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Free Vibrations of Thin Microstructured Plates

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Abstract

In this paper it is presented a problem of free vibrations of thin microstructured plates, which can be treated as made of functionally graded material on the macrolevel. The size of the microstructure of the plates is of an order of the plate thickness. In order to obtain averaged governing equations of these plates the tolerance modelling technique is applied, cf. [14, 15, 7]. The derived tolerance model equations have the terms dependent of the microstructure size. Hence, the tolerance model describes the effect of the microstructure size. In order to evaluate results, the asymptotic model is introduced. Obtained results can be compared to those calculated by using the finite element method.

Keywords: thin functionally graded plates, microstructure, tolerance modelling

1. Introduction

The objects under consideration are thin functionally graded plates with microstructure in planes parallel to the plate midplane along one, i.e. the x_1-axis direction. All plate properties along the perpendicular direction are assumed to be constant. Moreover, let the size of the microstructure be of an order of the plate thickness. An example of these plates is shown in Figure 1.

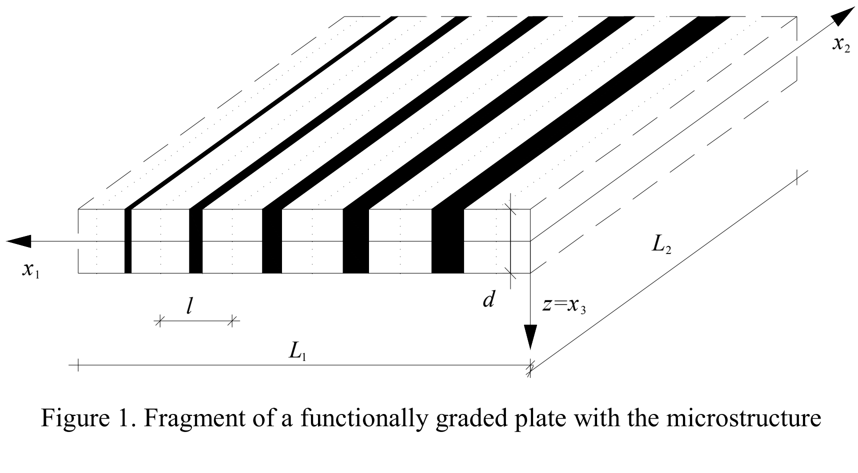

Figure 1. Fragment of a functionally graded plate with the microstructure

Plates of this kind are consisted of many small elements along the x_1-axis, called the cells, which have a span equal l, cf. Figure 2, ($x\equiv x_1$). This length l is called the microstructure parameter and describes the size of the microstructure.

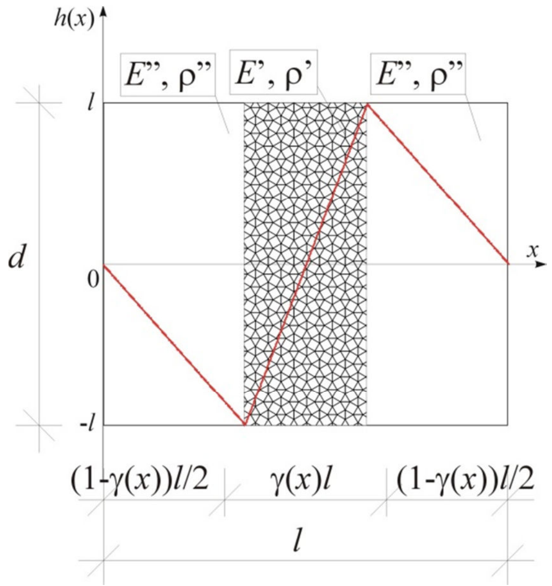

Figure 2. Element of a functionally graded plate with an example of the fluctuation shape function

A description of various thermomechanical problems of functionally graded structures or composites is often made using averaging approaches for macroscopically homogeneous structures, cf. Jędrysiak [6]. Between them it has to be mentioned models for periodic plates, cf. Kohn and Vogelius [11], based on the asymptotic homogenization method. Other method, used to describe various problems of thermoelasticity for beams, plates and shells are shown in a lot of papers, cf. [4, 13, 1, 2]. Unfortunately, model equations, obtained in this way, do not take into account the effect of the microstructure size.

However, in order to describe this effect the tolerance averaging technique can be used, cf. [14, 15, 6]. This method is applied to model various periodic structures in a series of papers, [5, 3]. Moreover, the tolerance method is also adopted to functionally graded structures like transversally tolerance-periodic plates, cf. [6, 7, 8, 9, 10] and for longitudinally functionally graded structures [12].

2. Modelling foundations

Let $Ox_1x_2x_3$ stand orthogonal Cartesian coordinate system and t be the time coordinate. Denote: $\mathbf{x}\equiv(x_1,x_2)$, $z\equiv x_3$ and the region of the undeformed plate by $\Omega\equiv\{(\mathbf{x},z):-d/2\leq z\leq d/2,\mathbf{x}\in\Pi\}$, with the midplane Π and the plate thickness d. Let $\Delta\equiv[-$

$l/2,l/2]$ be the "basic cell" in the interval $\Lambda=(-L_1/2,L_1/2)$ on the x_1-axis, and l be the length of cell Δ, called *the microstructure parameter*. It is assumed that this parameter l satisfies the conditions $d\sim l<<L_1$.

Let us introduce tolerance-periodic functions in x: a mass density μ, a rotational inertia ϑ and bending stiffnesses $d_{\alpha\beta\gamma\delta}$ given by:

$$\begin{aligned} &\mu(x)\equiv\int_{-d/2}^{d/2}\rho(x,z)dz, \qquad \vartheta(x)\equiv\int_{-d/2}^{d/2}\rho(x,z)z^2dz, \\ &d_{\alpha\beta\gamma\delta}(x)\equiv\int_{-d/2}^{d/2}c_{\alpha\beta\gamma\delta}(x,z)z^2dz \end{aligned} \tag{1}$$

Using the Kirchhoff-type plates theory assumptions we can write the equation for deflection $w(\mathbf{x},t)$ of functionally graded plates under consideration in the following form:

$$\partial_{\alpha\beta}(d_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}w)+\mu\ddot{w}-\partial_\alpha(\vartheta\partial_\beta\ddot{w})\delta_{\alpha\beta}=0\,, \tag{2}$$

which describes free vibrations of these plates.

The above equation has highly oscillating, tolerance-periodic, non-continuous coefficients being functions in x.

3. Modelling concepts

Averaged equations for functionally graded plates will be obtained using the tolerance averaging technique. The basic concepts of the modelling procedure are defined in books, cf. [14, 15, 6].

Let $\Delta(x)\equiv x+\Delta$, $\Lambda_\Delta=\{x\in\Lambda\colon \Delta(x)\subset\Lambda\}$, be a cell at $x\in\Lambda_\Delta$. The averaging operator for an arbitrary integrable function f is defined by

$$<f>(x,x_2)=\tfrac{1}{l}\int_{\Delta(x)}f(y,x_2)dy, \quad x\in\Lambda_\Delta. \tag{3}$$

If a function f is tolerance-periodic in x, then averaged value by (3) is a slowly-varying function in x.

Following the aforementioned books let us denote a set of tolerance-periodic functions by $TP_\delta^{\alpha}(\Lambda,\Delta)$, a set of slowly-varying functions by $SV_\delta^{\alpha}(\Lambda,\Delta)$, a set of highly oscillating functions by $HO_\delta^{\alpha}(\Lambda,\Delta)$, where $\alpha\geq 0$, δ is a tolerance parameter. Denote by $h(\cdot)$ a continuous highly oscillating function, $h\in FS_\delta^{1}(\Lambda,\Delta)$, having a piecewise continuous and bounded gradient $\partial^1 h$. Function $h(\cdot)$ is called *the fluctuation shape function* of the 1-st kind, if it depend on l as a parameter and satisfies conditions: $\partial^k h\in O(l^{\alpha-k})$ for $k=0,1,\ldots,\alpha$, $\partial^k h\equiv h$, and $<\mu h>(x)\approx 0$ for every $x\in\Lambda_\Delta$, $\mu>0$, $\mu\in TP_\delta^{1}(\Lambda,\Delta)$.

4. Fundamental assumptions of the tolerance modelling

The tolerance modelling is based on two fundamental modelling assumptions, cf. the book edited by Woźniak et al. [14,15] and for thin functionally graded plates by [6,7].

The first assumption of this procedure is *the micro-macro decomposition*, where the plate displacements are decomposed as:

$$u_3(\mathbf{x},z,t)=w(\mathbf{x},t),$$
$$u_\alpha(\mathbf{x},z,t)=-z[\partial_\alpha w(\mathbf{x},t)+h(x)v_\alpha(\mathbf{x},t)],\quad \alpha=1,2. \tag{4}$$

Functions $w(\cdot,x_2,t)\in SV_\delta^2(\Lambda,\Delta), v_\alpha(\cdot,x_2,t)\in SV_\delta^1(\Lambda,\Delta)$ are basic kinematic unknowns, called *the macrodeflection* and *the fluctuation amplitudes*, respectively; $h(\cdot)$ is the known *fluctuation shape function*, cf. Figure 2.

The second assumption is *the tolerance averaging approximation*, i.e. terms of an order of $O(\delta)$ are treated as negligibly small, cf. [14, 15, 7], e.g. for $f\in TP_\delta^2(\Lambda,\Delta)$, $h\in FS_\delta^1(\Lambda,\Delta)$, $F\in SV_\delta^a(\Lambda,\Delta), a=1,2,$ in: $<f>(x)=<\bar{f}>(x)+O(\delta)$, $<fF>(x_1)=<f>(x_1)F(x_1)+O(\delta)$, $<f\partial_\alpha(hF)>(x)=<f\partial_\alpha h>(x)F(x)+O(\delta)$.

5. The outline of the tolerance modelling procedure

The tolerance modelling procedure is shown in the books [14, 15, 7]. Here, there is presented only an outline of this method.

In the tolerance modelling two basic steps can be introduced. In the first step micro-macro decomposition (4) is applied. In the second step averaging operator (3) is used to the resulting formula. Hence, the tolerance averaged lagrangean $<\Lambda_h>$ is obtained:

$$\begin{aligned}<\Lambda_h>=&\tfrac{1}{2}(<\mu>\dot{w}\dot{w}+<\vartheta>\partial_\alpha\dot{w}\partial_\beta\dot{w}\delta_{\alpha\beta}+<\vartheta hh>\dot{v}_\alpha\dot{v}_\beta\delta_{\alpha\beta})-\\&-\tfrac{1}{2}(<b_{\alpha\beta\gamma\delta}>\partial_{\alpha\beta}w\partial_{\gamma\delta}w+2<b_{\alpha\beta1\delta}\partial_1 h>\partial_{\alpha\beta}wv_\delta+\\&+<b_{1\beta1\delta}\partial_1 h\partial_1 h>v_\beta v_\delta+<b_{2\beta2\delta}hh>\partial_2 v_\beta\partial_2 v_\delta).\end{aligned} \tag{5}$$

From the principle of stationary applied to (5) the Euler-Lagrange equations for $w(\cdot,x_2,t)$ and $v_\alpha(\cdot,x_2,t)$ can be derived:

$$\begin{aligned}&\frac{\partial}{\partial t}\frac{\partial<\Lambda_h>}{\partial\dot{w}}-\frac{\partial}{\partial t}\partial_\alpha\frac{\partial<\Lambda_h>}{\partial\partial_\alpha\dot{w}}-\partial_{\alpha\beta}\frac{\partial<\Lambda_h>}{\partial\partial_{\alpha\beta}w}-\frac{\partial<\Lambda_h>}{\partial w}=0,\\&\frac{\partial}{\partial t}\frac{\partial<\Lambda_h>}{\partial\dot{v}_\alpha}+\partial_2\frac{\partial<\Lambda_h>}{\partial\partial_2 v_\alpha}-\frac{\partial<\Lambda_h>}{\partial v_\alpha}=0.\end{aligned} \tag{6}$$

6. Governing equations of presented models

In this section equations of two models are presented: *the tolerance model*, *the asymptotic model*.

Substituting $<\Lambda_h>$, (5), to the Euler-Lagrange equations (6), after some manipulations we arrive at the following system of equations for $w(\cdot,x_2,t)$ and $v_\alpha(\cdot,x_2,t)$:

$$\begin{aligned}&\partial_{\alpha\beta}(<b_{\alpha\beta\gamma\delta}>\partial_{\gamma\delta}w+<b_{\alpha\beta\gamma1}\partial_1 h>v_\gamma)+<\mu>\ddot{w}-<\vartheta>\partial_{\alpha\beta}\ddot{w}\delta_{\alpha\beta}=0,\\&<b_{\alpha1\gamma\delta}\partial_1 h>\partial_{\gamma\delta}w+<b_{\alpha1\gamma1}\partial_1 h\partial_1 h>\varphi_\gamma-\underline{<b_{\alpha2\gamma2}hh>}\partial_{22}v_\gamma+\underline{<\vartheta hh>}\ddot{v}_\alpha=0,\end{aligned} \tag{7}$$

where all coefficients are slowly-varying functions in x. Equations (7) together with micro-macro decomposition (4) stand *the tolerance model of thin functionally graded* *plates with the microstructure size of an order of the plate thickness*. These equations describe free vibrations of these plates. There are the underlined terms, which depend on the microstructure parameter l, in these equations. Hence, the effect of the microstructure size on dynamic problems of these plates is taken into account. All coefficients of equations (7) are slowly-varying functions in $x \equiv x_1$ in contrast to equation (2), in which there are non-continuous, highly oscillating and tolerance-periodic coefficients. The basic unknowns w, v_α, α=1,2, are slowly-varying functions in $x \equiv x_1$. It can be observed that boundary conditions have to be formulated for *the macrodeflection* w on all edges, and for *the fluctuation amplitudes* v_α only for edges normal to the x_2-axis.

The asymptotic modelling procedure is shown in [15, 6, 7]. However, in order to obtain equations of an approximate model, which does not take into account the effect of the microstructure size, the underlined terms in equations (7) can be neglected. Hence, we arrive at the following equations of *the asymptotic model*:

$$\begin{aligned} &\partial_{\alpha\beta}(<b_{\alpha\beta\gamma\delta}>\partial_{\gamma\delta}w+<b_{\alpha\beta\gamma 1}\partial_1 h>v_\gamma)+<\mu>\ddot{w}-<\vartheta>\partial_{\alpha\beta}\ddot{w}\delta_{\alpha\beta}=0, \\ &<b_{\alpha 1\gamma\delta}\partial_1 h>\partial_{\gamma\delta}w+<b_{\alpha 1\gamma 1}\partial_1 h\partial_1 h>v_\gamma=0. \end{aligned} \tag{8}$$

These equations have smooth, slowly-varying coefficients in the contrast to equation (2). The asymptotic model equations describe free vibrations of thin plates under consideration on the macrolevel only.

7. Final remarks

Two modelling procedures are applied to the known differential equation of Kirchhoff-type plates with functionally graded macrostructure and the microstructure size of an order of the plate thickness in this note. Using these procedures the governing equation with non-continuous, tolerance-periodic functional coefficients of x_1 can be replaced by the systems of differential equations with slowly-varying, continuous coefficients of x_1 for each model.

Using the tolerance model, where the effect of the microstructure size is taken into account, not only macrovibrations can be investigated, but also microvibrations, which are related to the microstructure of the functionally graded plates.

The the tolerance model equations have a physical sense for unknowns w, v_α, which are slowly-varying functions in x_1. However, these conditions can be treated as a certain *a posteriori* criterion of physical reliability for the model.

The second presented model, the asymptotic model, with the governing equations neglecting the aforementioned effect, can describe only macrovibrations of these plates under consideration.

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