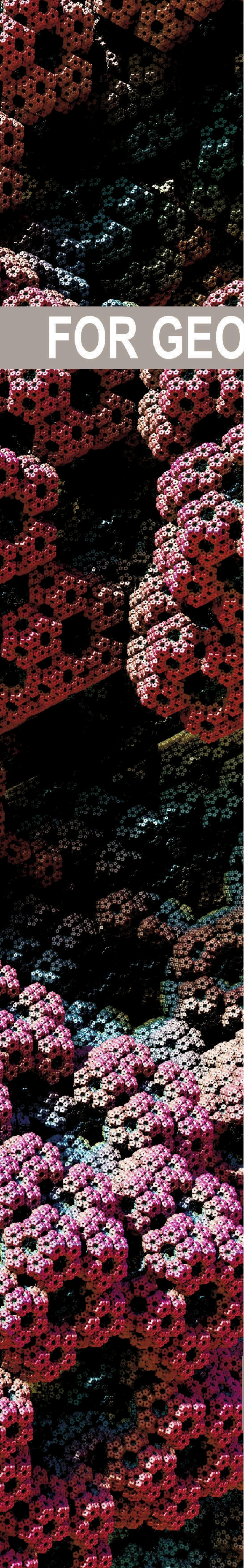

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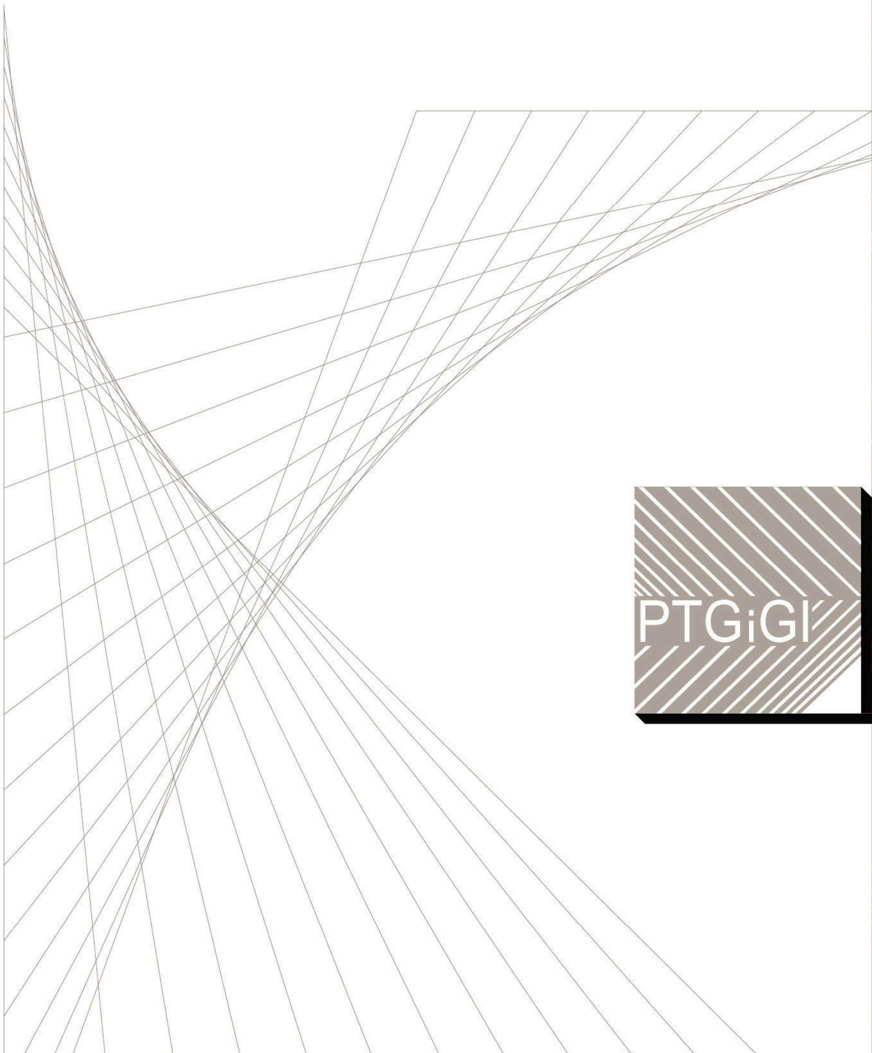

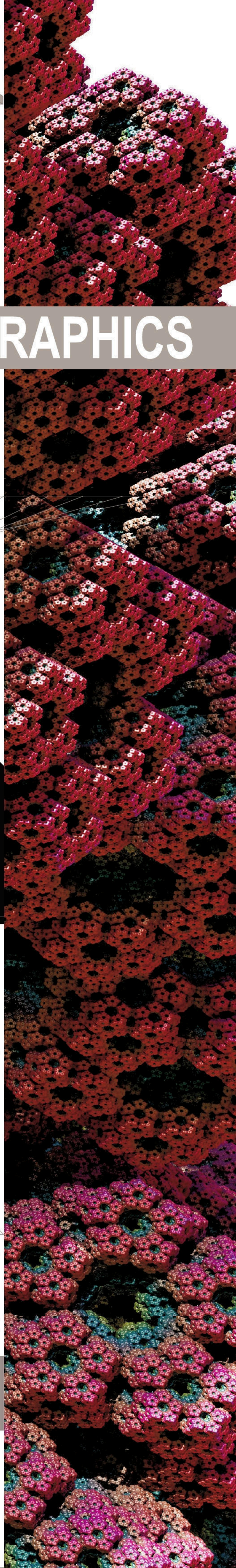

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CONTENTS

PLANES TANGENT SIMULTANEOUSLY TO THREE SPHERES

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Abstract. The authors show in simple steps how to reduce the problem of determing planes simultaneously tangent to three spheres for easier construction of tangent planes at the same time to two cones. The method presented here is of great educational importance: it develops spatial thinking by combining different types of surfaces (sphere and cones) an d shows how to use previously known properties to analyze new problems.

Keywords: common tangents to two circles, axis, vertex angle of a cone, planes tangent to cones/spheres, cone circumscribed on a sphere.

1 Introduction

Descriptive Geometry is part of the first year course curriculum of the Civil Engineering Faculty of Warsaw University of Technology. It is a subject taught over a span of two semesters, with 15 hours of lectures and 15 hours of tutorials in each, without final exams.

The first semester begins with central projection (4h + 4h), then continues with parallel projection (1h+1h) and ends with orthogonal projection. Commencing the study of geometry from the direct method of central projection is unfortunate (the order is imposed, we do not have influence on it). Many students, not well prepared for college courses, are shocked. Luckily, they can easily receive conditional course credit and repeat the class the following year. Multiview projection is chiefly aimed at learning to read such projections through representation of polyhedrons, stressing the incidence relation. The introduced measurement constructions consist of rabatment (which causes difficulties for students) and transformation (a method students think to be good for everything).

In the second semester the first half is dedicated to surfaces (of revolution and ruled), and the other one covers the rudiments of map projection. In that semester students are more oriented about what is happening. Problems related with surfaces are a great opportunity to develop spatial imagination and also show that some methods from the first semester cannot be used automatically – for example, to determine a plane or a sphere tangent simultaneously to two surfaces of revolution transformation is not the first choice.

The organization of the classes of each semester as 15 separate units does not pose a greater difficulty for the lectures, however during the tutorials it is troubling. Considering all of the organizational matters and tests, we are left with twelve 45-minute meetings to impart the material assumed in the program. It is quite challenging to manage. In practice 2 or 3 exercises are solved during the tutorials, which are an illustration to the lecture. Thus, the consultation hours become an essential complementary element that is eagerly taken advantage of by the students.

The title issue of planes tangent simultaneously to three spheres has been brought up by a student during consultations. Analyzing whether the spheres lie on the same side or different sides of the tangent plane, he came to the correct conclusion that there might be up to eight of these planes, however did not know how to find them.

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In this article we present a proposition of solving this task in a way that will be accessible to more students. To properly prepare for the title issue, students must first understand the properties of planes tangent to two cones.

2 Planes tangent simultaneously to two cones with parallel axes

It is evident that the necessary condition is the same vertex angle of the cones. The problem is an exemplary task solved during tutorials as it is presented in Fig. 1. Students are required first to individually determine a system defining the planes and next to choose and draw tangent polygons with marking the visibility.

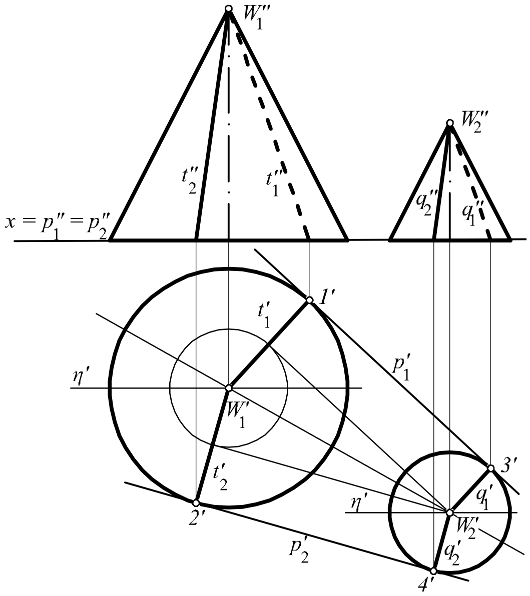

Figure 1

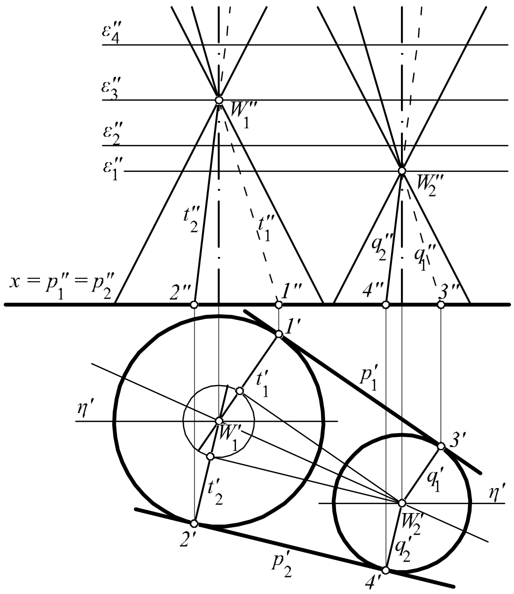

Figure 2

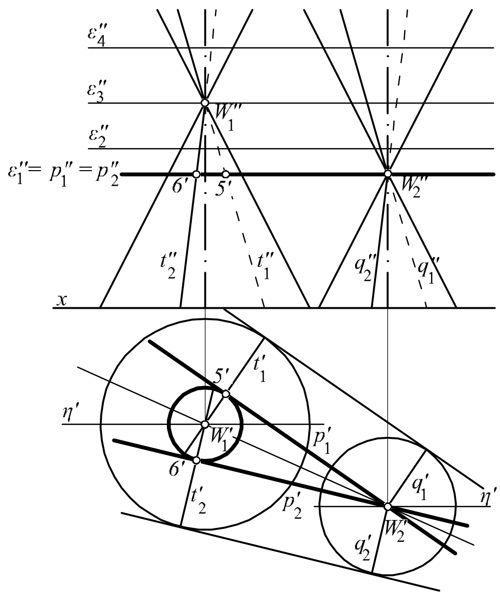

Figure 3

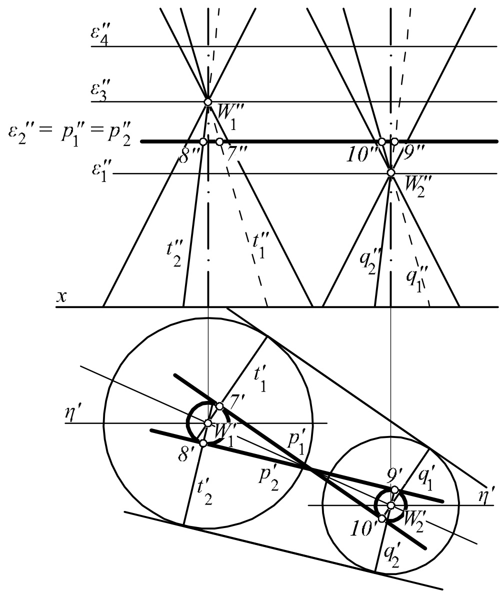

Figure 4

A difficulty usually arises if one of the cones is “inverted”. It is not easy to imagine relations between the two-napped cones. A way of handling the problem is to consider sections of the tangent planes and given cones by planes perpendicular to the axes of the cones. Fig. 2 – Fig.6 show how horizontal lines in the tangent planes change: common exterior tangents to the cut circles become common interior tangents and inversely.

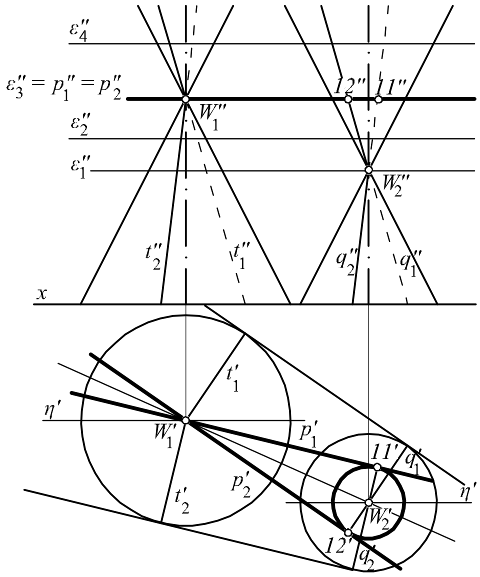

Figure 5

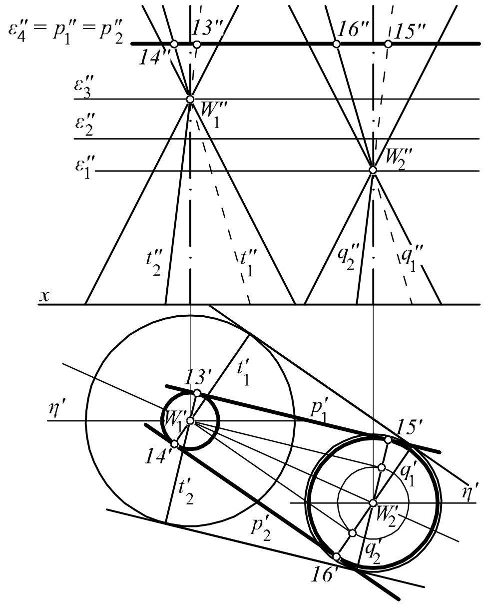

Figure 6

The above determined sections facilitate the solving process for students.

3 Existence of planes simultaneously tangent to two cones with parallel axes and the same vertex angle

Analyzing Fig.2 – Fig.6 one can also arrive at the following conclusions:

I. If the vertex of a cone is outside of the other cone, there are two common tangent planes.

II. If the vertex of a cone is lying in the surface of the other cone, there is only one solution in the case of $W_1 \neq W_2$.

III. If $W_1 = W_2$ the cones coincide, then it is evident that the number of solutions is infinite.

IV. If W_1 or W_2 is inside the other cone, there is no solution.

4 Planes tangent simultaneously to a cone and a hemisphere

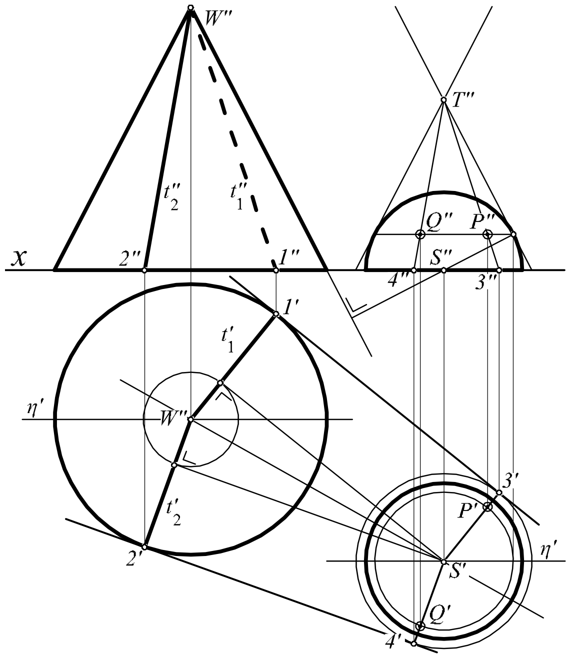

Figure 7

Such tangency problem is one of the exercises explained during the tutorials. It is reduced to the previous case. We consider a cone circumscribed on the given hemisphere with the axis parallel to the axis of the given cone and the vertex angle equal to that of the given cone. The solution, as is explained to students, is shown in Fig.7.

5 Planes tangent simultaneously to three spheres

Consider three spheres Σ_1 (O_1, r_1), Σ_2 (O_2, r_2), Σ_3 (O_3, r_3) in general position (i.e. with arbitrary centers O_1, O_2, O_3, and radii r_1, r_2, r_3, without common points). The problem of construction of the planes tangent simultaneously to these spheres can be reduced to the construction of planes tangent simultaneously to two cones with parallel axes and the same vertex angle.

In fact, any plane tangent simultaneously to two spheres is tangent to the cone circumscribed on these spheres and reversely. Hence, consider one cone Λ_1 circumscribed on two spheres out of the three (for example on Σ_1 and Σ_2; the choice is not important) and the other cone Λ_2 circumscribed on Σ_3 with the axis parallel to the axis of Λ_1 and the vertex angle equal to that of Λ_1.

Therefore, a plane α is simultaneously tangent to Σ_1, Σ_2 and Σ_3 if and only if it is tangent simultaneously to Λ_1 and Λ_2.

The four possible cases are presented by orthogonal projection onto the plane defined by the centres of the spheres as it is presented in Fig.8.

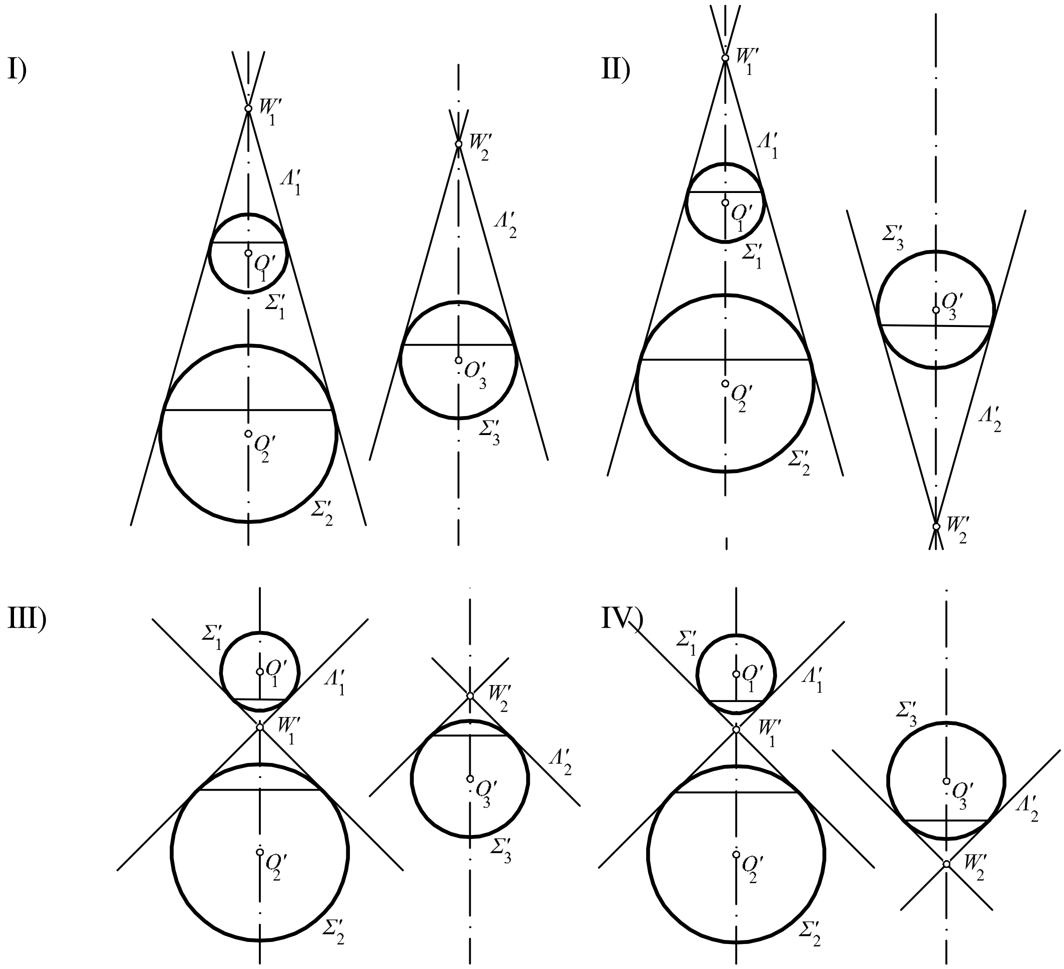

Figure 8

The construction of the required planes can be done as in Section 2. The existence of solutions follows from Section 3. Therefore there are 8 planes tangents simultaneously to three spheres in general.

6 Some special positions

There are specific situations when the location of the three spheres impose conditions on the amount of solutions of planes tangent simultaneously to all three spheres.

Remark that if $r_1 = r_2$ then the cones considered in cases I and II (Fig. 8) become cylinders and these cases present the same solutions (6 tangent planes in total instead of 8).

As an excellent exercise of imagination students have to find examples with 0, 1, 2, 3, 4, 5, 6, 7, 8 solutions. Fig. 9 shows some chosen positions.

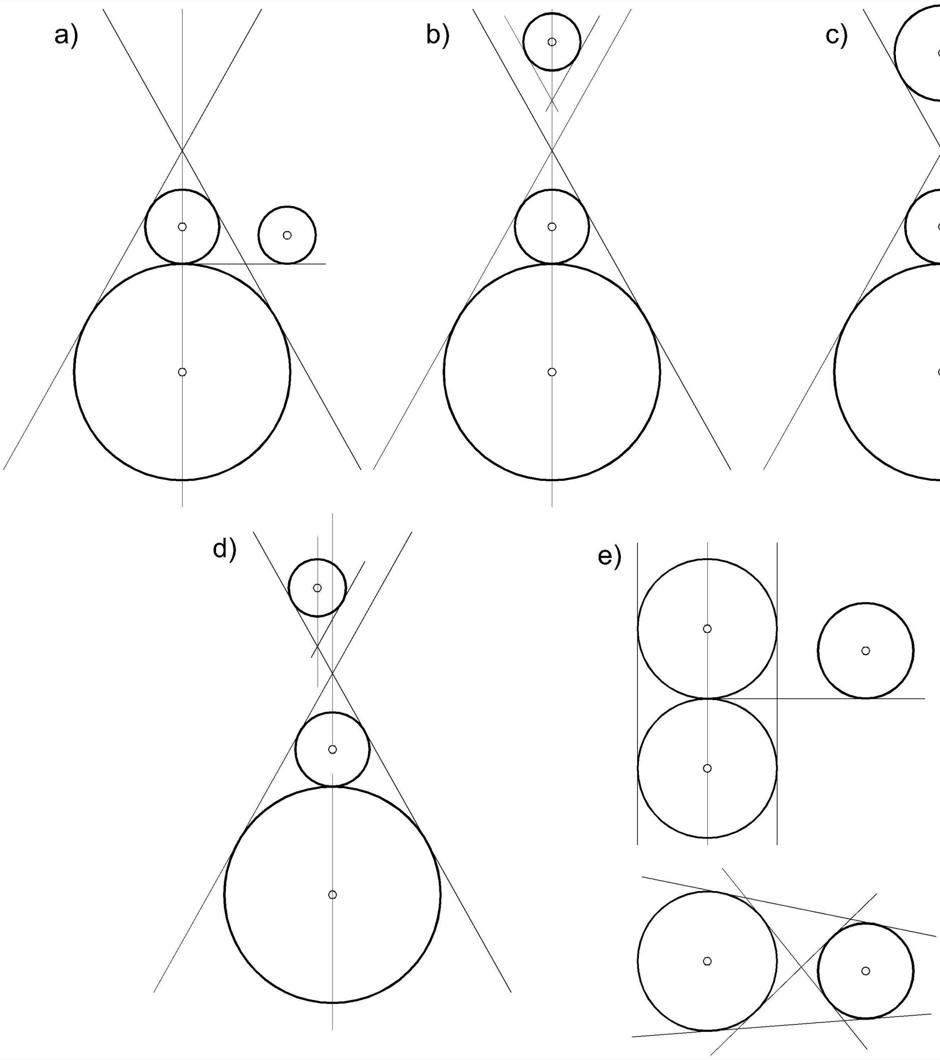

Figure 9

In the case depicted in Fig. 9a there are 5 possible common tangency planes, in 9b there are none, while in 9c there is an infinite number of such planes. If the spheres are chosen as portrayed in Fig. 9d there is only one plane simultaneously tangent to all of them. If two of the spheres have the same radius and have a common point in their surface as in Fig. 9e, then 5 planes exist meeting the condition of tangency.

7 Transformation after all

Constructions using an oblique auxiliary projection plane are easier for students to understand and to imagine the actual situation.

It follows from the analysis in Section 5 that every required plane is passing through vertices W_1 and W_2 of the considered cones, so it is passing through the line w defined by theses points. Therefore a convenient auxiliary projection plane is the plane perpendicular to this common line w. Fig. 10 shows a simplified case of a plane tangent simultaneously to three hemispheres found with the aid of the transformation method.

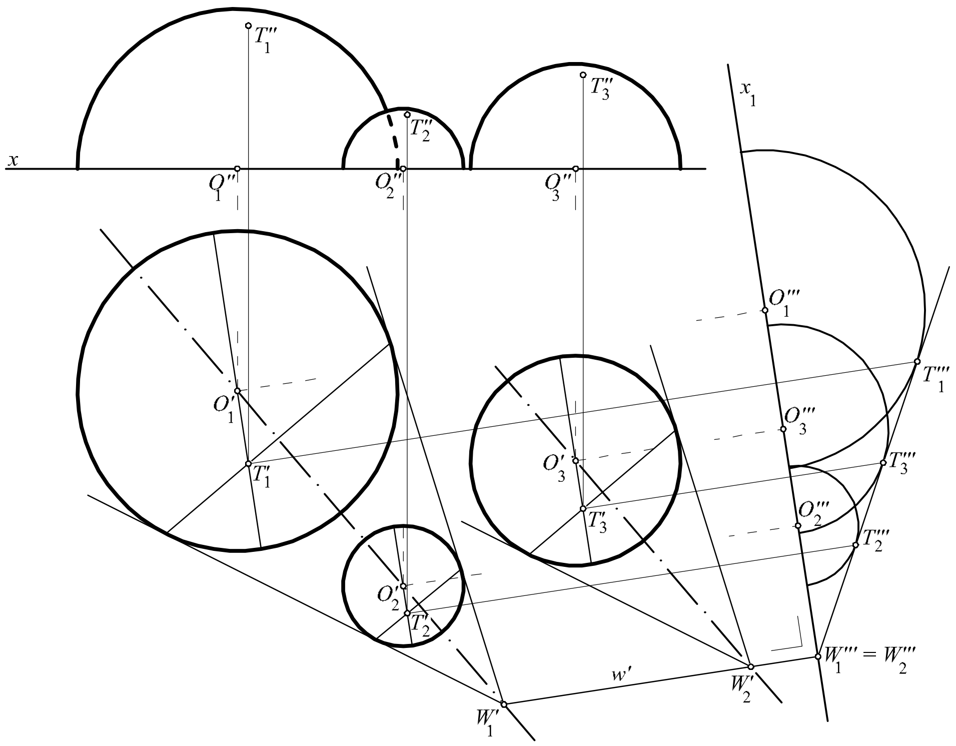

Figure 10

Students can observe that the tangency points can be constructed without drawing the auxiliary projection. In fact, these points are lying on the tangency circles and the great circles cut by the planes perpendicular to the line W_1W_2.

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PŁASZCZYZNY JEDNOCZEŚNIE STYCZNE DO TRZECH KUL

Autorki pokazują w kolejnych krokach jak sprowadzić problem wyznaczania płaszczyzn stycznych jednocześnie do trzech danych kul do łatwiejszej konstrukcji płaszczyzn jednocześnie stycznych do dwóch stożków. Prezentowana metoda ma duże walory dydaktyczne: rozwija myślenie przestrzenne analizując związki pomiędzy różnymi rodzajami powierzchni (łącząc stożki i kule) oraz pokazuje, jak w analizie nowego problemu wykorzystać własności poznane wcześniej.