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Practical and asymptotic stabilities for a class of delayed fractional discrete-time linear systems

A. RUSZEWSKI*

Bialystok University of Technology, Faculty of Electrical Engineering, 45D Wiejska St., 15-351 Białystok, Poland.

Abstract. The practical and asymptotic stabilities of delayed fractional discrete-time linear systems described by the model without a time shift in the difference are addressed. The D-decomposition approach is used for stability analysis. New necessary and sufficient stability conditions are established. The conditions in terms of the location of eigenvalues of the system matrix in the complex plane are given.

Key words: fractional, discrete-time, stability, time-delay.

1. Introduction

Fractional calculus is the branch of mathematics in which non-integer order integrals and derivatives are considered. It is used in many fields of science and engineering, such as signal and image processing, physics, mechanics, control systems, biology, etc. Many fractional models of physical phenomena have been presented and used in real world applications [1]. The review of fractional modeling and applications can be found in many monographs and articles, e.g. [2–7].

The stability is the most important property of dynamic systems. The state-space model describes the behavior of a dynamic system. For the discrete-time systems, the model with a time shift in the difference is the most well-known. The stability problem of this model with fractional order has been considered. The asymptotic stability and the so-called practical stability for a given length of practical implementation have been analysed. The asymptotic stability conditions and the stability domains have been presented in [8-12]. The conditions of practical stability have been established in [8, 13]. The practical stability problem for the discrete-time fractional model of the heat transfer process has been presented in [14].

The discrete-time fractional order state-space model without a time shift in the difference has been introduced in [15]. The solution of state-space equations and the realization problem have been presented in [15, 16], respectively. The asymptotic stability conditions have been established in [17, 18]. The stability testing of this model with state delays is less advanced. Recently, the sufficient condition for asymptotic stability of this system with one delay has been established in [19]. In the case of scalar systems, the stability conditions for systems with pure delay and multiple delays have been presented in [20, 21], respectively.

In this paper the stability of delayed fractional discrete-time system (model without a time shift in the difference) will be investigated. New necessary and sufficient conditions for the practical and asymptotic stabilities will be proposed.

The paper is organized as follows. In Section 2 the formulation of the problem is given. Necessary and sufficient conditions for practical and asymptotic stabilities are established in Section 3. Concluding remarks are given in Section 4.

2. Problem formulation

Let us consider a discrete-time linear system described by the fractional homogeneous state equation

$$\Delta^{\alpha} x_i = A_1 x_{i-1}, \qquad i = 0, 1, \ldots, \tag{1}$$

with the initial condition $x_{-l} \in \Re^n$ $(l = 0, 1)$. Moreover, $x_i \in \Re^n$ is the state vector, $A_1 \in \Re^{n\times n}$ is the state matrix and α is the fractional order $\alpha \in (0, 1)$. In this paper we use the following fractional difference of the discrete-time function x_i [3]

$$\Delta^{\alpha} x_i = \sum_{k=0}^{i} c_k(\alpha) x_{i-k}, \qquad i = 0, 1, \ldots, \tag{2}$$

where

$$c_k(\alpha) = (-1)^k \binom{\alpha}{k} = \begin{cases} 1 & \text{for } k = 0, \\ (-1)^k \dfrac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!} & \text{for } k > 0. \end{cases} \tag{3}$$

Using fractional difference (2), equation (1) can be written in the following form

$$x_i = A_1 x_{i-1} - \sum_{k=1}^{i} c_k(\alpha) x_{i-k}, \qquad i = 0, 1, \ldots, \tag{4}$$

where coefficients $c_k(\alpha)$ are defined by (3).

*e-mail: a.ruszewski@pb.edu.pl

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The sequence of coefficients (3) for $k > 0$ can be calculated by the following recursive formula

$$c_{k+1}(\alpha) = c_k(\alpha)\frac{k-\alpha}{k+1}, \quad k = 1,2,\ldots, \tag{5}$$

where $c_1(\alpha) = -\alpha$.

It is easy to see that coefficients (3) for $k > 0$ are negative and absolute values of coefficients $c_k(\alpha)$ quickly decrease for increasing k. Therefore, in equation (4) we can limit upper bound of summation by the natural number L, which is called the length of the practical implementation [13]. Thus, equation (4) can be written in the form

$$x_i = A_1 x_{i-1} - \sum_{k=1}^{L} c_k(\alpha) x_{i-k}, \quad i = 0, 1, \ldots. \tag{6}$$

Equation (6) is called the practical realization of fractional system (1).

The definition of practical stability for fractional discrete-time systems have been introduced in [13]. Regarding equations (1) and (6) this definition takes the following form.

Definition 1. The fractional system (1) is called practically stable for given length L of practical implementation if the system (6) is asymptotically stable.

If system (6) is asymptotically stable for $L \to \infty$ then fractional system (1) is called asymptotically stable.

It is well-known that discrete-time linear systems are asymptotically stable if all roots of their characteristic polynomials have absolute values less than 1, i.e. lie inside the unit circle of the complex z-plane. From the above we can formulate the following theorem.

Theorem 1. The fractional system (1) with given length L of practical implementation is practically stable if and only if

$$w(z) \neq 0, \qquad |z| \geq 1, \tag{7}$$

where

$$w(z) = \det\left\{ I - A_1 z^{-1} + \sum_{k=1}^{L} c_k(\alpha) z^{-k} \right\}. \tag{8}$$

is the characteristic polynomial of system (6), where I is the $n \times n$ identity matrix.

The asymptotic stability of system (6), i.e. the practical stability of fractional system (1), can be checked using well-known stability tests for discrete-time systems with delays. However, these methods may be inconvenient, because the degree of polynomial (8) can be very high for a large length L of practical implementation.

3. Solution of the problem

3.1. Practical stability. The characteristic equation $w(z) = 0$ of system (6) with polynomial (8) has the form

$$\det\left\{ I - A_1 z^{-1} + \sum_{k=1}^{L} c_k(\alpha) z^{-k} \right\} = 0. \tag{9}$$

By multiplying both sides of equation (9) by z we obtain

$$\det\left\{ \left(z + \sum_{k=1}^{L} c_k(\alpha) z^{1-k} \right) I - A_1 \right\} = 0. \tag{10}$$

The multiplication by z adds the root at 0, but this does not change the stability of the discrete-time system (6). Equation (10) can be written as

$$\prod_{i=1}^{n} w_i(z) = 0, \tag{11}$$

where

$$w_i(z) = z + \sum_{k=1}^{L} c_k(\alpha) z^{1-k} - \lambda_i(A_1) \tag{12}$$

and $\lambda_i(A_1)$ denotes i-th eigenvalue of A_1 $(i = 1,2,\ldots,n)$.

Fractional system (1) with given length L of practical implementation is practically stable if all roots of all equations $w_i(z) = 0$ $(i = 1,2,\ldots,n)$ are stable, i.e. have absolute values less than 1. Therefore, we analyse the stability problem of roots of the equation

$$z + \sum_{k=1}^{L} c_k(\alpha) z^{1-k} - \rho = 0 \tag{13}$$

in terms of $\rho = \lambda_i(A_1)$.

We apply the D-decomposition method [22] to stability analysis. According to this method, the parameter plane is partitioned into a finite number of regions. For a point from each region the characteristic polynomial has fixed number of unstable and stable roots. This number does not change in each region. If the number of unstable roots is zero, thus this region is the stability region.

Substituting $z = \exp(j\omega)$, $\omega \in [0, 2\pi]$, i.e. boundary of the unit circle in the complex z-plane, in equation (13) we have

$$\rho(\omega) = e^{j\omega} + \sum_{k=1}^{L} c_k(\alpha) e^{j\omega(1-k)}, \qquad \omega \in [0, 2\pi]. \tag{14}$$

Equation (14) determines the boundary of asymptotic stability region in the complex ρ-plane for given values of α and L. The closed curve (14) partitions the complex ρ-plane into two regions, one bounded and one unbounded. The bounded region will be denoted by $S(\alpha, L)$.

It is easy to check that (14) for $\alpha = 1$ describes the circle of radius 1 centered at point $(-1, 0)$.

Lemma 1. All roots of equation (13) are stable if and only if $\rho \in S(\alpha, L)$.

Proof. According to the D-decomposition method, we need only to show that there exists at least one point in the region $S(\alpha, L)$ such that all roots of equation (13) are stable. It is easy to check that for all $\alpha \in (0, 1)$ and $L \geq 2$ the point $\rho = 0$ lies in bounded region $S(\alpha, L)$.

Equation (13) for $\rho = 0$ can be written in the form

$$z\left(z^L + \sum_{k=1}^{L} c_k(\alpha) z^{L-k}\right) = 0. \tag{15}$$

Thus, equation (15) has one stable root $z = 0$ and L roots of equation

$$z^L + \sum_{k=1}^{L} c_k(\alpha) z^{L-k} = 0. \tag{16}$$

To prove that all roots of (16) are stable we use the fact presented in [23], i.e. the polynomial $z^n + a_{n-1}z^{n-1} + \dots + a_1 z + a_0$ has all roots of absolute values less than 1 if

$$1 > |a_{n-1}| + \dots + |a_1| + |a_0|. \tag{17}$$

Notice that for $\alpha \in (0, 1)$ coefficients $c_k(\alpha) < 0$ for $k = 1, 2, \dots, n$, therefore $\left|\sum_{k=1}^{L} c_k(\alpha)\right| = -\sum_{k=1}^{L} c_k(\alpha)$.

Considering above, condition (17) for the left-hand side of equation (16) has the form

$$1 > -\sum_{k=1}^{L} c_k(\alpha). \tag{18}$$

Taking into account that $\sum_{k=1}^{\infty} c_k(\alpha) = -1$ [3], we have

$$1 + \sum_{k=1}^{L} c_k(\alpha) > 1 + \sum_{k=1}^{\infty} c_k(\alpha) = 0. \tag{19}$$

Thus, condition (18) holds.

For example, for $L = 2$ equation (16) has the form

$$z^2 + c_1(\alpha) z + c_2(\alpha) = 0, \tag{20}$$

where $c_1(\alpha) = -\alpha$ and $c_2(\alpha) = \dfrac{\alpha(\alpha - 1)}{2}$.

The characteristic equation (20) has only real roots since

$$c_1(\alpha)^2 - 4c_2(\alpha) = 2\alpha - \alpha^2 > 0 \quad \text{for} \quad \alpha \in (0, 1). \tag{21}$$

It easy to check that the real roots of equation (20)

$$z_1 = \frac{\alpha - \sqrt{2\alpha - \alpha^2}}{2}, \qquad z_2 = \frac{\alpha + \sqrt{2\alpha - \alpha^2}}{2} \tag{22}$$

have absolute values less than 1 for all $\alpha \in (0, 1)$.

It follows that equation (15) for $\rho = 0$ has one root $z = 0$ and L roots with absolute values less than 1. Since $\rho = 0 \in S(\alpha, L)$, then $S(\alpha, L)$ is the stability region for equation (15) and the proof is completed. □

Theorem 2. The fractional system (1) with given length L of practical implementation is practically stable if and only if all eigenvalues $\lambda_i(A_1)$ $(i = 1, 2, \dots, n)$ lie in the stability region $S(\alpha, L)$, i.e. $\lambda_i(A_1) \in S(\alpha, L)$ for all $i = 1, 2, \dots, n$.

Proof. The proof directly follows from Lemma 1 and equations (11) and (12).

Substituting $\omega = 0$ and $\omega = \pi$ in (14), we obtain

$$\rho(0) = 1 + \sum_{k=1}^{L} c_k(\alpha), \tag{23}$$

$$\rho(\pi) = -1 - \sum_{k=1}^{L} c_k(\alpha)(-1)^k. \tag{24}$$

□

Lemma 2. If all eigenvalues $\lambda_i(A_1)$ are real, then the fractional system (1) with given length L of practical implementation is practically stable if and only if

$$\rho(\pi) < \lambda_i(A_1) < \rho(0), \qquad i = 1, 2, \dots, n. \tag{25}$$

Proof. Note that $\rho(\pi) < \rho(0)$ and from Theorem 2 we have that the interval $(\rho(\pi), \rho(0))$ of the real axis lies in the stability region $S(\alpha, L)$ for $\alpha \in (0, 1)$. This completes the proof. Lemma 2 also follows from [20]. □

Fig. 1 shows the practical stability regions $S(\alpha, L)$ on the plane of eigenvalues of A_1 for $L = 100$ and some values of fractional order $\alpha \in (0, 1)$. The practical stability regions $S(\alpha, L)$ for $\alpha = 0.5$ and some values of length L of practical implementation are shown in Fig. 2. Notice that for fixed α and different values of L, the values of $\rho(\pi)$ are almost the same, while the values of $\rho(0)$ differ to a greater extent.

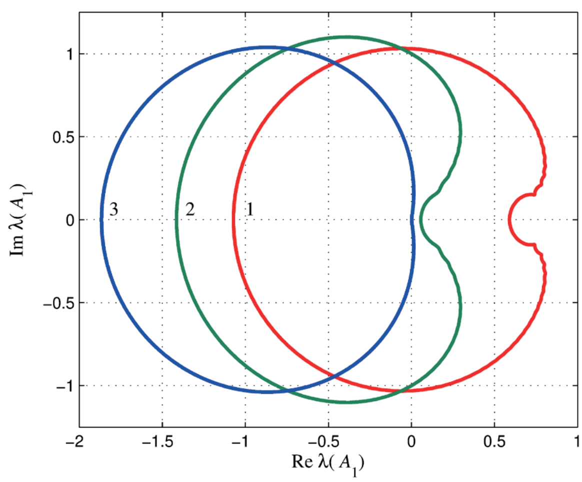

Fig. 1. Regions $S(\alpha, L)$ for $L = 100$ and $\alpha = 0.1$ (boundary 1), $\alpha = 0.5$ (boundary 2) and $\alpha = 0.9$ (boundary 3)

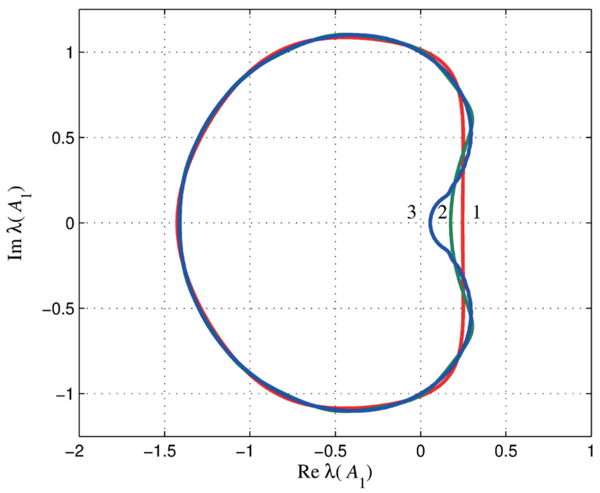

Fig. 2. Regions $S(\alpha,L)$ for $\alpha = 0.5$ and $L = 5$ (boundary 1), $L = 10$ (boundary 2) and $L = 100$ (boundary 3)

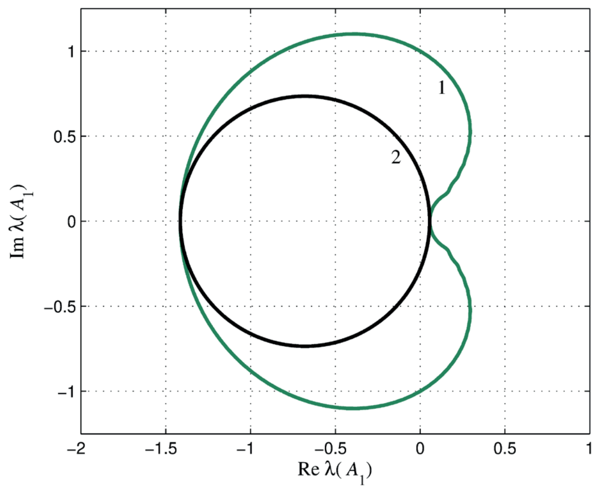

Fig. 3. Region $S(\alpha,L)$ for $\alpha = 0.5$ and $L = 100$ (boundary 1) and circle D_1 (boundary 2)

From Figs. 1 and 2 and Lemma 2 it follows that for fixed $\alpha \in (0, 1)$ and L we can inscribe a circle $D_1 = D_1(\rho_1, r_1)$ with the centre

$$\rho_1 = 0.5(\rho(0) + \rho(\pi)) = \sum_{k=1,3,\ldots}^{L} c_k(\alpha) \tag{26}$$

and radius

$$r_1 = 0.5(\rho(0) - \rho(\pi)) = 1 + \sum_{k=2,4,\ldots}^{L} c_k(\alpha) \tag{27}$$

into the stability region $S(\alpha,L)$.

From the above we have the following sufficient condition for the practical stability.

Lemma 3. The fractional system (1) with given length L of practical implementation is practically stable if all eigenvalues of A_1 lie in circle $D_1 = D_1(\rho_1, r_1)$.

Region $S(\alpha,L)$ with $\alpha = 0.5$, $L = 100$ and circle D_1 are shown in Fig. 3. From (26) and (27) for $\alpha = 0.5$ and $L = 100$ we have $\rho_1 = -0.6788$ and $r_1 = 0.73521$.

Example 1. Check the practical stability of fractional system (1) with length $L = 100$ of practical implementation and the matrix

$$A_1 = \begin{bmatrix} -0.01 & -1.82 & 0.04 & -0.52 \\ 0.95 & -2.24 & -1.21 & 0.81 \\ 0.17 & -0.75 & 0.75 & -1.55 \\ 0.34 & -0.54 & 0.48 & -0.71 \end{bmatrix}. \tag{28}$$

The matrix A_1 has the following eigenvalues:

$$\lambda_{1,2} = -1.197 \pm j0.2767, \qquad \lambda_{3,4} = 0.092 \pm j0.7039. \tag{29}$$

Practical stability regions $S(\alpha,L)$ with $L = 100$ for $\alpha = 0.1$, $\alpha = 0.5$, $\alpha = 0.9$ and eigenvalues (29) are shown in Fig. 4. From Fig. 4 it follows that eigenvalues (29) lie in the practical stability region for $\alpha = 0.5$. Thus, the considered fractional system with $L = 100$ is practically stable for $\alpha = 0.5$, while it is not practically stable for $\alpha = 0.1$ and $\alpha = 0.9$. Analysing practical stability regions for all fixed $\alpha \in (0, 1)$ we obtain that fractional system (1), (28) is practically stable for $\alpha \in (0.31, 0.65)$. The circle D_1 for $L = 100$ and $\alpha = 0.5$ is also shown in Fig. 4. It is easy to see that not all eigenvalues (29) lie in circle D_1. This means that the sufficient condition of Lemma 3 is not satisfied for system (1), (28) with $L = 100$ and $\alpha = 0.5$.

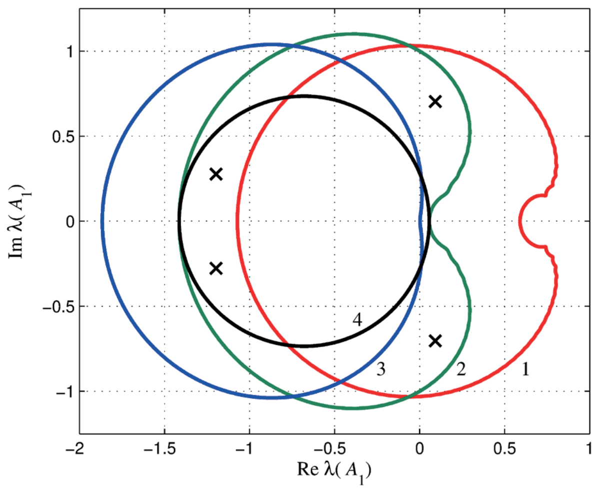

Fig. 4. Regions $S(\alpha,L)$ for $L = 100$ and $\alpha = 0.1$ (boundary 1), $\alpha = 0.5$ (boundary 2), $\alpha = 0.9$ (boundary 3), circle D_1 for $L = 100$ and $\alpha = 0.5$ (boundary 4) and eigenvalues (29) (x)

Notice that the presented method of practical stability test does not require calculation of $L = 100$ roots of the characteristic equation.

3.2. Asymptotic stability. We will analyse system (6) with $L \to \infty$ to formulate asymptotic stability conditions of system (1). Firstly, we prove the following important lemma.

Lemma 4. The following formula is true

$$\sum_{k=1}^{\infty} c_k(\alpha) z^{-k} = (z-1)^{\alpha} z^{-\alpha} - 1, \tag{30}$$

where coefficients $c_k(\alpha)$ are defined by (3).

Proof. Using Newton's generalized binomial formula

$$(a+b)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} a^{\alpha-k} b^k, \tag{31}$$

where α is a real number, for $a = z$ and $b = -1$ we obtain

$$\begin{aligned}(z-1)^{\alpha} &= \sum_{k=0}^{\infty} \binom{\alpha}{k} z^{\alpha-k}(-1)^k \\ &= z^{\alpha} + \sum_{k=1}^{\infty} \binom{\alpha}{k} z^{\alpha-k}(-1)^k.\end{aligned} \tag{32}$$

Multiplying both sides of (32) by $z^{-\alpha}$ leads to

$$(z-1)^{\alpha} z^{-\alpha} = 1 + \sum_{k=1}^{\infty} \binom{\alpha}{k} z^{-k}(-1)^k. \tag{33}$$

Finally, from (33) we have

$$\sum_{k=1}^{\infty} \binom{\alpha}{k} z^{-k}(-1)^k = (z-1)^{\alpha} z^{-\alpha} - 1. \tag{34}$$

Using (3), from (34) we obtain (30). This completes the proof. □

Fractional system (1) is called asymptotically stable if system (6) is practically stable for $L \to \infty$. Characteristic function (10) for $L \to \infty$ has the form

$$\det\left\{\left(z + \sum_{k=1}^{\infty} c_k(\alpha) z^{1-k}\right) I - A_1\right\} = 0. \tag{35}$$

Using (30) and (35), we obtain

$$\det\left\{(z-1)^{\alpha} z^{1-\alpha} I - A_1\right\} = 0. \tag{36}$$

Equation (36) can be written in the form (11), where

$$w_i(z) = (z-1)^{\alpha} z^{1-\alpha} - \lambda_i(A_1) \tag{37}$$

and $\lambda_i(A_1)$ denotes i-th eigenvalue of A_1 $(i = 1, 2, \ldots, n)$.

From the above it follows that fractional system (1) is asymptotically stable if and only if all roots of all equations $w_i(z) = 0$ $(i = 1, 2, \ldots, n)$ are stable, where $w_i(z)$ has the form (37). Thus, we consider the stability problem of roots of the equation

$$(z-1)^{\alpha} z^{1-\alpha} - \eta = 0 \tag{38}$$

in terms of $\eta = \lambda_i(A_1)$.

Substituting $z = \exp(j\omega)$, $\omega \in [0, 2\pi]$, in equation (38) we obtain the parametric description of boundary of the stability region in the complex η-plane

$$\eta(\omega) = (e^{j\omega} - 1)^{\alpha} (e^{j\omega})^{1-\alpha}, \qquad \omega \in [0, 2\pi]. \tag{39}$$

It is easy to check that (39) for $\alpha = 1$ describes the circle of radius 1 centered at point $(-1, 0)$.

The closed curve (39) partitions the complex η-plane into two regions (bounded and unbounded). The bounded region will be denoted by $S(\alpha)$.

Lemma 5. If η is real, then all roots of equation (38) are stable if and only if

$$-2^{\alpha} < \eta < 0. \tag{40}$$

Proof. The system (1) where the matrix A_1 is a scalar a_1 has been considered in [20]. It has been shown that this system is asymptotically stable if and only if $-2^{\alpha} < a_1 < 0$. Thus, (40) is the necessary and sufficient condition for asymptotic stability of real η roots of (38) and the proof is completed. □

Lemma 6. All roots of equation (38) are stable if and only if $\eta \in S(\alpha)$.

Proof. We need only to show that there exists at least one point in the region $S(\alpha)$ such that all roots of (38) are stable. Using (39) for $\omega = 0$ and $\omega = \pi$ we obtain $\eta(0) = 0$ and $\eta(\pi) = -2^{\alpha}$. Note that $\eta(\pi) < \eta(0)$ and from Lemma 5 we have that the interval $(-2^{\alpha}, 0)$ of the real axis is the stability region for all real η roots of (38). Hence, there exist such points η in $S(\alpha)$ for which all roots of (38) are stable. This completes the proof. □

Asymptotic stability regions $S(\alpha)$ for $\alpha = 0.1$, $\alpha = 0.5$ and $\alpha = 0.9$ are shown in Fig. 5.

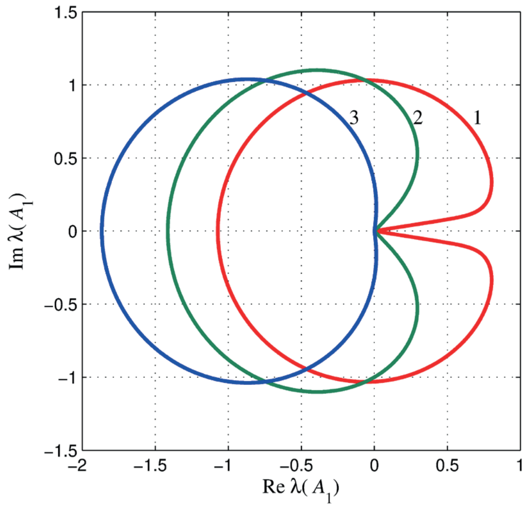

Fig. 5. Regions $S(\alpha)$ for $\alpha = 0.1$ (boundary 1), $\alpha = 0.5$ (boundary 2) and $\alpha = 0.9$ (boundary 3)

Theorem 3. The fractional system (1) is asymptotically stable if and only if all eigenvalues of the matrix A_1 lie in the stability region $S(\alpha)$ with boundary (39).

Proof. The proof directly follows from Lemma 6 and equations (11) and (37). □

Lemma 7. If all eigenvalues $\lambda_i(A_1)$ are real, then the fractional system (1) is asymptotically stable if and only if

$$-2^{\alpha} < \lambda_i(A) < 0, \qquad i = 1, 2, \ldots, n. \tag{41}$$

Proof. The proof directly follows from Lemma 5 and equations (11) and (37). □

Theorem 4. The fractional system (1) is asymptotically stable if and only if

$$\phi_i \in \left[\alpha\frac{\pi}{2}, 2\pi - \alpha\frac{\pi}{2}\right], \quad \wedge \quad |\lambda_i| < |m_i|, \quad i = 1, 2, \ldots, n, \tag{42}$$

where $|\lambda_i|$ and ϕ_i are the modulus and argument, respectively, of the i-th eigenvalue λ_i of the matrix A_1 and

$$|m_i| = \left(2\left|\sin\frac{\phi_i - \alpha\frac{\pi}{2}}{2-\alpha}\right|\right)^{\alpha}, \qquad i = 1, 2, \ldots, n. \tag{43}$$

Proof. The fractional system described by state equation

$$\Delta^{\alpha} x_{i+1} = A_1 x_i \tag{44}$$

has been considered in [8] for $\alpha \in (0, 1)$ and [11, 12] for $\alpha \in (0, 2)$. Note that asymptotic stability regions of system (44) with $\alpha \in (0, 1)$ presented in [8, 11, 12] are the same as obtained for system (1). Thus, the analytical asymptotic stability condition given in [12] for system (44) is also true for system (1). This completes the proof. □

Similar results for asymptotic stability of system (1) have been obtained in [19] using Z-transform method.

Similarly as in [8] from Fig. 5 it follows that for any fixed $\alpha \in (0, 1)$ we can inscribe a circle $D_2 = D_2(\eta_2, r_2)$ with the centre $\eta_2 = 0.5(\eta(0) + \eta(\pi)) = -2^{\alpha-1}$ and radius $r_2 = 0.5(\eta(0) - \eta(\pi)) = 2^{\alpha-1}$ into the stability region $S(\alpha)$. For example, for $\alpha = 0.5$ we obtain $\eta_2 = -2^{-0.5} = -0.7071$ and $r_2 = 0.7071$.

Region $S(\alpha)$ for $\alpha = 0.5$ and circle D_2 are shown in Fig. 6.

From the above we have the following sufficient condition for asymptotic stability.

Lemma 8. The fractional system (1) is asymptotically stable if all eigenvalues of A_1 lie in the circle $D_2 = D_2(\eta_2, r_2)$.

Example 2. Check asymptotic stability of fractional system (1), (28) considered in Example 1.

Asymptotic stability regions $S(\alpha)$ for $\alpha = 0.1$, $\alpha = 0.5$, $\alpha = 0.9$, eigenvalues (29) and circle D_2 for $\alpha = 0.5$ are shown in Fig. 7. We can see that all eigenvalues (29) lie in the stability region $S(\alpha)$ for $\alpha = 0.5$, but not all eigenvalues (29) lie in circle D_2. This means that the sufficient condition of Lemma 8 is not satisfied for system (1), (28) with $\alpha = 0.5$. Analysing asymptotic stability regions for all fixed $\alpha \in (0, 1)$ we obtain that fractional system (1), (28) is asymptotically stable for $\alpha \in (0.301, 0.659)$.

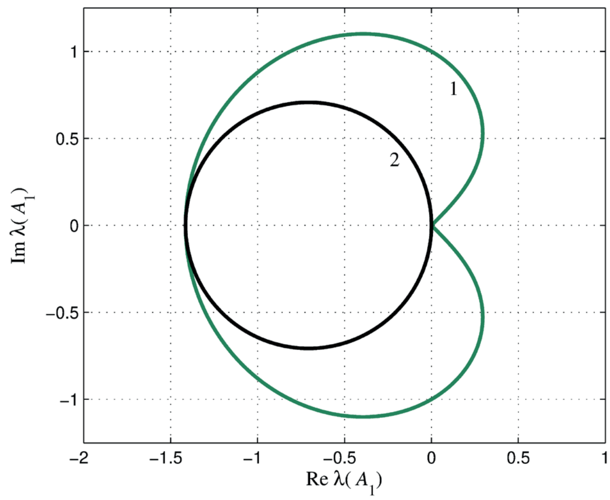

Fig. 6. Region $S(\alpha)$ for $\alpha = 0.5$ (boundary 1) and circle D_2 (boundary 2)

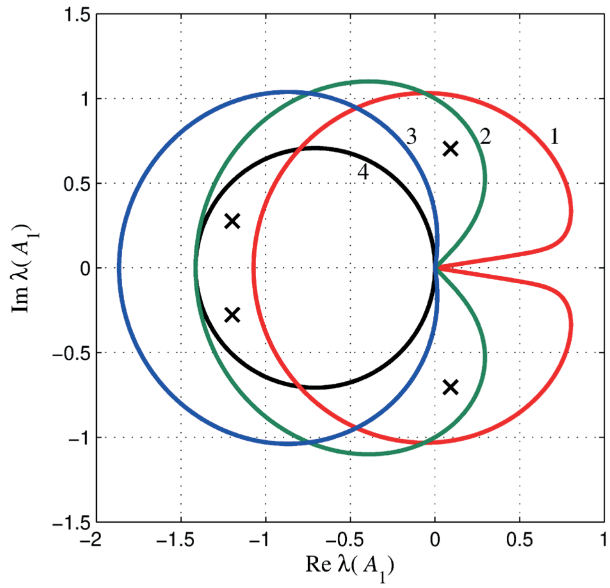

Fig. 7. Regions $S(\alpha)$ for $\alpha = 0.1$ (boundary 1), $\alpha = 0.5$ (boundary 2), $\alpha = 0.9$ (boundary 3), circle D_2 for $\alpha = 0.5$ (boundary 4) and eigenvalues (29) (x)

Example 3. Check asymptotic stability of fractional system (1) with the matrix

$$A_1 = \begin{bmatrix} -2.71 & 0.45 & 0.37 & 1.67 \\ -3.89 & 0.36 & 0.61 & 3.61 \\ 0.42 & -1.11 & -0.47 & 1.03 \\ -1.28 & 0.02 & 0.62 & 0.72 \end{bmatrix}. \tag{45}$$

The matrix A_1 has the following eigenvalues: $\lambda_1 = -1.0178$, $\lambda_2 = -0.9352$, $\lambda_3 = -0.3119$ and $\lambda_4 = 0.1649$. Note that the considered system has only real eigenvalues and one of them is positive. According to Lemma 7 this system is not asymptotically stable for any $\alpha \in (0, 1)$, because condition (41) does not hold.

4. Concluding remarks

The practical and asymptotic stabilities of the fractional order $\alpha \in (0, 1)$ discrete-time delayed linear system have been analysed. The state equation (1) is the fractional model without a time shift in the difference.

Necessary and sufficient conditions for practical stability and for asymptotic stability have been established. The conditions in terms of the location of eigenvalues of the state matrix in the complex plane have been given. Parametric descriptions of the boundary of practical stability and asymptotic stability regions have been presented. Necessary conditions for the practical stability and the asymptotic stability have been also proposed in Lemmas 3 and 8, respectively. These conditions give more restrictive stability regions than conditions in Theorems 2 and 3, but they are easier to check.

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