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ON THE STRUCTURE OF COMPACT GRAPHS

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Abstract. A simple graph G is called a compact graph if G contains no isolated vertices and for each pair x, y of non-adjacent vertices of G, there is a vertex z with $N(x) \cup N(y) \subseteq N(z)$, where $N(v)$ is the neighborhood of v, for every vertex v of G. In this paper, compact graphs with sufficient number of edges are studied. Also, it is proved that every regular compact graph is strongly regular. Some results about cycles in compact graphs are proved, too. Among other results, it is proved that if the ascending chain condition holds for the set of neighbors of a compact graph G, then the descending chain condition holds for the set of neighbors of G.

Keywords: compact graph, vertex degree, cycle, neighborhood.

Mathematics Subject Classification: 05C07, 05C38, 68R10.

1. INTRODUCTION

Throughout this paper, a graph G is an undirected simple graph with the vertex set $V = V(G)$ and the edge set $E = E(G)$. If vertices x and y are adjacent we write $x - y$. By $\overline{G}$, we mean the complement graph of G. Define the *neighborhood* $N_G(x)$ of a vertex x to be the set of all vertices adjacent to x and let $\overline{N(z)} = N(z) \cup \{z\}$ denote the *closure* of $\overline{N_G(z)}$. When there is no risk of confusion, we denote $N_G(v)$ by $N(v)$. The degree of a vertex x is denoted by $deg(x)$. Vertices with zero degree are called *isolated* vertices. The graph G is said to be *r-regular*, if the degree of each vertex is r. Also, G is called a *refinement of a star graph* if there exists a vertex adjacent to all other vertices. The complete graph of order n is denoted by K_n. The *girth* of a graph G, denoted by $girth(G)$, is the length of a shortest cycle contained in G. If G does not contain any cycle, its girth is defined to be infinity. The *diameter* of a connected graph G, denoted by $diam(G)$, is the maximum distance between any pair of the vertices of G. We denote by P_n and C_n a path and a cycle of order n, respectively. Every connected graph with no cycle is called a *tree*. For a subset $U \subseteq V(G)$, the subgraph of G *induced* by U, denoted $G[U]$, is the graph with $V(G[U]) = U$ and edge

$\{u, v\} \in E(G[U])$ if and only if $\{u, v\} \in E(G)$. The *join* $G_1 \vee G_2$ of graphs G_1, G_2 is the union of the two graphs with additional edges $v_1 - v_2$ for all $v_1 \in V(G_1), v_2 \in V(G_2)$. The *complete n-partite* graph is $K_{p_1,p_2,\ldots,p_n} = \bigvee_{i=1}^{n} \overline{K_{p_i}}$. A complete 2-partite graph is called *complete bipartite*. The graph $K_{1,p}$ is called a *star graph*, and the vertex adjacent to p other vertices is the *center* of the star graph. For undefined terminologies the reader is referred to [2] and [3].

As it was defined by Lu and Wu in [7], a graph G is called a *compact graph* if G contains no isolated vertices and for each pair x, y of non-adjacent vertices of G, there is a vertex z with $N(x) \cup N(y) \subseteq N(z)$. It was proved in [7, Theorem 3.1] that a simple graph G is the zero-divisor graph of a poset if and only if G is a compact graph. Therefore, compact graphs play an important role in the study of zero-divisor graphs of posets. To find some kinds of zero-divisor graphs we refer the reader to [1, 5–7] and [8]. In this article, we continue the study of graph-theory properties of compact graphs by determining when an arbitrary graph is a compact graph. We find the lowest possible bound on the number of edges $e(G) = |E(G)|$ in a graph G that guarantees G is either compact or a refinement of a star graph. We prove a necessary and sufficient condition for determining if a regular graph is a compact graph. Finally, some results about the cycles in compact graphs are proved.

The following theorem from [7] reflects some properties of compact graphs and will be used in this paper, frequently.

Theorem 1.1 ([7, Proposition 2.1])**.** *Let G be a compact graph. Then for any pair of distinct vertices x, y of G, either $N(x) \cap N(y) \neq \varnothing$ or each vertex in $N(x)$ is adjacent to all the vertices of $N(y)$. In particular, G is connected with diameter at most* 3.

2. HOW MANY EDGES COMPACT GRAPHS CAN HAVE?

It is straightforward to see that complete graphs K_n and complete graphs K_n with p removed edges are always compact graphs if p is very small relative to n. In this section, we show that all connected graphs with a sufficient number of edges are compact. For this reason, we characterize compact graphs by the number of edges in their complements.

We start with recalling the definition of compact graphs.

Definition 2.1 ([7])**.** A graph G is called a compact graph if G contains no isolated vertices and for each pair x, y of non-adjacent vertices of G, there is a vertex z with $N(x) \cup N(y) \subseteq N(z)$.

Recall that a *wheel graph* W_n is a graph with n vertices ($n \geq 4$), formed by connecting a single vertex to all vertices of a C_{n-1}.

Example 2.2. From Theorem 1.1, we know that any compact graph G is connected with diameter at most 3. If moreover, G contains a cycle then $\mathrm{girth}(G) \leq 4$. So,

(i) P_n is compact if and only if $n \in \{2, 3\}$,
(ii) C_n is compact if and only if $n \in \{3, 4\}$,
(iii) W_n is compact if and only if $n \in \{4, 5\}$.

To prove the main result of this section (Theorem 2.6), the following lemma is needed.

Lemma 2.3. *Let $\{G_\alpha\}_{\alpha\in\Lambda}$ be a family of disjoint graphs. Then $G = \bigvee_{\alpha\in\Lambda} G_\alpha$ is compact if and only if every G_α is compact.*

Proof. First suppose that G is a compact graph and choose two nonadjacent vertices x and y in G_α. Then it is clear that x and y are not adjacent in G and so there exists a vertex z of G such that $N_G(x) \cup N_G(y) \subseteq N_G(z)$. If $z \in V(G_\beta)$ with $\beta \neq \alpha$, then $z \in N_G(x)$ which is impossible. Thus we can assume that $z \in V(G_\alpha)$. Since $N_G(a) = N_{G_\alpha}(a) \cup (\bigcup_{\beta\neq\alpha} V(G_\beta))$, for every $a \in V(G_\alpha)$, we deduce that $N_{G_\alpha}(x) \cup N_{G_\alpha}(y) \subseteq N_{G_\alpha}(z)$. Conversely, assume that every G_α is compact. If x, y are two nonadjacent vertices of $G = \bigvee_{\alpha\in\Lambda} G_\alpha$, then $x, y \in V(G_\alpha)$ and so there exists a vertex z of G_α such that $N_{G_\alpha}(x) \cup N_{G_\alpha}(y) \subseteq N_{G_\alpha}(z)$. Hence

$$\begin{aligned} N_G(x) \cup N_G(y) &= N_{G_\alpha}(x) \cup N_{G_\alpha}(y) \cup (\bigcup_{\beta\neq\alpha} V(G_\beta)) \\ &\subseteq N_{G_\alpha}(z) \cup (\bigcup_{\beta\neq\alpha} V(G_\beta)) = N_G(z), \end{aligned}$$

and this completes the proof. □

Now, we have the following immediate corollaries.

Corollary 2.4. *Let G be a connected graph which is not the refinement of a star graph, and let $H_1, \ldots, H_m$ be the connected components of the complement graph $\overline{G}$. Then G is a compact graph if and only if each induced subgraph $G[V(H_i)]$ is a compact graph.*

Corollary 2.5. *Any complete k-partite graph is compact.*

Clearly, the graphs $P_4, P_5, \overline{K_{1,2}} \vee P_4$ and the graph G_1 in Figure 1 are neither compact nor refinement of star graphs.

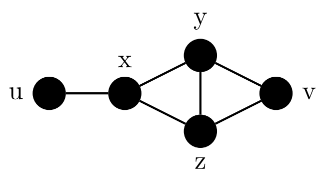

Fig. 1. The plane of graph G_1

In the following theorem, it is shown that every noncompact graph with a sufficient number of edges is the join of one of the above graphs with some copies of $\overline{K_2}$. By $\mathcal{K}_{n,p}$, we mean the set of all connected graphs with n vertices and $e(K_n) - p$ edges. Note that a complete graph K_n has $e(K_n) = \frac{n(n-1)}{2}$ edges.

Theorem 2.6. *Let $G \in \mathcal{K}_{n,p}$ be a graph which is not a refinement of a star graph. Then the following statements hold.*

(1) *If $p \leq \frac{n}{2} + 1$ and n is even, then G is compact if and only if*

$$G \ncong P_4 \vee \underbrace{\overline{K_2} \vee \cdots \vee \overline{K_2}}_{\frac{(n-4)}{2} \text{ times}}.$$

(2) *If $p \leq \lceil \frac{n}{2} \rceil + 1$ and n is odd, then G is compact if and only if G is not isomorphic to the following graphs:*

$$G_1 \vee \underbrace{\overline{K_2} \vee \cdots \vee \overline{K_2}}_{\frac{(n-5)}{2} \text{ times}},$$

$$\overline{K_{1,2}} \vee P_4 \vee \underbrace{\overline{K_2} \vee \cdots \vee \overline{K_2}}_{\frac{n-7}{2} \text{ times}},$$

$$P_5 \vee \underbrace{\overline{K_2} \vee \cdots \vee \overline{K_2}}_{\frac{n-5}{2} \text{ times}},$$

where G_1 is the graph in Figure 1.

(3) *For every integer p with $\lceil \frac{n}{2} \rceil + 1 < p \leq \frac{(n-1)(n-2)}{2}$, there exists a graph in $\mathcal{K}_{n,p}$ which is not compact.*

Proof. Choose $G \in \mathcal{K}_{n,p}$ with $p \leq \lceil \frac{n}{2} \rceil + 1$. If $p \leq \lfloor \frac{n-1}{2} \rfloor$, then we show that G has a vertex which is adjacent to every other vertex of G. Suppose to the contrary, every vertex of G has degree at most $n-2$. Then we have $e(G) = \frac{n(n-1)}{2} - p \leq \frac{n(n-2)}{2}$ and so $p \geq \frac{n}{2}$, a contradiction. Hence G is a refinement of a star graph and by hypothesis, we can assume that $\lfloor \frac{n-1}{2} \rfloor < p \leq \lceil \frac{n}{2} \rceil + 1$. Thus it is clear that for even n, $p = \frac{n}{2}, \frac{n}{2} + 1$ and for odd n, $p = \frac{n+1}{2}, \frac{n+3}{2}$. Now, we follow the proof in the following cases:
Case 1. n is even and $p = \frac{n}{2}$. In order to G be not the refinement of a star graph, every vertex of G must have degree $n-2$, which implies that all of p edges in $\overline{G}$ must be single component edges. So, in this case, Corollary 2.4 implies that G is a compact graph.

Case 2. n is even and $p = \frac{n}{2} + 1$. In this case, G has one of the degree sequence either $\{n-3, n-3, n-2, \ldots, n-2\}$ or $\{n-4, n-2, \ldots, n-2\}$. If G has the second degree sequence, then every connected component of $\overline{G}$ is either $K_{1,2}$ or K_2, and so G is compact, by Corollary 2.4. Thus we can assume that G has the first degree sequence. Choose $u, v \in V(G)$ with $\deg(u) = \deg(v) = n-3$. If u and v are adjacent, then every connected component of $\overline{G}$ is either $K_{1,2}$ or K_2; therefore, G is compact, by Corollary 2.4. On the other hand, if u and v are nonadjacent vertices, one of the connected components of $\overline{G}$ is

$$P_4: \ x - u - v - y,$$

for some vertices x, y, and any other connected component of $\overline{G}$ is K_2. Thus

$$G \cong P_4 \vee \underbrace{\overline{K_2} \vee \cdots \vee \overline{K_2}}_{\frac{(n-4)}{2} \text{ times}},$$

which is not compact, by Lemma 2.3.

Case 3. n is odd and $p = \frac{n+1}{2}$. In this case, G can only have degree sequence $\{n-3,,n-2,\dots,n-2\}$. Thus every connected component of $\overline{G}$ is either $K_{1,2}$ or K_2; so G is compact.

Case 4. n is odd and $p = \frac{n+3}{2}$. Then G has one of the degree sequences $S_1 = \{n-5, n-2,\dots,n-2\}$, $S_2 = \{n-4, n-3, n-2,\dots,n-2\}$, or $S_3 = \{n-3, n-3, n-3, n-2,\dots,n-2\}$; so $\overline{G}$ has at least $p-6$ single component edges. In case that G has the degree sequence S_1, every connected component of $\overline{G}$ must be either $K_{1,4}$ or K_2, and so G is compact, by Corollary 2.4. Now, assume that one of S_2 and S_3 is the degree sequence of G.

Subcase 1. Let G has the degree sequence S_2 and choose $u,v \in V(G)$ with $\deg(u) = n-4$ and $\deg(v) = n-3$. If $u \in N(v)$, then one of the connected components of G is the noncompact graph H, pictured in Figure 2, for some vertices $x,y,z \in V(G)$ and any other connected component of G is K_2.

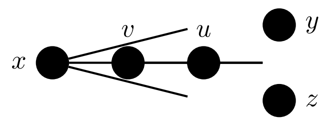

Fig. 2. The plane of graph H

So,

$$G \cong G_1 \vee \underbrace{\overline{K_2} \vee \cdots \vee \overline{K_2}}_{\frac{(n-5)}{2} \text{ times}}.$$

Now, assume that $u \in N(v)$. Then every connected component of $\overline{G}$ is $K_{1,3}$, $K_{1,2}$ or K_2 and hence G is compact, again by Corollary 2.4.

Subcase 2. Let G has the degree sequence S_3 and choose $u,v,w \in V(G)$ with $\deg(u) = \deg(v) = \deg(w) = n-3$. If $G[\{u,v,w\}] = K_3$, then every connected component of $\overline{G}$ is either $K_{1,2}$ or K_2; moreover, $G[\{u,v,w\}] = \overline{K_3}$ implies that every connected component of $\overline{G}$ is either K_3 or K_2, and so G is compact. Otherwise, we can assume that $G[\{u,v,w\}]$ is either $K_{1,2}$ or $K_2 \cup K_1$. If $G[\{u,v,w\}] = K_{1,2}$, then two connected components of $\overline{G}$ are $K_{1,2}$ and P_4 and all of the other components of $\overline{G}$ are single edges. Thus

$$G \cong \overline{K_{1,2}} \vee P_4 \vee \underbrace{\overline{K_2} \vee \cdots \vee \overline{K_2}}_{\frac{n-7}{2} \text{ times}}.$$

Finally, if $G[\{u,v,w\}] = K_1 \cup K_2$, then a similar argument shows that

$$G \cong P_5 \vee \underbrace{\overline{K_2} \vee \cdots \vee \overline{K_2}}_{\frac{n-5}{2} \text{ times}}.$$

The proofs of (1) and (2) follow from Cases 1 and 2. Next, we prove (3). The assumption $\lceil \frac{n}{2} \rceil + 1 < p \le \frac{(n-1)(n-2)}{2}$ implies that $n > 4$. Let G be a graph with the vertex set $\{v_1, v_2, \dots, v_n\}$ and $n-1$ edges $v_1 - v_3, v_1 - v_4, \dots, v_1 - v_{n-1}, v_2 - v_4, v_2 - v_n$. Add edges to G so that it has $e(K_n) - \max\{n, p\}$ edges, but the n edges $v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n - v_1$ are not in G. If $p < n$, then also add in the edges $v_{4+2j} - v_{5+2j}$, for $0 \leq j \leq n-p$. Then v_1, v_2 are nonadjacent and $\overline{N(v_1)} \cup \overline{N(v_2)} = \{v_3, v_4, \ldots, v_n\}$. From construction and for $i = 1, 2, 3$, we have $v_{i-1}, v_{i+1} \notin \overline{N(v_i)}$, and for $3 < i < n$, either $v_{i-1} \notin \overline{N(v_i)}$ or $v_{i+1} \notin \overline{N(v_i)}$. Thus $N(v_1) \cup N(v_2) \nsubseteq N(v_i)$, for every $1 \leq i \leq n$ and so, the constructed graph is not compact. Note that $v_4 \in N(v_1) \cap N(v_2)$. Therefore, the path $v_1 - v_4 - v_2$ connects v_1 and v_2. Since $V(G) = \overline{N(v_1)} \cup \overline{N(v_2)}$, we deduce that G is connected. Hence $G \in \mathcal{K}_{n,p}$ and the proof is complete. □

Remark 2.7. We note that if G is a refinement of a star graph with n vertices, then G is compact if and only if $G \cong K_r \vee H$, for some positive integer r and some compact graph H with $n - r$ vertices. So, G is compact if and only if the graph obtained from G after removing all vertices with degree $n - 1$ is compact.

3. DEGREES OF THE VERTICES IN COMPACT GRAPHS

The main aim of this section is to show that regular compact graphs are strongly regular. First, we need the following lemma.

Lemma 3.1. *Let G be a compact graph. Then every vertex of G has finite degree if and only if G is a finite graph.*

Proof. Suppose to the contrary, G is an infinite graph. Let x be a vertex of G. Then by hypothesis, there exists an infinite subset $\{y_i\}_{i\geq 1}$ of $V(G)$ such that x and y_i are not adjacent in G, for every $i \geq 1$. Since G is compact, for every $i \geq 1$, there exists a vertex z_i such that $N(x) \cup N(y_i) \subseteq N(z_i)$. We show that $\{z_i\}_{i\geq 1}$ is a finite set. Since G is connected (see Theorem 1.1), we can choose $a \in N(x)$. Thus $a \in N(z_i)$, for every $i \geq 1$. Now, from this fact that $\deg(a) < \infty$, we deduce that $\{z_i\}_{i\geq 1}$ is a finite set. Thus there exists a positive integer n such that $z_m = z_n$, for every $m \geq n$. Hence we have $\bigcup_{i=1}^{\infty} N(y_i) \subseteq N(z_n)$. From the finiteness of $\deg(z_n)$, we conclude that $|\bigcup_{i=1}^{\infty} N(y_i)| < \infty$. Therefore, there exists $t \in \bigcup_{i=1}^{\infty} N(y_i)$ and $k \geq 1$ such that t is adjacent to y_i, for every $i \geq k$, a contradiction. The converse is clear. □

An r-regular graph G is said to be *strongly regular* if there are integers λ and μ such that every two adjacent vertices have λ common neighbors and every two non-adjacent vertices have μ common neighbors. If G is a strongly regular graph with parameters r, λ, μ, then we write $G = srg(|V(G)|, r, \lambda, \mu)$.

Theorem 3.2. *Let G be an r-regular graph, for some positive integer r. If G is compact, then $|V(G)| \leq 2r$ and $G = \mathrm{srg}(|V(G)|, r, 2r - |V(G)|, r)$.*

Proof. Since $\deg(v) = r < \infty$, for every vertex of G, Lemma 3.1 implies that $|V(G)| = n$, for some positive integer n. If G is a complete graph, then there is nothing to prove. Thus we can assume that G is not complete. Now, we prove the assertion in the following steps:

Step 1. For every two non-adjacent vertices x and y, we show that $N(x) = N(y)$. Since G is compact, there exists a vertex, say z, such that $N(x) \subseteq N(z)$ and $N(y) \subseteq N(z)$.

So, the regularity implies that $N(x) = N(y) = N(z)$. Thus x and y have r common neighbors.

Step 2. For every two adjacent vertices x and y, we show that $|N(x) \cap N(y)| = 2r - n$. Let $\{y_1, \ldots, y_{n-r-1}\} \subseteq V(G)$ such that x and y_i are not adjacent, for every $1 \leq i \leq n-r-1$. Since G is compact, by Step 1, $N(x) = N(y_i)$, for every $1 \leq i \leq n-r-1$. Clearly, $y \in N(x) = N(y_i)$, for every $1 \leq i \leq n-r-1$. Thus $r = \deg(y) \geq n - r$ and so $r = \lceil \frac{n}{2} \rceil + k$, for some $k \geq 0$. Hence $k = r - \lceil \frac{n}{2} \rceil$. Now, one can easily check that

$$n = |V(G)| = \deg(x) + \deg(y) - |N(x) \cap N(y)|.$$

Therefore, we have $|N(x) \cap N(y)| = 2r - n$, as desired.

Moreover, since G is not a null graph, we should have $n \geq 2r$. □

Example 3.3. The Petersen graph (see [4, p. 9]) is a 3-regular graph with 10 vertices. By Theorem 3.2, the Petersen graph is not a compact graph.

4. CYCLES IN COMPACT GRAPHS

We start this section with the following proposition which shows that compact and bipartite graphs are complete bipartite.

Proposition 4.1. *Let G be a graph. Then G is a compact bipartite graph if and only if G is complete bipartite.*

Proof. Let G be a bipartite graph with parts X and Y. We show that G is complete bipartite. Suppose to the contrary, $x \in X$ and $y \in Y$ be two non-adjacent vertices of G. Since G is compact, there exists a vertex z such that $N(x) \subseteq N(z)$ and $N(y) \subseteq N(z)$. So, $z \in X \cap Y = \varnothing$, a contradiction. The converse is a special case of Corollary 2.5. □

From the previous proposition, we have the following immediate corollary.

Corollary 4.2. *Let G be a compact graph. Then G is a tree if and only if G is a star graph.*

Now we give a necessary and sufficient condition under which a complete r-partite graph ($r \geq 3$) is compact.

Definition 4.3 (see also [7, Definition 2.9])**.** Let r be an integer with $r \geq 2$. We call a complete r-partite graph, together with some end vertices, a complete r-partite graph with horns.

Theorem 4.4. *Let G be a complete r-partite graph $(r \geq 3)$ with horns. Then G is compact if and only if each part of the complete r-partite subgraph, which is connected to a horn, contains exactly one vertex.*

Proof. Let G be a complete r-partite graph with horns. Assume that $X_1, X_2, \ldots, X_r$ are parts of the complete r-partite subgraph of G such that for every i, $1 \leq i \leq k \leq r$, there exists $s_i \in X_i$ joined with a horn H_i. First suppose that $X_i = \{s_i\}$, for every

$1 \leq i \leq k$. Choose $x_i \in X_i$, for every $1 \leq i \leq n$. If x and y are two non-adjacent vertices of G, then we have:

$$N(x) \cup N(y) = \begin{cases} \bigcup_{i \neq j} X_i \subseteq N(x_j); & \text{if there exists } j \text{ such that } k+1 \leq j \leq n \\ & \text{and } x, y \in X_j, \\ \{s_i\} \subseteq N(x_i); & \text{if there exists } i \text{ such that } 1 \leq i \leq k \\ & \text{and } x, y \in H_i, \\ \{s_i, s_j\} \subseteq N(x_k); & \text{if } x \in H_i, y \in H_j \text{ and } i \neq j, \\ \bigcup_{k \neq j} X_k \subseteq N(x_j); & \text{if } x \in H_i, y \in X_j \text{ and } i \neq j. \end{cases}$$

Therefore, G is compact. Conversely, assume that G is compact. We show that $X_i = \{s_i\}$, for every $1 \leq i \leq k$. Suppose to the contrary and with no loss of generality that $X_1 \neq \{s_1\}$. Choose $x_1 \in X_1 \setminus \{s_1\}$ and $h_1 \in H_1$. Then x_1 and h_1 are not adjacent and we have $N(x_1) \cup N(h_1) = \bigcup_{k=1}^{r} X_k$. Since G is compact, there exists a vertex, say z, such that $\bigcup_{k=1}^{r} X_k \subseteq N(z)$, a contradiction. □

Lemma 4.5. *Let G be a compact graph. If $a - x - b$ is a path in G, then either $N(y) \subseteq N(x)$, for every $y \notin N(x)$ or $a - x - b$ is contained in a cycle of length ≤ 4.*

Proof. Let $a - x - b$ be a path in G. Then either $N(a) \cap N(b) = \{x\}$ or there exists a vertex $x \neq c \in N(a) \cap N(b)$. First suppose that the first case occurs. If y is a non-adjacent vertex with x, then since G is compact, there exists a vertex, say u, such that $N(x) \cup N(y) \subseteq N(z)$. Thus $z \in N(a) \cap N(b)$ and so $u = x$. This implies that $N(y) \subseteq N(y)$. Now, assume that the second case occurs. Then $a - x - b - c - a$ is a cycle of length ≤ 4. □

Proposition 4.6. *Assume G is a compact graph with at least three vertices and for any vertex x of G there exists a vertex $y \notin N(x)$ such that $N(y) \nsubseteq N(x)$. Then any edge in G is contained in a cycle of length ≤ 4, and therefore G is a union of triangles and squares.*

Proof. Let $a - x$ be an edge in G. Since G is connected and contains at least three vertices, there exists a vertex b in G with $a - x - b$ or $x - a - b$ paths in G. In either case, Lemma 4.5 implies that x is contained in a cycle of length ≤ 4, so $a - x$ is an edge of either a triangle or a rectangle. □

The following example shows that the compactness of G in the previous proposition is necessary.

Example 4.7. The following figure gives a graph whose every vertex belongs to a cycle, but where not every pair of vertices is contained in a cycle (see Figure 3).

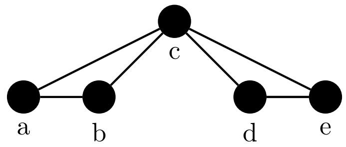

Fig. 3. The pair $\{b, d\}$ contained in no cycle

Clearly, this graph is not compact. Moreover, b and d are not adjacent vertices and $N(d) \nsubseteq N(b)$.

Theorem 4.8. *Let G be a compact graph with at least three vertices. If for every vertex x of G, there is a vertex y with $y \notin N(x)$ such that $N(y) \nsubseteq N(x)$, then every pair of vertices in G is contained in a cycle of length ≤ 6.*

Proof. Let a, b be vertices of G. If a —b is an edge in G , then a —b is the edge of a triangle or rectangle by Proposition 4.6. If $a - x$ —b is a path in G, for some vertex x, then a —x —b is contained in a cycle of length ≤ 4, by Lemma 4.5. If a —x —y —b is a path in G, for some vertices x, y, then by Lemma 4.5, we can find cycles a —x —y —c —a and b —y —x —d —b, where $c \neq x$ and $d \neq y$. This gives a cycle a —x —d —b —y —c —a of length ≤ 6. $\square$

Example 4.9. The bound six in Theorem 4.8 is best possible. Consider the graph as in Figure 4.

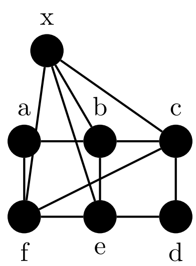

Fig. 4. The pair $\{a, d\}$ contained in no cycle with length less than 6

In this graph the vertices a and d are contained in a cycle of length 6 but not any cycle of shorter length.

5. SOME FURTHER RESULTS

In this section some further properties of compact graphs are given. The following result was proved in [7].

Theorem 5.1 ([7, Proposition 2.6])**.** *For a compact graph G, the following statements are equivalent:*

(1) *G contains no infinite cliques;*
(2) *The ascending chain condition holds for neighborhoods of G;*
(3) *$\omega(G) = n$ for some positive integer $n < \infty$.*

In addition, if one of the three conditions hold, then $\omega(G) = n$ is the number of mutually distinct maximal neighborhoods.

Now, we show that for a compact graph G if the ascending chain condition holds for the set of neighbors of G, then the descending chain condition holds.

Theorem 5.2. *Let G be a compact graph. If the ascending chain condition holds for the set of neighbors of G, then the descending chain condition holds for the set of neighbors of G.*

Proof. Assume to the contrary, there exists an infinite subset $\{x_1, x_2, \ldots\}$ of $V(G)$ such that $N(x_1) \supset N(x_2) \supset \cdots$. Then there exists $y_i \in N(x_i) \setminus N(x_{i+1})$, for every $i \geq 1$. Since G is compact, there exists a_i such that $N(x_{i+1}) \cup N(y_i) \subseteq N(a_i)$, for $i \geq 1$. Since $x_i \in N(y_i)$, we deduce that $a_i \in N(x_i) \setminus N(x_{i+1})$, for $i \geq 1$. Thus if $i > j \geq 1$, then we have $a_i \in N(x_j) \subseteq N(a_j)$. Thus the set $\{a_i\}_{i \geq 1}$ is an infinite clique in G and this contradicts Proposition 2.6 of [7]. □

The following example shows that the converse of Theorem 5.2 does not hold.

Example 5.3. Let $X = \{x_n |\, n \in \mathbb{N}\}$ be a set. Assume that G is a graph with the vertex set $V(G) = \mathbb{N} \cup X$ and with the edge set $E(G) = \{ij |\, i, j \in \mathbb{N}\} \cup \{ix_j |\, i, j \in \mathbb{N} \text{ and } i \geq j\}$. Then it is not hard to check that G is a compact graph and the descending chain condition holds for the set of neighbors of G; but however, $N(x_1) \subset N(x_2) \subset \ldots$ is a strict non-stationary chain of neighbors of G.

In the sequel, we show that the complement graph of a compact graph is not compact.

Theorem 5.4. *If G is a compact graph, then $\overline{G}$ is not compact.*

Proof. Suppose to the contrary, both of G and $\overline{G}$ are compact graphs. Thus G is not a complete graph and so there exist two non-adjacent vertices in G, say x and y. Therefore, there exists a vertex z such that $N_G(x) \cup N_G(y) \subseteq N_G(z)$. If $x \neq z$, then x and z are not adjacent in G and so they are adjacent in $\overline{G}$. Thus we have:

$$x \in N_{\overline{G}}(z) \subseteq N_{\overline{G}}(x) \cap N_{\overline{G}}(y),$$

which is impossible. Hence $x = z$. A similar proof shows that $y = z$. So $x = y$, a contradiction. □

The converse of the previous theorem does not hold in general. For instance if $G = P_4$, then $G = \overline{G}$ is not a compact graph.

We close this paper with the next result on disjunctive product of compact graphs. Let G and H be two graphs. The *disjunctive product* of G and H is a graph whose vertex set is $V(G) \times V(H)$ and two distinct vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1u_2 \in E(G)$ or $v_1v_2 \in E(H)$.

Theorem 5.5. *Let G and H be two graphs. Then the disjunctive product of G and H is a compact graph if and only if both of G and H are compact graphs.*

Proof. Let K be the disjunctive product of G and H. First assume that K is compact. If u_1 and u_2 are non-adjacent vertices of G, then (u_1, v) and (u_2, v) are two non-adjacent vertices of K, for every vertex v of H. Since K is compact, there exists a vertex, say (u_3, w) of K such that $N_K((u_1, v)) \cup N_K((u_2, v)) \subseteq N_K((u_3, w))$. Now, we show that $N_G(u_1) \cup N_G(u_2) \subseteq N_G(u_3)$. To see this, with no loss of generality, it is enough to show that $N_G(u_1) \subseteq N_G(u_3)$. Choose $x \in N_G(u_1)$. Then it is clear that $(x, v) \in N_K((u_1, v)) \subseteq N_K((u_3, w))$. Since w is not adjacent to v, we deduce that $x \in N_G(u_3)$. Hence G is compact. Similarly, it is proved that H is compact, too. Conversely, let (u_1, v_1) and (u_2, v_2) be two distinct non-adjacent vertices of K, then neither $u_1u_2 \in E(G)$ nor $v_1v_2 \in E(H)$. Since both of G and H are compact graphs, there are $u_3 \in V(G)$ and $v_3 \in V(H)$ such that $N_G(u_1) \cup N_G(u_2) \subseteq N_G(u_3)$ and $N_H(v_1) \cup N_H(v_2) \subseteq N_H(v_3)$. One can easily check that $N_K((u_1, v_1)) \cup N_K((u_2, v_2)) \subseteq N_K((u_3, v_3))$. So, the proof is complete. $\square$

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