an automorphism, for $X = [x_{ij}],\ Y = [y_{ij}] \in M_n(L),\ \alpha \in L$ we have

$$\tau(X+Y) = \tau([x_{ij}+y_{ij}]) = [\sigma(x_{ij}+y_{ij})] = [\sigma(x_{ij}) + \sigma(y_{ij})] =$$
$$= [\sigma(x_{ij})] + [\sigma(y_{ij})] = \tau(X) + \tau(Y),$$
$$\tau(XY) = \left[\sigma\left(\sum_{k=1}^{m} x_{ik}y_{kj}\right)\right] = \left[\sum_{k=1}^{m} \sigma(x_{ik})\sigma(y_{kj})\right] = \tau(X)\tau(Y),$$
$$\tau(\alpha X) = \tau([\alpha x_{ij}]) = [\sigma(\alpha x_{ij})] = [\sigma(\alpha)\sigma(x_{ij})] = \sigma(\alpha)\tau(X). \tag{24}$$

Both A and I are over K and σ fixes K elementwise, so $\tau(A) = A,\ \tau(I) = I$ and (24) implies that for every polynomial $P(x) = p_0 + \sum_{i=1}^{n} p_i x^i \in L[x]$ we have

$$\tau(P(A)) = \sigma(p_0)I + \sum_{i=1}^{n} \sigma(p_i)A. \tag{25}$$

Now, because $A = S(A) + N(A)$, we have

$$A = \tau(A) = \tau(S(A) + N(A)) = \tau(S(A)) + \tau(N(A)).$$

$N(A)$ is nilpotent, that is $(N(A))^m = 0$ for some m and therefore (24) implies

$$(\tau(N(A)))^m = \tau\left((N(A))^m\right) = \tau(0) = 0$$

so $\tau(N(A))$ is also nilpotent. Moreover, $S(A)$ (which is over L) is semisimple and L contains all roots of $\chi_A(x)$, which (because of (16)) means that $S(A)$ is diagonalizable over L, that is there exists a diagonal matrix $D \in M_n(L)$ and an invertible matrix $C \in M_n(L)$ such that $S(A) = CDC^{-1}$ and (24) implies

$$\tau(S(A)) = \tau(CDC^{-1}) = \tau(C)\tau(D)(\tau(C))^{-1}.$$

Since $\tau(D)$ is still diagonal (as $\sigma(0)=0$), $\tau(S(A))$ is diagonalizable, whence semisimple. By assumption the decomposition of A is unique hence $\tau(S(A))-S(A)=0$, $\tau(N(A))-N(A)=0$. In other words, if $S(x):=\sum_{i=0}^{m} s_i x^i$, then

$$T(x) := \sum_{i=0}^{m} \sigma(s_i)x^i - \sum_{i=0}^{m} s_i x^i = \sum_{i=0}^{m} (\sigma(s_i) - s_i)x^i$$

is such that $T(A) = 0$. But by assumption $\deg(T(x)) \leqslant \deg(S(x)) < \deg(\rho_A(x))$ and $\rho_A(x)$ is a polynomial of the smallest possible positive degree such that $\rho_A(A) = 0$ (Definition 3). Thus $T(x)$ has to be a zero polynomial, that is $\sigma(s_i) = s_i$ for each i, which in turn means that s_i are fixed points of σ. Similarly, coefficients of $N(x)$, n_i's say, are also fixed points of σ. But σ was an arbitrary automorphism in G, so it follows that s_i, n_i lies in a fixed field of $G(L/K)$, which is K, for each i. Therefore $S(x), N(x) \in K[x]$, which finishes the proof for A and hence for f. □

Corollary 1. *Let A be a square matrix over a perfect field K. Then there exist exactly one pair of commuting matrices – semisimple A_s and nilpotent A_n – such that $A = A_s + A_n$.* □

Observation 1. The theorem (and Corollary 1) states that the Jordan-Chevalley decomposition always exists if a field is perfect. Moreover, I is semisimple over any field, 0 is nilpotent over any field and $I = I + 0$, so the decomposition may exists even over non-perfect fields. But it may not exist either.

Indeed, let K be the smallest extension field of $\mathbb{Z}_2$ containing T^2 for some T transcendental over $\mathbb{Z}_2$. In particular $T \notin K$. Now take $A = \begin{bmatrix} 0 & T^2 \\ 1 & 0 \end{bmatrix} \in M_2(K)$. Then $\chi_A(x) = x^2 - T^2 = (x - T)(x + T) = (x - T)^2$ (since $-1 = 1$ in $\mathbb{Z}_2$). This polynomial is irreducible over K (since $T \notin K$ and the factorization is unique) and has a double root (over $\mathbb{Z}_2(T)$ which is an extension field of K). By definition it means that K is not perfect. Note that A is not semisimple. Indeed, otherwise there exists an invertible matrix $C = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \mathbb{Z}_2(T)$ such that

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 0 & T^2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

whence in particular $c = aT, d = bT$ and thus $ad - bc = 0$, a contradiction.

Now, assume, contrary to our claim, that the Jordan-Chevalley decomposition of A exists, that is $A = S + N$, with $S \in M_2(K)$ semisimple, $N \in M_2(K)$ nilpotent and $SN = NS$. Since A is not semisimple, $A \neq S$ whence $N \neq 0$. Moreover, $NA = NS + N^2 = SN + N^2 = AN$, so for $N = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $a, b, c, d \in K$ we have

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 0 & T^2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & T^2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

and hence (by multiplying and comparing matrices) $N = \begin{bmatrix} a & bT^2 \\ b & a \end{bmatrix}, a, b \in K$. Since N is nilpotent, $\chi_N(x) = x^2$ (Fact 8) that is $x^2 = \chi_N(x) = (x-a)^2 - b^2T^2 = x^2 + a^2 - b^2T^2$ and so $a^2 - b^2T^2 = 0$. Hence a is a root of $x^2 - b^2T^2 = (x - bT)^2$, so $a = bT$. Since $T \notin K$, this is possible only if $a = b = 0$ and consequently $N = 0$, a contradiction. Thus the Jordan-Chevalley decomposition of A does not exist.

Observation 2. Parts 1 and 2 of the proof of the Theorem can be pursued in more compact way (like e.g. in [3]). Namely, if $\chi_f(x) = \prod (x - \lambda_i)^{n_i}$, then from the structural theorem for modules over principal ideal rings $V = \bigoplus Ker(f - \lambda_i Id)^{n_i}$ (which is the first equality in (5)). Moreover, each of these subspaces is invariant under f (that is asserted by the fact that P_i's are actually projection operators (by (11)) and follows from the second equality in (5)) and f acts on each as a multiplication by λ_i (which we show in (16)). Since $(x - \lambda_i)^{n_i}$, $(x - \lambda_j)^{n_j}$ are coprime for $i \neq j$, then the system of congruences[1]

$$P(x) \equiv \lambda_i \pmod{(x - \lambda_i)^{n_i}} \tag{26}$$

has a solution by the Chinese Remainder Theorem[2] and $f_s = P(A)$ ($P(x)$ given in (4) satisfies this system of congruences; the difference is that $P_i(x)$ determined by ChRT satisfy the congruence $\sum P_i(x) \equiv 1 \pmod{\chi_A(x)}$ rather than equality

[1] $f(x) \equiv g(x) \pmod{h(x)} \overset{\text{def}}{\Longleftrightarrow} h(x) | (f(x) - g(x))$; for details see e.g. [1, Chapter II, §1].

[2] Works over any commutative ring, e.g. $\mathbb{Z}$; for more details see e.g. [1, Chapter II, §2].

(2), which is of no importance since it is no needed explicitly and gives the same operator f_s). Thus our proof is essentially the same but we use basic facts only.

Definition 11. *Let* $A = [a_{ij}] \in M_m(K)$. *By the* ***diagonal part of*** A *we mean the matrix* $\operatorname{diag}(A) := [c_{ij}] \in M_m(K)$, *where* $c_{ij} = \begin{cases} a_{ii} & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$

Example: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Then $\operatorname{diag}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$.

Corollary 2. *Let K be a perfect field, $A \in M_n(K)$ and L be a root field of $\chi_A(x)$ over K. Therefore there exist a Jordan matrix $J \in M_n(L)$ and an invertible matrix $C \in M_n(L)$ such that $A = CJC^{-1}$. Let $D := \operatorname{diag}(J)$, $N := J - D$. Then $CDC^{-1}, CNC^{-1} \in M_n(K)$ and*

$$A = CDC^{-1} + CNC^{-1} \tag{27}$$

is the Jordan-Chevalley decomposition of A.

Proof. Since L contains all roots of $\chi_A(x)$, a Jordan normal form of A exists (Fact 6). If $D = \operatorname{diag}(J)$, then D is diagonal and the only nonzero elements of $N = J - D$ are the 1's lying on the superdiagonal of J. Thus N is upper-triangular with zeros on the diagonal, which means that it is nilpotent. Also, the diagonal of D contains all roots of $\chi_A(x)$ and in that case we may choose the basis B given by (14) in such a manner that columns of C are the vectors in B (in different order maybe).

Now, suppose $J = \bigoplus_{i=1}^{r} J_i$ and for each Jordan block J_i we have $J_i = D_i + N_i$, where $D_i := \operatorname{diag}(J_i)$, $N_i := J_i - D_i$ whence $D = \bigoplus_{i=1}^{r} D_i$, $N = \bigoplus_{i=1}^{r} N_i$. Now, if J_i corresponds to λ_i, then

$$D_i N_i = N_i D_i = [c_{ij}], \quad \text{where} \quad c_{ij} = \begin{cases} \lambda_i & j = i+1, \\ 0 & j \neq i+i. \end{cases}$$

Therefore $DN = \bigoplus_{i=1}^{r} D_i N_i = \bigoplus_{i=1}^{r} N_i D_i = ND$ and hence

$$(CDC^{-1})(CNC^{-1}) = C(DN)C^{-1} = C(ND)C^{-1} = (CNC^{-1})(CDC^{-1}).$$

Moreover, CDC^{-1} is semisimple (since D is diagonal), CNC^{-1} is nilpotent (since $N^m = 0$, then $(CNC^{-1})^m = CN^mC^{-1} = 0$). This means that CDC^{-1}, CNC^{-1}

are precisely $S(A), N(A)$, respectively, given in 6-th part of the proof of the Theorem, whence both belong to $M_n(K)$. Finally,

$$A = CJC^{-1} = C(D + N)C^{-1} = CDC^{-1} + CNC^{-1}$$

so it is a Jordan-Chevally decomposition of A as required. □

Remark. The above proof is in fact another version of the existence part of the proof of the Theorem (but it does not preserve uniqueness).

Observation 3. Let A be a linear operator on V (or a square matrix) over K. 0 is both nilpotent and semisimple and commutes with every matrix. So if A is semisimple (respectively nilpotent), then $A = A + 0$ (respectively $A = 0 + A$) is the Jordan-Chevalley decomposition of A.

4. How to determine the decomposition?

We shall focus on determining the Jordan-Chevalley decomposition over perfect fields. If f is a linear operator, take A to be its matrix representation with respect to the standard basis. Thus we assume that a square matrix A over some perfect field is given as a start point and our aim is to evaluate A_s, A_n.

Case 1: Take a look at A. If A is semisimple, then $A_s := A,\ A_n := 0$. If A is nilpotent then $A_s := 0,\ A_n := A$.
Sometimes it is enough to take a look at A to determine whether it is semisimple, for instance if

1. A is diagonal (by definition);
2. A is triangular (upper- or lower-) with pairwise distinct elements on the diagonal (since then J is diagonal);
3. A is a direct sum of a finite number of semisimple matrices (if $A = \bigoplus A_i$ and $A_i = C_i D_i C_i^{-1}$ over some extension field of K with D_i diagonal for each i, then $A = CDC^{-1}$, where $C = \bigoplus C_i,\ D = \bigoplus D_i,\ C^{-1} = \bigoplus C_i^{-1}$ by Fact 5).

Moreover, if A upper-triangular or lower-triangular with only zeros on the diagonal then A is surely nilpotent.

If this is not the case, then consider

Case 2: Determine $\chi_A(x)$.
Note that if $\chi_A(x)$ is irreducible over K, then (since K is perfect) all roots of $\chi_A(x)$ are distinct, so J is diagonal. The same situation is if $\chi_A(x)$ is a product of pairwise distint irreducible factors over K, since then J is a direct sum of diagonal matrices. In either case A is semisimple and $A_s := A, A_n := 0$.

If this is still not the case, then consider the general

Case 3: Determine A_s, A_n.
The previous section provides a few methods for evaluating A_s, A_n namely

Method 1, which follows from the proof of the Theorem:

Step 1 Factorize the characteristic polynomial $\chi_A(x) = \prod_{i=1}^{r}(x-\lambda_i)^{n_i}$.

Step 2 Take $W_i(x) := \prod_{k\neq i}(x-\lambda_k)^{n_k}$ and find Q_i's such that $\sum_{i=1}^{r} Q_i(x)W_i(x) = 1$. It can be done by the repeated application of the Inverse Euclidean Algorithm. Also, we may assume that $\deg(Q_i(x)) < n_i$ (see e.g. [1, Chapter V, Theorem 8]), so $Q_i(x) = \sum_{j=1}^{n_i-1} q_{ij}x^j$ for each i and it is enough to determine the coefficients.

Step 3 Take $A_s := \sum_{i=1}^{r} \lambda_i Q_i(A)W_i(A),\ A_n := A - A_s$.

Method 2, which follows from Observation 2:

Step 1 Factorize the characteristic polynomial $\chi_A(x) = \prod_{i=1}^{r}(x-\lambda_i)^{n_i}$.

Step 2 Determine any polynomial $P(x)$ satisfying the system of congruences $P(x) \equiv \lambda_i \pmod{(x-\lambda_i)^{n_i}},\ i = 1, 2, .., r$ by the Chinese Remainder Theorem

Step 3 Take $A_s := P(A),\ A_n := A - A_s$

Method 3, which follows from Corollary 2:

Step 1 Factorize the characteristic polynomial $\chi_A(x) = \prod_{i=1}^{r}(x-\lambda_i)^{n_i}$.

Step 2 Determine a Jordan matrix J and and invertible matrix C such that $A = CJC^{-1}$.

Step 3 Take $D := \mathrm{diag}(J),\ N := J - D$ and $A_s := CDC^{-1},\ A_n := CNC^{-1}$

Note that if A has only two eigenvalues, then Methods 1 and 2 works the same. In general, usually, Method 2 works faster than 1 and Method 3 is the slowest.

Finally, observe that if $A = \bigoplus A_i$ and $A_i = S_i + N_i$ is the Jordan-Chevalley decomposition of A_i for each i, then $A = \bigoplus S_i + \bigoplus N_i$ is the Jordan-Chevalley decomposition of A (which follows from Corollary 2). So can say we have

The general method

Step 0 Write A as a direct sum of some $A_i,\ i = 1, 2, .., a$ (possibly with $a = 1$)

Steps 1-3 For each A_i use one of the three methods given above to determine the semisimple part A_{is} and the nilpotent part A_{in}.

Step 4 Take $A_s := \bigoplus_{i=1}^{a} A_{is},\ A_n := \bigoplus_{i=1}^{a} A_{in}$

Now we shall give a few examples to show how it works.

Example 1. Let $K = \mathbb{C}$, $V = \mathbb{C}^4$ and

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 1 & 6 & 0 & 0 \\ 2 & 0 & 10 & 0 \end{bmatrix}.$$

A is lower-triangular, whence nilpotent, so $A = 0 + A$ is the Jordan-Chevalley decomposition of A.

Example 2. Let $K = \mathbb{Z}_3$, and

$$A = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Since "taking a look" at A does not give a clue, we evaluate the characteristic polynomial $\chi_A(x) = \det(xId - A) = x^3 + 2x + 1$. It has no roots in $\mathbb{Z}_3$ since $\chi_A(0) = \chi_A(1) = \chi_A(2) = 1$, and therefore $\chi_A(x)$ has no factor of degree 1 over $\mathbb{Z}_3$. So it cannot have a factor of degree 2 either, that is $\chi_A(x)$ is irreducible. Hence A is semisimple and $A = A + 0$ is the Jordan-Chevalley decomposition of A.

Example 3. Let $K = \mathbb{R}$ and

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 9 & 6 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

Then $A=\bigoplus_{i=1}^{3} A_i$, where $A_1=\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, A_2=\begin{bmatrix} 4 & 9 & 6 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, A_3=\begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$. Thus it is enough to consider each A_i separately. $\chi_{A_1}(x) = (x-1)^2+1$ is irreducible over $\mathbb{R}$, so A_1 is semisimple and $S_1=A_1, N_1=0$. A_2 is also semisimple (having distint elements on the diagonal), whence $S_2 = A_2, N_2 = 0$. A_3 is a Jordan matrix, so $S_3 = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}, N_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Thus $A = \bigoplus_{i=1}^{3} S_i + \bigoplus_{i=1}^{3} N_i$ is the Jordan-Chevalley decomposition of A.

Example 4. Let $K = \mathbb{Q}$, $V = \mathbb{Q}^6$ and

$$f(x_1, x_2, x_3, x_4, x_5, x_6) = (3x_1, x_1 + 3x_2, 0, x_3, 7x_5, 3x_6).$$

Then f is linear and its matrix representation with respect to the standard basis is

$$A = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

Thus A is a Jordan matrix, whence $A_s = \operatorname{diag}(A)$, $A_n = A - A_s$ and therefore

$$f = f_s + f_n \quad \text{where} \quad \begin{cases} f_s(x_1, x_2, x_3, x_4, x_5, x_6) = (3x_1, 3x_2, 0, 0, 7x_5, 3x_6), \\ f_n(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1, 0, 0, x_3, 0, 0). \end{cases}$$

Example 5. Let $K = \mathbb{R}$ and

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then $\chi_A(x) = (x^2 + 1)^2$.

None of the tricks will work, so we have to go through the whole process (e.g. Method 1). $(x^2+1)^2 = (x+i)^2(x-i)^2$, so we have to find $Q_1(x), Q_2(x)$ such that $Q_1(x)(x-i)^2 + Q_2(x)(x+i)^2 = 1$. By the Euclidean Algorithm we have

$$x^2 + 2ix - 1 = 1 \cdot (x^2 - 2ix - 1) + 4ix, \quad x^2 - 2ix - 1 = \left(-\frac{1}{4}ix - \frac{1}{2}\right) 4ix - 1$$

hence the Inverse Euclidean Algorithm yields

$$1 = \left(-\frac{1}{4}ix - \frac{1}{2}\right) 4ix - (x^2 - 2ix - 1) =$$

$$= \left(-\frac{1}{4}ix - \frac{1}{2}\right) \Big((x^2 + 2ix - 1) - (x^2 - 2ix - 1)\Big) - (x^2 - 2ix - 1) =$$

$$= \left(-\frac{1}{4}ix - \frac{1}{2}\right) (x^2 + 2ix - 1) + \left(\frac{1}{4}ix - \frac{1}{2}\right) (x^2 - 2ix - 1).$$

Therefore we take $Q_1(x) = \left(\frac{1}{4}ix - \frac{1}{2}\right)$, $Q_2(x) = \left(-\frac{1}{4}ix - \frac{1}{2}\right)$. Now simplify:

$$-i(x-i)^2 \left(\frac{1}{4}ix - \frac{1}{2}\right) + i(x+i)^2 \left(-\frac{1}{4}ix - \frac{1}{2}\right) = \frac{1}{2}x^3 + \frac{3}{2}x.$$

Thus $A_s = \frac{1}{2}A^3 + \frac{3}{2}A$, so

$$A = \begin{bmatrix} 0 & -1 & 0 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and this is the Jordan-Chevalley decomposition of A.

References

1. Lang S.: *Algebra.* Addison-Wesley, Reading 1971.
2. Bäuerle G.G.A., de Kerf E.A.: *Lie algebras, Part 1: Finite and infinite dimensional Lie algebras and applications in physics.* North-Holland, Amsterdam 1990.
3. `http://www.hausdorff-center.uni-bonn.de/people/perrin/chap6.pdf` (A part of the lecture "Introduction to Lie algebras" by prof. N. Perrin).

Omówienie

Operator liniowy na przestrzeni wektorowej nad danym ciałem K nazywa się półprostym, jeśli jest diagonalizowalny nad pewnym rozszerzeniem tego ciała. Operator nazywa się nilpotentnym, jeśli pewna jego potęga jest operatorem zerowym.

W niektórych przypadkach łatwo rozłożyć operator na sumę operatorów półprostego i nilpotentnego, ale z reguły nie są one przemienne.

Jeśli jednak ciało K jest doskonałe, to rozkład dowolnego operatora na sumę dwóch przemiennych operatorów – półprostego i nilpotentnego – zawsze istnieje i jest jednoznaczny (fakt ten najczęściej nazywa się twierdzeniem o rozkładzie Jordana-Chevalleya). Rozkład ten jest istotny np. w badaniu algebr Liego.

W głównej części artykułu (rozdział 4) zostały omówione różnego rodzaju metody wyznaczania postulowanego rozkładu, na podstawie wyników z rozdziału 3, dla których punktem wyjścia jest dowód twierdzenia o rozkładzie Jordana-Chevalleya. Warto zauważyć, że do zrozumienia rozważań, przeprowadzonych w pracy, wystarczy podstawowa wiedza z zakresu przestrzeni liniowych, ciał oraz pierścieni wielomianów (niektóre mniej podstawowe fakty i definicje zostały przypomniane w rozdziale 2).