

On Some Qualitative Properties of Integrable Solutions for Cauchy-type Problem of Fractional Order

Mohamed M. A. Metwali

ABSTRACT: The paper discusses the existence of solutions for Cauchy-type problem of fractional order in the space of Lebesgue integrable functions on bounded interval. Some qualitative properties of solutions are presented such as monotonicity, uniqueness and continuous dependence on the initial data. The main tools used are measure of weak (strong) noncompactness, Darbo fixed point theorem and fractional calculus.

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1. Introduction

The differential equations involving Riemann-Liouville differential operators of fractional orders are widely used in modeling several physical phenomena (see e.g. [22, 23, 25]). The problems with this approach is in that they seem to deserve an independent study of their theory parallel to the well-known theory of ordinary differential equations.

The aim of this paper is to propose the solvability of the following nonlinear Cauchy-type problem:

$$\begin{cases} D^\alpha u(t) = u(t) \cdot \int_0^1 k(t, s) f(s, u(s)) ds, & \alpha \in (0, 1), \\ u(0) = u_0, \end{cases} \quad (1.1)$$

where D^α is the fractional derivative (in the sense of Riemann-Liouville) of order α . The existence result for solutions to the hybrid differential equation

$$\begin{cases} \frac{d^\alpha}{dt^\alpha} [u(t) - f(t, u(t))] = g(t, u(t)), & a.e. \ t \in [t_0, t_0 + a], \\ u(0) = u_0, \end{cases}$$

has been discussed in [19] with the Riemann-Liouville fractional derivative and in [16] studied the existence of mild solutions for the above hybrid differential equation using the Caputo fractional derivative. This was extended by [24] to give approximation of solutions to Caputo fractional order hybrid differential equations.

The authors (see [15]) studied the Cauchy-type problem:

$$\begin{cases} D^\alpha u(t) = f(t, u(\varphi(t))), \\ t^{1-\alpha} u(t)|_{t=0} = b, \quad b > 0, \end{cases}$$

where D^α is the Riemann-Liouville fractional derivative and the function $f(t, u)$ was assumed to be continuous. The existence of L_1 -solution and investigated the behavior of solutions was proved in [12], where the operators were considered to be compact in L_p -spaces and this approach does not work for quadratic equations (see [10, Example 9.2]). Our approach overcomes this difficulty.

Let us recall that the quadratic integral equations were discussed in many different functions spaces (see [3, 6, 7, 8, 20]) and have numerous applications in the theories of radiative transfer, neutron transport and in the kinetic theory of gases [3, 6, 7].

Using the equivalence of the fractional Cauchy-type problem with the corresponding quadratic integral equation, we prove the existence of L_1 -solution of equation (1.1).

Moreover, we discuss the monotonicity, uniqueness and Continuous dependence on the initial condition of the solution. To achieve our goal we use the technique of measure of weak (strong) noncompactness associated with Darbo fixed point theorem.

2. Notation and auxiliary facts

Let \mathbb{R} be the field of real numbers, J be the interval $[0, 1]$ and $L_1(J)$ be the space of Lebesgue integrable functions (equivalence classes of functions) on a measurable subset J of \mathbb{R} , with the standard norm

$$\|x\|_{L_1(J)} = \int_J |x(t)| dt.$$

By $L_\infty(J)$ we denote the Banach space of essentially bounded measurable functions with the essential supremum norm (denoted by $\|x\|_{L_\infty}$). We will write L_1 and L_∞ instead of $L_1(J)$ and $L_\infty(J)$ respectively.

Definition 2.1. [2] Assume that a function $f : J \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Carathéodory conditions i.e. it is measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in J$. Then to every function $x(t)$ being measurable on J we assign

$$(Fx)(t) = f(t, x(t)), \quad t \in J.$$

The operator F is called the superposition (Nemytskii) operator.

Theorem 2.1. [2] *Let f satisfies the Carathéodory conditions. The operator F maps continuously the space L_1 into itself if and only if*

$$|f(t, x)| \leq a(t) + q \cdot |x|,$$

for all $t \in J$ and $x \in \mathbb{R}$, where $a \in L_1$ and $q \geq 0$.

Let $S = S(J)$ denote the set of measurable (in Lebesgue sense) functions on J and let $meas$ stands for the Lebesgue measure in \mathbb{R} . Identifying the functions equal almost everywhere the set S furnished with the metric

$$d(x, y) = \inf_{a>0} [a + meas\{s : |x(s) - y(s)| \geq a\}],$$

becomes a complete metric space. Moreover, the convergence in measure on J is equivalent to the convergence with respect to the metric d (Proposition 2.14 in [26]). The compactness in such a space is called a "compactness in measure".

Theorem 2.2. [20] *Let X be a bounded subset of L_1 and suppose that there is a family of measurable subsets $(\Omega_c)_{0 \leq c \leq 1}$ of the interval J such that $meas \Omega_c = c$ for every $c \in J$ and for $x \in X$*

$$x(t_1) \geq x(t_2), \quad (t_1 \in \Omega_c, \quad t_2 \notin \Omega_c).$$

Then X is compact in measure subset of L_1 .

Theorem 2.3. [18, Theorem 6.2] *The linear integral operator $K : L_1 \rightarrow L_1$ given by a formula $(Ku)(t) = \int_J k(t, s)u(s) ds$ preserve the monotonicity of functions iff*

$$\int_0^l k(t_1, s) ds \geq \int_0^l k(t_2, s) ds$$

for $t_1 < t_2$, $t_1, t_2 \in J$ and for any $l \in J$.

Assume that $(E, \|\cdot\|)$ is an arbitrary Banach space with zero element θ . Denote by $B(x, r)$ the closed ball centered at x and with radius r . The symbol B_r stands for the ball $B(\theta, r)$. Denote by \mathcal{M}_E the family of all nonempty and bounded subsets of E and by $\mathcal{N}_E, \mathcal{N}_E^W$ its subfamilies consisting of all relatively compact and weakly relatively compact sets, respectively. The symbols \bar{X}, \bar{X}^W stand for the closure and the weak closure of a set X , respectively and $conv X$ will denote the convex closure of X .

Definition 2.2. [4] A mapping $\mu : \mathcal{M}_E \rightarrow [0, \infty)$ is called a regular measure of noncompactness in E if the following conditions hold:

- (i) $\mu(X) = 0 \Leftrightarrow X \in \mathcal{N}_E$.
- (ii) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
- (iii) $\mu(\bar{X}) = \mu(conv X) = \mu(X)$.

- (iv) $\mu(\lambda X) = |\lambda| \mu(X)$, for $\lambda \in \mathbb{R}$.
- (v) $\mu(X + Y) \leq \mu(X) + \mu(Y)$.
- (vi) $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$.
- (vii) If X_n is a sequence of nonempty, bounded, closed subsets of E , $X_n = \bar{X}_n$ such that $X_{n+1} \subset X_n$, $n = 1, 2, 3, \dots$, and $\lim_{n \rightarrow \infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

An example of such a mapping is the following:

Definition 2.3. [4] Let X be a nonempty and bounded subset of E . The Hausdorff measure of noncompactness $\chi(X)$ is defined as

$$\chi(X) = \inf\{r > 0 : \text{there exists a finite subset } Y \text{ of } E \text{ such that } x \subset Y + B_r\}.$$

Definition 2.4. [4] A mapping $\mu : \mathcal{M}_E \rightarrow [0, \infty)$ is called a regular measure of weak noncompactness in E if it satisfies conditions (ii)-(vi) of Definition 2.2 and the following two conditions (being counterparts of (i) and (vii)) hold:

- (i') $\mu(X) = 0 \Leftrightarrow X \in \mathcal{N}_E^W$.
- (vii') If X_n is a sequence of nonempty, bounded, closed subsets of E , $X_n = \bar{X}_n^W$ such that $X_{n+1} \subset X_n$, $n = 1, 2, 3, \dots$, and $\lim_{n \rightarrow \infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

Another regular measure was defined in the space L_1 (cf. [5]). For any $\varepsilon > 0$, let c be a measure of equiintegrability of the set X [2, p. 39] i.e.

$$c(X) = \lim_{\varepsilon \rightarrow 0} \left\{ \sup_{x \in X} \left\{ \sup_D \int_D |x(t)| dt, D \subset J, \text{meas} D \leq \varepsilon \right\} \right\}.$$

It forms a regular measure of noncompactness if restricted to the family of subsets being compact in measure (cf. [14]).

Theorem 2.4. [4] Let Q be a nonempty, bounded, closed, and convex subset of E and let $H : Q \rightarrow Q$ be a continuous transformation which is a contraction with respect to the measure of noncompactness μ , i.e. there exists $\gamma \in [0, 1)$ such that

$$\mu(H(X)) \leq \gamma \mu(X),$$

for any nonempty subset X of E . Then H has at least one fixed point in the set Q .

Next, we introduce a short note about fractional calculus theory.

Definition 2.5. [17] The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, \infty) \rightarrow \mathbb{R}$ is given by

$$D^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t - s)^{\alpha - n + 1}} ds$$

provided that the right side is pointwise defined on $(0, \infty)$, where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α .

Definition 2.6. [17] The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \rightarrow \mathbb{R}$ is defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds$$

provided that the right side is pointwise defined on $(0, \infty)$.

Lemma 2.1. [13, 17]. Let $f \in L_1$ and let $\alpha \in (0, 1)$, then

- (a) $D^\alpha I^\alpha f(t) = f(t)$.
- (b) $I^\alpha D^\alpha f(t) = f(t) - f(0)$.
- (c) The operator I^α maps L_1 into itself continuously.
- (d) The operator I^α maps the monotonic nondecreasing function into functions of the same type.

3. Main result

First we need to prove the equivalence of (1.1) with the corresponding quadratic integral equation:

$$u(t) = u_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left(u(s) \cdot \int_0^1 k(s, v) f(v, u(v)) dv \right) ds, \quad t \in (0, 1). \quad (3.1)$$

Indeed, let $u(t)$ be a solution of (1.1). Applying the operator I^α to both sides of (1.1). By Lemma 2.1, we have

$$I^\alpha D^\alpha u(t) = u(t) - u(0) = I^\alpha \left(u(t) \cdot \int_0^1 k(t, s) f(s, u(s)) ds \right),$$

where $u(0) = u_0$, we have

$$u(t) = u_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left(u(s) \cdot \int_0^1 k(s, v) f(v, u(v)) dv \right) ds.$$

Thus equation (3.1) holds.

Conversely, let $u(t)$ be a solution of equation (3.1). Then applying D^α on both sides of (3.1), we obtain (1.1). Finally, put $t = 0$ in equation (3.1), we get $u(0) = u_0$. Then problem (1.1) and equation (3.1) are equivalent to each other.

3.1. Existence of monotonic integrable solution

Rewrite (3.1) as

$$u = Tu,$$

where

$$Tu(t) = u_0 + I^\alpha(Au)(t), \quad (Au)(t) = u(t) \cdot (KFu)(t),$$

$$(Ku)(t) = \int_0^1 k(t, s)u(s)ds, \quad Fu = f(t, u), \text{ and } I^\alpha \text{ is as in Definition 2.6.}$$

To solve equation (3.1) it is necessary to find a fixed point of the operator T . For facilitating our discussion, we shall treat (3.1) under the following assumptions listed below:

- (i) $f : J \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies Carathéodory conditions and $f(t, u) \geq 0$ for $u \geq 0$. Moreover, f is assumed to be nondecreasing with respect to both variables t and u separately.
- (ii) There is a positive function $a \in L_1$ and a constant $q \geq 0$ such that

$$|f(t, u)| \leq a(t) + q|u|,$$

for all $t \in J$ and $u \in \mathbb{R}$.

- (iii) $k(t, s) : J \times J \rightarrow \mathbb{R}$ is measurable with respect to both variables. The linear integral operator K associated with the kernel k maps L_1 into L_∞ and is continuous.
- (iv) For any $0 < l < 1$ the following condition holds true

$$t_1 < t_2 \quad \Rightarrow \quad \int_0^l k(t_1, s) ds \geq \int_0^l k(t_2, s) ds.$$

- (v) Let $r > 0$ be such that $\|K\|_{L_\infty} (\|a\|_{L_1} + q \cdot r) < \Gamma(\alpha + 1)$.

Theorem 3.1. *Let assumptions (i)-(v) be satisfied, then the Cauchy-type problem (1.1) has at least one solution in L_1 a.e. nondecreasing on J .*

Proof. First of all observe that by assumptions (i), (ii) and Theorem 2.1 we have that the superposition operator F maps continuously L_1 into itself. From assumption (iii) the operator (KF) maps L_1 into L_∞ . From the Hölder inequality the operator A maps L_1 into itself and is continuous. Finally, since the operator I^α maps L_1 into itself continuously, then we can deduce that the operator T maps L_1 into itself and is continuous.

Thus for $u \in L_1$, we have

$$\begin{aligned}
 \|Tu\|_{L_1} &= \int_0^1 |(Tu)(t)| dt \leq \int_0^1 u_0 dt + \int_0^1 I^\alpha(Au)(t) dt \\
 &\leq u_0 \cdot t \Big|_0^1 + \int_0^1 \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left(u(s) \cdot \int_0^1 k(s, v) f(v, u(v)) dv \right) ds \right| dt \\
 &\leq u_0 + \int_0^1 \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left(u(s) \cdot \int_0^1 k(s, v) (a(v) + q|u(v)|) dv \right) ds \right| dt \\
 &\leq u_0 + \int_0^1 \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s) \cdot \|K\|_{L_\infty} (\|a\|_{L_1} + q\|u\|_{L_1}) ds \right| dt \\
 &\leq u_0 + \|K\|_{L_\infty} (\|a\|_{L_1} + q\|u\|_{L_1}) \int_0^1 \frac{t^{\alpha-1}}{\Gamma(\alpha)} dt \int_0^1 |u(s)| ds \\
 &\leq u_0 + \|K\|_{L_\infty} (\|a\|_{L_1} + q\|u\|_{L_1}) \frac{t^\alpha}{\alpha\Gamma(\alpha)} \Big|_0^1 \cdot \|u\|_{L_1} \\
 &= u_0 + \frac{\|K\|_{L_\infty}}{\Gamma(\alpha+1)} (\|a\|_{L_1} + q\|u\|_{L_1}) \cdot \|u\|_{L_1},
 \end{aligned}$$

so the function Tu is bounded in J . This allows us to infer that the operator T transforms L_1 into itself. Moreover, this estimate the following

$$\|Tu\|_{L_1} \leq u_0 + \frac{\|K\|_{L_\infty}}{\Gamma(\alpha+1)} (\|a\|_{L_1} + q \cdot r) \cdot r \leq r.$$

As a domain for the operator T we will consider the ball B_r , where r is the positive solution of the equation

$$u_0 + \frac{\|K\|_{L_\infty}}{\Gamma(\alpha+1)} (\|a\|_{L_1} + q \cdot r) \cdot r = r.$$

Let us remark, that the above inequality is of the form $\hat{a} + (\hat{b} + \hat{v}r)\hat{c}r \leq r$ with $\hat{a}, \hat{b}, \hat{c}, \hat{v} > 0$. Then $\hat{v}\hat{c} > 0$ and by assumption (v), we have that $\hat{b}\hat{c} - 1 < 0$ and that the discriminant is positive, then Viète's formulas imply that the quadratic equation has two positive solutions $r_1 < r_2$. So there exists a positive number $r > 0$ satisfying this inequality such that T maps the ball B_r into itself and is continuous.

Further, let Q_r is a subset of B_r which has the functions a.e. nondecreasing on J . This set is nonempty, bounded (by r), convex and closed in L_1 . In view of Theorem 2.2 the set Q_r is compact in measure.

Now, we will show that T preserve the monotonicity of functions. Take $u \in Q_r$, then $u(t)$ is a.e. nondecreasing on J and consequently F is also of the same type by virtue of the assumption (i). Further, $Ku(t)$ is a.e. nondecreasing on J due to assumption (iv). Since the pointwise product of a.e. monotone functions is still of the same type, the operator $A = u \cdot (KFu)$ is a.e. nondecreasing on J . Moreover, the operator I^α maps a.e. nondecreasing functions into functions of the same type

(thanks to Lemma 2.1). Thus we can deduce that Tu is also a.e. nondecreasing on J . Then T maps continuously Q_r into Q_r .

From now we will assume that X is a nonempty subset of Q_r and the constant $\varepsilon > 0$ is arbitrary, but fixed. Then for an arbitrary $u \in X$ and for a set $D \subset J$, $measD \leq \varepsilon$ we obtain

$$\begin{aligned}
\|Tu\|_{L_1(D)} &= \int_D |Tu(t)| dt \leq \int_J u_0 \cdot \chi_D(t) dt + \int_D I^\alpha(Au)(t) dt \\
&\leq \|u_0 \cdot \chi_D\|_{L_1} \\
&\quad + \int_D \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left(u(s) \cdot \int_0^1 k(s, v) f(v, u(v)) dv \right) ds \right| dt \\
&\leq \|u_0 \cdot \chi_D\|_{L_1} \\
&\quad + \int_D \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (u(s) \cdot \|K\|_{L_\infty} [\|a\|_{L_1} + q\|u\|_{L_1}]) ds \right| dt \\
&\leq \|u_0 \cdot \chi_D\|_{L_1} + \|K\|_{L_\infty} (\|a\|_{L_1} + q \cdot r) \int_0^1 \frac{t^{\alpha-1}}{\Gamma(\alpha)} dt \cdot \int_D |u(s)| ds \\
&\leq \|u_0 \cdot \chi_D\|_{L_1} + \frac{\|K\|_{L_\infty} (\|a\|_{L_1} + q \cdot r)}{\Gamma(\alpha + 1)} \cdot \int_D |u(s)| ds.
\end{aligned}$$

Hence, taking into account the equality

$$\lim_{\varepsilon \rightarrow 0} \{\sup\{\|u_0 \cdot \chi_D\|_{L_1} : D \subset J, measD \leq \varepsilon\}\} = 0,$$

and by the definition of $c(X)$, we get

$$c(TX) \leq \frac{\|K\|_{L_\infty} (\|a\|_{L_1} + q \cdot r)}{\Gamma(\alpha + 1)} \cdot c(X).$$

Recall that $\|K\|_{L_\infty} (\|a\|_{L_1} + q \cdot r) < \Gamma(\alpha + 1)$ and the inequality obtained above with the properties of the operator T and since the set Q_r is compact in measure we are able to apply Theorem 2.4 which completes the proof. \square

3.2. Uniqueness of the solution

We will discuss the uniqueness of solutions in the following theorem:

Theorem 3.2. *Let assumptions of Theorem 3.1 be satisfied, but instead of assumptions (ii) and (v) consider the following condition:*

(vi) *Assume that, there is a constant $q \geq 0$ such that for a.e. $t \in J$*

$$|f(t, u) - f(t, v)| \leq q|u - v|, \text{ and } |f(t, 0)| \leq a(t),$$

where $u, v \in \mathbb{R}$ and $a \in L_1$.

Moreover, let

$$\left(\frac{\|K\|_{L_\infty} (\|a\|_{L_1} + 2q \cdot r)}{\Gamma(\alpha + 1)} \right) < 1.$$

Then the Cauchy-type problem (1.1) has an unique solution in $L_1(J)$.

Proof. From assumption (vi), we have

$$\begin{aligned} ||f(t, u) - f(t, 0)|| &\leq |f(t, u) - f(t, 0)| \leq q|u| \\ \Rightarrow |f(t, u)| &\leq |f(t, 0)| + q|u| \leq a(t) + q|u|, \end{aligned}$$

which imply that assumption (ii) and (v) of Theorem 1.1 are satisfied.

For the uniqueness solution of (1.1), let $x(t)$ and $y(t)$ be any two solutions of (3.1) in B_r , then we have

$$\begin{aligned} |x(t) - y(t)| &\leq \left| I^\alpha \left(x(t) \cdot \int_0^1 k(t, v) f(v, x(v)) dv \right) \right. \\ &\quad \left. - I^\alpha \left(y(t) \cdot \int_0^1 k(t, v) f(v, y(v)) dv \right) \right| \\ &\leq \left| I^\alpha \left(x(t) \cdot \int_0^1 k(t, v) f(v, x(v)) dv \right) \right. \\ &\quad \left. - I^\alpha \left(x(t) \cdot \int_0^1 k(t, v) f(v, y(v)) dv \right) \right| \\ &\quad + \left| I^\alpha \left(x(t) \cdot \int_0^1 k(t, v) f(v, y(v)) dv \right) \right. \\ &\quad \left. - I^\alpha \left(y(t) \cdot \int_0^1 k(t, v) f(v, y(v)) dv \right) \right| \\ &\leq I^\alpha \left(|x(t)| \cdot \int_0^1 |k(t, v)| |f(v, x(v)) - f(v, y(v))| dv \right) \\ &\quad + I^\alpha \left(|x(t) - y(t)| \cdot \int_0^1 |k(t, v)| |f(v, y(v))| dv \right) \\ &\leq I^\alpha \left(|x(t)| \cdot \int_0^1 |k(t, v)| q|x(v) - y(v)| dv \right) \\ &\quad + I^\alpha \left(|x(t) - y(t)| \cdot \int_0^1 |k(t, v)| [a(v) + q|y(v)|] dv \right). \end{aligned}$$

Therefore,

$$\begin{aligned} \|x - y\|_{L_1} &\leq \int_0^1 \left\{ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left(|x(s)| \cdot \int_0^1 |k(s, v)| q|x(v) - y(v)| dv \right) ds \right\} dt \\ &\quad + \int_0^1 \left\{ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \right. \\ &\quad \left. \left(|x(s) - y(s)| \cdot \int_0^1 |k(s, v)| [a(v) + q|y(v)|] dv \right) ds \right\} dt \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\|x\|_{L_1} \cdot q \|K\|_{L_\infty}}{\Gamma(\alpha + 1)} \cdot \|x - y\|_{L_1} + \frac{\|K\|_{L_\infty} (\|a\|_{L_1} + q \|y\|_{L_1})}{\Gamma(\alpha + 1)} \cdot \|x - y\|_{L_1} \\
&\leq \left(\frac{q \cdot r \cdot \|K\|_{L_\infty} + \|K\|_{L_\infty} (\|a\|_{L_1} + q \cdot r)}{\Gamma(\alpha + 1)} \right) \cdot \|x - y\|_{L_1}.
\end{aligned}$$

The above inequality yields

$$\left(1 - \frac{\|K\|_{L_\infty} (\|a\|_{L_1} + 2q \cdot r)}{\Gamma(\alpha + 1)} \right) \cdot \|x - y\|_{L_1} \leq 0,$$

which implies that

$$\|x - y\|_{L_1} = 0 \Rightarrow x = y.$$

This complete the proof. \square

3.3. Continuous dependence on the initial condition

In the present section sufficient conditions are obtained under which the solution $u(t, 0, u_0)$ of problem (1.1) depends continuously on the initial conditions.

Definition 3.1. [11] We say that the solution $u(t, 0, u_0)$ of problem (1.1) depends continuously on the initial conditions for $t \in J$, if for every two positive numbers ϵ and η there exists a number $\delta = \delta(\epsilon, \eta) > 0$ such that if $|u_0 - \tilde{u}_0| < \delta$ then $\|u(t, 0, u_0) - \tilde{u}(t, 0, \tilde{u}_0)\|_{L_1} < \epsilon$ for $t \in J$ and $|t| > \eta$.

Theorem 3.3. *Let assumptions of Theorem 3.2 be satisfied, then the solution of the Cauchy-type problem (1.1) depends continuously on the initial condition in J .*

Proof. According to Theorem 3.2, the problem (1.1) has an unique solution. We will prove that this solution depends continuously on the initial condition in J .

Let $u(t)$ be a solution of

$$u(t) = u_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left(u(s) \cdot \int_0^1 k(s, v) f(v, u(v)) dv \right) ds,$$

and let $\tilde{u}(t)$ be a solution of the above equation such that $\tilde{u}(0) = \tilde{u}_0$, then

$$\begin{aligned}
|u(t) - \tilde{u}(t)| &= |u_0 - \tilde{u}_0| \\
&\quad + \left| I^\alpha \left(u(t) \cdot \int_0^1 k(t, v) f(v, u(v)) dv \right) \right. \\
&\quad \left. - I^\alpha \left(\tilde{u}(t) \cdot \int_0^1 k(t, v) f(v, \tilde{u}(v)) dv \right) ds \right| \\
&\leq |u_0 - \tilde{u}_0| + I^\alpha \left(|u(t)| \cdot \int_0^1 |k(t, v)| |f(v, u(v)) - f(v, \tilde{u}(v))| dv \right) \\
&\quad + I^\alpha \left(|u(t) - \tilde{u}(t)| \cdot \int_0^1 |k(t, v)| |f(v, \tilde{u}(v))| dv \right) \\
&\leq |u_0 - \tilde{u}_0| + I^\alpha \left(|u(t)| \cdot \int_0^1 |k(t, v)| q|u(v) - \tilde{u}(v)| dv \right) \\
&\quad + I^\alpha \left(|u(t) - \tilde{u}(t)| \cdot \int_0^1 |k(t, v)| [|a(v) + q|\tilde{u}(v)|] dv \right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|u - \tilde{u}\|_{L_1} &\leq \int_0^1 |u_0 - \tilde{u}_0| dt \\
&\quad + \int_0^1 \left\{ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left(|u(s)| \cdot \int_0^1 |k(s, v)| q|u(v) - \tilde{u}(v)| dv \right) ds \right\} dt \\
&\quad + \int_0^1 \left\{ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \right. \\
&\quad \left. \left(|u(s) - \tilde{u}(s)| \cdot \int_0^1 |k(s, v)| [|a(v) + q|\tilde{u}(v)|] dv \right) ds \right\} dt \\
&\leq |u_0 - \tilde{u}_0| + \left(\frac{q \cdot r \cdot \|K\|_\infty + \|K\|_{L_\infty} (\|a\|_{L_1} + q \cdot r)}{\Gamma(\alpha + 1)} \right) \cdot \|u - \tilde{u}\|_{L_1}.
\end{aligned}$$

Then we get

$$\|u - \tilde{u}\|_{L_1} \leq \left(1 - \frac{\|K\|_{L_\infty} (\|a\|_{L_1} + 2q \cdot r)}{\Gamma(\alpha + 1)} \right)^{-1} \cdot |u_0 - \tilde{u}_0|.$$

Therefore, if $|u_0 - \tilde{u}_0| < \delta(\epsilon)$, then $\|u - \tilde{u}\|_{L_1} < \epsilon$, where

$$\delta(\epsilon) = \epsilon \cdot \frac{\Gamma(\alpha + 1) - \|K\|_{L_\infty} (\|a\|_{L_1} + 2q \cdot r)}{\Gamma(\alpha + 1)}.$$

Now from the equivalence of (1.1) and (3.1), we get that the solution of the Cauchy-type problem (1.1) depends continuously on the initial condition in J . \square

3.4. Remarks

We need to stress on some aspects of obtained results. First of all we can observe, that our solutions are not necessarily continuous as in almost all previously investigated cases (cf. [1, 21]). So we need not to assume, that the Hammerstein operator transforms the space $C(I)$ into itself. Our solutions belong to the space L_1 , for the examples and conditions related to Hammerstein operators in L_1 we refer the readers to [20, 27].

We need to emphasize that the following Cauchy-type problem is also strictly related to quadratic equations (cf. [8])

$$\left(\frac{x(t) - g(t)}{f_1(t, x)} \right)' = f_2(t, x(t)), \quad x(0) = 0,$$

where $f_1 : I \times \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\}$.

It can be easily checked that under some typical assumptions this problem is equivalent to the integral equation [9]

$$x(t) = g(t) + f_1(t, x(t)) \cdot \left(\int_0^t f_2(s, x(s)) ds - \frac{g(0)}{f_1(0, 0)} \right).$$

Nevertheless, when we are looking for continuous solutions for integral equation, for differential one we obtain classical solutions, i.e. x is continuously differentiable. In the case presented above we investigate Carathéodory solutions for the Cauchy problem.

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Mohamed M.A. Metwali
email: m.metwali@yahoo.com
Department of Mathematics
Faculty of Sciences
Damanhour University
EGYPT

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