Tolerance Modelling of Nonstationary Problems of Microheterogeneous Media and Structures

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Abstract

In this note a certain review of applications of a non-asymptotic modelling approach, called *the tolerance modelling,* is presented. Some objects and thermomechanical problems are shown, with a general outline of this method and an example of application for nonlinear vibrations of periodic beams.

Keywords: *microheterogeneous media and structures, tolerance modelling, non-stationary problems, nonlinear vibrations, microstructure effect*

1. Introduction

Microheterogeneous media and structures are more and more widely used in modern engineering. It is very important to combine different materials into one heterogeneous object, which material properties are outstanding when compared to 'classic' homogeneous materials. However, a proper model of such structures is needed, which can be used in design and optimization process.

The aim of this note is to show a scope of investigations carried out by the Research Team from Łódź. Presented problems were analysed by authors and their co-workers.

Objects under consideration are composites and structures with microheterogeneity. These media can be divided on three kinds:

- periodic media, which are composed from many identical small elements, called periodicity cells; their microstructure is regular and described by a diameter of the cell, called the microstructure parameter l; examples of them are shown in Fig. 1;

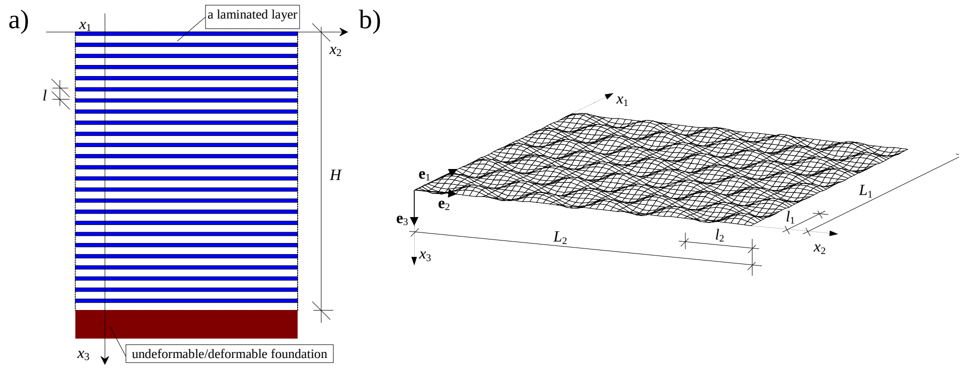

Figure 1. Examples of periodic media: a) a periodically stratified layer; b) a periodic wavy-type plate

- multiperiodic media, in which it is difficult to distinguish one, repetitive small periodicity element; however, they can have more than one period size along one axis of a coordinate system; in this case there are many different microstructure parameters along this axis; an example is presented in Fig. 2;

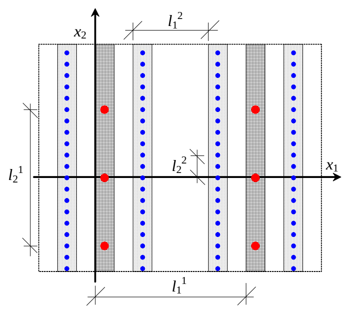

Figure 2. Example of multiperiodic medium

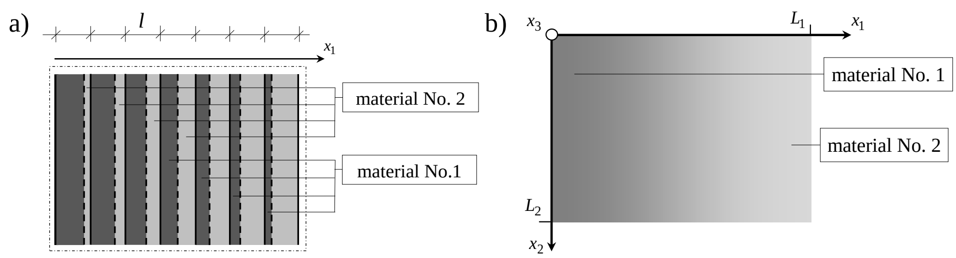

Figure 3. An example of functionally graded medium: a) on the micro-level; b) on the macro-level

- media with functionally graded properties, which macroproperties can be determined by slowly-varying, smooth functions, but their properties on the micro-level are described by highly-oscillating, non-continuous functions; hence, they have microstructure; examples of them can be found in Fig. 3.

Phenomena which take place in composites with periodic, multiperiodic or functionally graded structure are described by governing equations with functional, highly-oscillating, non-continuous coefficients. Hence, the direct exact description of these problems is rather very difficult. In order to analyse various thermomechanical problems of these composites different averaging methods are proposed and applied, which lead to differential equations with continuous, constant or slowly-varying coefficients. Fundamental references of them can be found in books by Woźniak and Wierzbicki [175], Woźniak, Michalak and Jędrysiak [173], Woźniak et al. [174]. Between many these approaches it is necessary to mention those, based on assumptions of the asymptotic homogenisation.

Unfortunately, methods used to describe periodic, multiperiodic or functionally graded structures lead usually to model equations, in which, so called, the effect of the microstructure size is neglected, i.e. the effect of the microstructure parameter l. In order to take into account this effect some averaged models, proposed by authors and their co-workers, are based on the tolerance modelling method, shown in [173-175]. Using concepts, assumptions and a proper modelling procedure, differential equations with non-continuous, highly-oscillating, functional coefficients can be replaced by systems of differential equations with constant coefficients (for periodic or multiperiodic composites and structures) or with continuous, smooth, slowly-varying coefficients (for functionally graded composites and structures with microheterogeneity).

A characteristic feature of the new equations is that they involve terms, which depend directly on the microstructure parameter l. Hence, governing equations of tolerance models describe the effect of the microstructure size on the overall behaviour of microheterogeneous media under consideration.

2. Investigated objects and thermomechanical problems

The tolerance modelling approach was applied to propose various tolerance models for different microheterogeneous structures.

There can be mentioned following problems of periodic composites and structures:

- wave propagation in lattice-type structures by: Cielecka [2], Cielecka et al. [6, 7];
- vibrations of lattice-type plates by: Cielecka [3], Cielecka and Jędrysiak [4, 5];
- vibrations of thin plates by: Jędrysiak and Woźniak [68], Jędrysiak [17-20, 22-24, 26, 28, 30, 47], Baron and Jędrysiak [1], Jędrysiak and Michalak [51], Jędrysiak and Wyrwa [72], Marczak and Jędrysiak [80];
- stability problems of thin plates by Jędrysiak [21, 24, 25];
- dynamical stability of thin plates by Jędrysiak [27, 29];
- non-linear vibrations of visco-elastic thin plates by Jędrysiak [42, 43];
- vibrations of wavy-type plates by: Michalak et al. [105], Michalak [87-92];
- stability of wavy-type plates by: Michalak [85, 86, 90, 93];

- vibrations of medium thickness plates by: Baron and Jędrysiak [1], Jędrysiak and Paś [54-56], Paś [119];
- dynamics of thin shells by: Tomczyk [136, 137, 141, 143, 145-151], Tomczyk and Litawska [152, 153], Tomczyk and Ślęzowski [157];
- stability of thin shells by: Tomczyk [138-140, 148];
- dynamical stability of thin shells by: Tomczyk [142, 144];
- dynamics of periodically laminated layers by: Jędrysiak and Juszczak [48];
- non-linear bending of thin plates by: Domagalski and Jędrysiak [11-13], Domagalski [8], Domagalski and Gajdzicki [10], Ma. Świątek and Domagalski [132];
- vibrations of periodic sandwich plates by: Jędrysiak and Zaleska [73], Marczak and Jędrysiak [81-84], Marczak [79];
- non-linear vibrations of beams by: Domagalski and Jędrysiak [14-16], Domagalski [9], Mi. Świątek and Domagalski [133];
- linear vibrations of beams by: Mi. Świątek et al. [134], Ma. Świątek et al. [135];
- vibrations of thin plates with uncertain distribution of properties by Jędrysiak and Ostrowski [53];
- heat transfer in periodic laminates with uncertain distribution of properties by Ostrowski and Jędrysiak [115].

For multiperiodic composites and structures it was investigated wave propagation in reinforced composite by Jędrysiak and Woźniak [69].

In the last years analyses of various thermomechanical problems for functionally graded composites and structures, using the tolerance modelling, were developed and presented. It can be mentioned:

- heat transfer in functionally graded composites by: Michalak and Woźniak [106], Jędrysiak and Radzikowska [61-65], Jędrysiak [32-34], Radzikowska and Jędrysiak [128, 129], Michalak et al. [107], Michalak [97], Ostrowski [108-112]; Ostrowski and Michalak [116, 117], Radzikowska [127], Radzikowska and Ostrowski [130], Radzikowska and Wirowski [131], Woźniak et al. [171];
- vibrations of functionally graded laminated plates by: Jędrysiak et al. [66];
- thermolelasticity problems of functionally graded composites by: Jędrysiak [36, 32], Jędrysiak and Pazera [57-60], Ostrowski [113, 114], Ostrowski and Radzikowska [118], Pazera and Jędrysiak [121, 122], Pazera [120];
- vibrations of thin plates with transversally graded structure and a thickness smaller than the size of the microstructure by: Jędrysiak [31, 32, 35], Wirowski et al. [168], Kaźmierczak and Jędrysiak [74-76], Kaźmierczak et al. [78], Jędrysiak and Kaźmierczak-Sobińska [49, 50];
- stability of thin plates with transversally graded structure by: Jędrysiak and Michalak [52], Kaźmierczak-Sobińska and Jędrysiak [77];
- vibrations of thin plates with longitudinally graded structure by: Michalak [95, 97], Michalak and Wirowski [102], Wirowski [160-167], Wirowski et al. [169];
- stability of thin plates with longitudinally graded structure by: Michalak [94, 96, 97], Perliński et al. [123];
- dynamics of microlayered functionally graded media by: Jędrysiak et al. [67], Jędrysiak and Pazera [59];

- dynamics of thin functionally graded shells by: Jędrysiak and Woźniak [70, 71], Jędrysiak [30, 37];
- dynamics of thin skeletonal functionally graded shells by: Michalak and Woźniak [103, 104], Woźniak and Michalak [172], Michalak [98];
- stability of thin skeletonal functionally graded shells by: Michalak [98];
- vibrations of thin transversally graded plates with a thickness of an order of the size of the microstructure by: Jędrysiak [38-41, 45];
- vibrations of medium thickness transversally graded plates with a thickness of an order of the size of the microstructure by: Jędrysiak [41, 44, 46];
- dynamics of thin-walled structures with dense system of ribs by: Michalak [99], Michalak and Rabenda [101], Rabenda [124], Rabenda and Michalak [125, 126], Wirowski et al. [170], Tomczyk and Woźniak [158];
- stablity of thin-walled structures with dense system of ribs by: Michalak and Ostrowski [100];
- dynamics of thin transversally graded shells by: Tomczyk and Szczerba [154-156].

In the next sections there are presented an outline of the tolerance modelling, an example of application to non-linear vibrations of periodic beams and a certain summary.

2. Outline of the tolerance modelling

2.1. *Introduction*

The tolerance modelling method (called also the tolerance averaging technique) was proposed and developed for periodic composites and structures by prof. Czesław Woźniak and His co-workers since the beginning of the 90s of the 20th century. The basic books, in which there are presented fundamental concepts and assumptions of this method, its modelling procedure and also various applications and some discussions and summaries of its development, are following:

- Woźniak C., Wierzbicki E. [175];
- Woźniak C., Michalak B., Jędrysiak J., [173];
- Woźniak C., et al., [174].

In general the whole modelling procedure with the use of the tolerance modelling method applies some introductory concepts, such as: an averaging operator, a slowly-varying function, a tolerance-periodic function or a highly-oscillating function. The idea standing behind those concepts, as well as a detailed description of this modelling approach, can be found in the above books.

Let $Ox_1x_2x_3$ be the orthogonal Cartesian co-ordinate system in the physical space and t be the time co-ordinate. Let subscripts α, β, …(i, j, …) run over 1, 2 (over 1, 2, 3) and indices $A, B,\ldots$ ($a, b,\ldots$) run over $1,\ldots,N$ $(1,\ldots,n)$. It is assumed that summation convention holds for all aforementioned indices. Let us introduce $\mathbf{x}\equiv(x_1,x_2,x_3)$. Let the region Ω be occupied by the undeformed microheterogeneous medium. Media under consideration are assumed to have a periodic (tolerance-periodic) structure with periods l_1, l_2, l_3, respectively, along the x_1-, x_2-, x_3-axis directions. By $\Delta\equiv[-l_1/2, l_1/2]\times[-l_2/2, l_2/2]\times[-l_3/2, l_3/2]$ *the basic cell* is denoted. The cell size is specified by a parameter l, which is a diameter of the cell and satisfies the condition $l \ll \min(L_\Omega)$, and l is called *the microstructure* *parameter*. Let $(\cdot)_{,i}\equiv\partial/\partial x_i$ denote the partial derivatives with respect to a space co-ordinate.

2.2. *Introductory concepts*

Following the above books some of the introductory concepts are reminded below.

Let cell at $\mathbf{x}\in\Omega_\Delta$ be denoted by $\Delta(\mathbf{x})=\mathbf{x}+\Delta$, where $\Omega_\Delta=\{\mathbf{x}\in\Omega\colon \Delta(\mathbf{x})\subset\Omega\}$. The averaging operator is the basic concept of the modelling technique, which is defined by

$$<\varphi>(\mathbf{x})=|\Delta|^{-1}\int_{\Delta(\mathbf{x})}\varphi(y_1,y_2)dy_1dy_2,\quad \mathbf{x}\in\Omega_\Delta,\quad \mathbf{y}\in\Delta(\mathbf{x}),\tag{1}$$

for any integrable function φ. For tolerance-periodic function φ of $\mathbf{x}$ the averaged value calculated from (1) is slowly-varying function in $\mathbf{x}$, but for periodic – is constant.

Denote by δ and X an arbitrary positive number and a linear normed space, respectively. Tolerance relation $\approx$ for a certain positive constant δ, called the tolerance parameter, is defined by

$$(\forall(\mathbf{x}_1,\mathbf{x}_2)\in \mathrm{X}^2)\quad [\mathbf{x}_1\approx\mathbf{x}_2 \Leftrightarrow \|\mathbf{x}_1-\mathbf{x}_2\|_{\mathrm{X}}\leq\delta].\tag{2}$$

Let $\partial^k\varphi$ be the kth gradient of function $\varphi=\varphi(\mathbf{x})$, $\mathbf{x}\in\Omega$, k=0, 1,…,α, $\alpha\geq 0$, and $\partial^0\varphi\equiv\varphi$. Let $\tilde{\varphi}^{(k)}=\tilde{\varphi}^{(k)}(\mathbf{x},\mathbf{y})$ be a function defined in $\overline{\Omega}\times\mathrm{R}^m$, and $\tilde{\varphi}\equiv\tilde{\varphi}^{(0)}$. Denote also $\Omega_{\mathbf{x}}\equiv\Omega\cap\bigcup_{\mathbf{z}\in\Delta(\mathbf{x})}\Delta(\mathbf{z})$, $\mathbf{x}\in\overline{\Omega}$.

Function $\varphi\in H^\alpha(\Omega)$ is the tolerance-periodic function, $\varphi\in TP^\alpha_\delta(\Omega,\Delta)$, (with respect to cell Δ and tolerance parameter δ), if for k=0, 1,…,α, it satisfies the following conditions

$$\begin{aligned}&(i)\quad (\forall\mathbf{x}\in\Omega)(\exists\tilde{\varphi}^{(k)}(\mathbf{x},\cdot)\in H^0(\Delta))[\|\partial^k\varphi|_{\Omega_{\mathbf{x}}}(\cdot)-\tilde{\varphi}^{(k)}(\mathbf{x},\cdot)\|_{H^0(\Omega_{\mathbf{x}})}\leq\delta],\\ &(ii)\quad \int_{\Delta(\cdot)}\tilde{\varphi}^{(k)}(\cdot,\mathbf{z})d\mathbf{z}\in C^0(\overline{\Omega});\end{aligned}\tag{3}$$

where $\tilde{\varphi}^{(k)}(\mathbf{x},\cdot)$ is the periodic approximation *of* $\partial^k\varphi$ *in* $\Delta(\mathbf{x})$, $\mathbf{x}\in\Omega$, k=0, 1,…,α.

Function $F\in H^\alpha(\Omega)$ is the slowly-varying function, $F\in SV^\alpha_\delta(\Omega,\Delta)$, if

$$\begin{aligned}&(i)\quad F\in TP^\alpha_\delta(\Omega,\Delta),\\ &(ii)\quad (\forall\mathbf{x}\in\Omega)[\tilde{F}^{(k)}(\mathbf{x},\cdot)|_{\Delta(\mathbf{x})}=\partial^k F(\mathbf{x}),\ k=0,\dots,\alpha].\end{aligned}\tag{4}$$

Function $\phi\in H^\alpha(\Omega)$ is the highly oscillating function, $\phi\in HO^\alpha_\delta(\Omega,\Delta)$, if

$$\begin{aligned}&(i)\quad \phi\in TP^\alpha_\delta(\Omega,\Delta),\\ &(ii)\quad (\forall\mathbf{x}\in\Omega)[\tilde{\phi}^{(k)}(\mathbf{x},\cdot)|_{\Delta(\mathbf{x})}=\partial^k\tilde{\phi}(\mathbf{x}),\ k=0,\dots,\alpha],\\ &(iii)\quad \forall F\in SV^\alpha_\delta(\Omega,\Delta)\quad \exists\varphi=\phi F\in TP^\alpha_\delta(\Omega,\Delta)\\ &\qquad\quad \tilde{\varphi}^{(k)}(\mathbf{x},\cdot)|_{\Delta(\mathbf{x})}=F(\mathbf{x})\partial^k\tilde{\phi}(\mathbf{x})|_{\Delta(\mathbf{x})},\ k=1,\dots,\alpha.\end{aligned}\tag{5}$$

Let us introduce a highly oscillating function $g(\cdot)$, $g\in HO^2_\delta(\Omega,\Delta)$, defined on $\overline{\Omega}$, being continuous together with gradient $\partial^1 g$ and with a piecewise continuous and bounded gradient $\partial^2 g$. Function $g(\cdot)$ is the fluctuation shape function of the 2nd kind, $FSF_\delta^2(\Omega,\Delta)$, if it depends on l as a parameter and holds the conditions:

$$\begin{array}{ll} (i) & \partial^k g \in O(l^{\alpha-k}) \text{ for } k=0,1,\ldots,\alpha,\ \alpha=2,\ \partial^0 g \equiv g, \\ (ii) & <g>(\mathbf{x}) \approx 0 \ \ \forall \mathbf{x} \in \Omega_\Delta, \end{array} \tag{6}$$

where l is the microstructure parameter. Condition (6 *ii*) can be replaced by $<\mu g>(\mathbf{x}) \approx 0$ for every $\mathbf{x}\in\Omega_\Delta$, where $\mu > 0$ is a certain tolerance-periodic function.

2.3. *Fundamental assumptions*

Using the above introductory concepts some fundamental modelling assumptions can be formulated, cf. [175, 173-174]. Let $w(\mathbf{x},t)$ be a basic unknown field in the problem under consideration, $\mathbf{x}\in\Omega$, $t\in(t_0,t_1)$.

The basic modelling assumption is so called the Conformability Assumption. It is assumed that the basic unknown field $w(\cdot,t)$ has to be conformable to a microstructure of the considered medium, i.e. it is a tolerance-periodic function

$$w(\cdot,t) \in TP_\delta^\alpha(\Omega,\Delta),$$

for any time t. This condition may be violated only near the boundary of the medium.

The next assumption is the Micro-Macro Decomposition, which stands, that the basic unknown field can be formulated as sum of macrofield $U(\boldsymbol{x},t)$ and products of fluctuation shape functions $h^A(\boldsymbol{x})$ and fluctuation amplitudes $V^A(\boldsymbol{x},t)$:

$$\begin{array}{l} w(\mathbf{x},t) = U(\mathbf{x},t) + h^A(\mathbf{x})V^A(\mathbf{x},t), \qquad \mathbf{x}\in\overline{\Omega}, \quad t\in(t_0,t_1), \\ U(\cdot,t), V^A(\cdot,t) \in SV_\delta^\alpha(\Omega,\Delta). \end{array} \tag{7}$$

Macrofield $U(\boldsymbol{x},t)$ and fluctuation amplitudes $V^A(\boldsymbol{x},t)$ are new basic unknowns, additionally assumed to be slowly-varying functions for every t.

The third assumption is the Tolerance Averaging Approximations. In the modelling procedure terms of an order of $O(\delta)$ can be treated as negligibly small in the following relations:

$$\begin{array}{ll} <f>(\mathbf{x}) = <\bar{f}>(\mathbf{x}) + O(\delta), & \mathbf{x}\in\Omega \\ <f\psi>(\mathbf{x}) = <f>(\mathbf{x})\psi(\mathbf{x}) + O(\delta) & \\ <f\partial(h\psi)>(\mathbf{x}) = <f\partial h>(\mathbf{x})\psi(\mathbf{x}) + O(\delta), & \forall \mathbf{x}\in\Omega \end{array} \tag{8}$$

where $f \in TP_\delta^\alpha(\Omega,\Delta)$ is tolerance-periodic function, $\bar{f}$ is a periodic approximation of f, $\psi \in SV_\delta^\alpha(\Omega,\Delta)$ is slowly-varying function, $h \in HO_\delta^\mu(\Omega,\Delta)$ is highly-oscillating function, $0 < \delta << 1$.

2.4. *Outline of the modelling procedure*

Here, it is presented only an outline of the tolerance modelling procedure, which is detaily shown in the book [174].

The starting point of the modelling is formulation of the action functional for the thermomechanical problem of the microstructured medium under consideration:

$$\Delta\,(w(\cdot)) = \int_{\Omega}\int_{t_0}^{t_1} \Lambda(\cdot)\, dtd\mathbf{y}, \tag{9}$$

where the lagrangian $\Lambda(\cdot)$ is defined by the unknown fields w and their derivatives in this problem:

$$\Lambda(\cdot) = \Lambda(\mathbf{y}, w(\mathbf{y},t), \dot{w}(\mathbf{y},t), \partial w(\mathbf{y},t)). \tag{10}$$

This lagrangian (10) is a tolerance-periodic function. Using to (10) the principle of stationary there can be derived the known Euler-Lagrange equations of the microstructured medium under consideration, which have periodic or tolerance-periodic, highly-oscillating, non-continuous coefficients.

However, introducing micro-macro decomposition (7) into lagrangian (10), averaging by (1) and using tolerance averaging approximations (8) the tolerance averaged action functional can be obtained

$$\Delta_h(U(\cdot),V^A(\cdot)) = \int_{\Omega}\int_{t_0}^{t_1} < \Lambda_h > (\cdot)dtd\mathbf{y}, \tag{11}$$

with the tolerance averaged lagrangian $< \Lambda_h > (\cdot)$ defined by the new basic unknown fields U, V^A and their derivatives:

$$\Delta_h(U(\cdot),V^A(\cdot)) = \int_{\Omega}\int_{t_0}^{t_1} < \Lambda_h > (\mathbf{y},\partial U,\dot{U},\dot{V}^A,U,V^A)dtd\mathbf{y}. \tag{12}$$

This lagrangian is slowly-varying function. Using the principle of stationary to (12) there can be derived the new averaged Euler-Lagrange equations of the microstructured medium under consideration, which have constant or slowly-varying, smooth coefficients.

3. Example: non-linear vibrations of periodic beams

3.1. *Introduction*

As an example a linearly elastic prismatic beam is considered. Let *Oxyz* be an orthogonal Cartesian coordinate system, the *Ox* be the axis of the beam, the cross section of the beam be symmetric with respect to the plane *Oxz*, the load act in the direction of the axis *Oz*. The problem can be treated as one-dimensional, so that the region occupied by the beam is defined as $\Omega\equiv[0, L]$, L stands for the beam length. Let $\partial^k = \partial^k/\partial x^k$ be the k-th derivative of a function with respect to the x coordinate, overdot stands for the derivative with respect to time.

The periodic beam is assumed to be made of many repetitive small elements – periodicity cells, $\Delta\equiv[-l/2,l/2]$, where $l<<L$ is the length of the cell, called the microstructure parameter.

Considerations are based on the Rayleigh theory of beams with von Kármán type nonlinearity, with neglected effect of axial inertia. Let $w = w(x,t)$ be the transverse deflection, $u_0 = u_0(x,t)$ the longitudinal displacement, $EA = E(x)A(x)$ and $E(x)J = EJ(x)$ tensile and flexural stiffness, $\mu = \mu(x)$ and $\vartheta = \vartheta(x)$ mass and rotational moment of inertia per unit length and $q = q(x,t)$ – the transverse load.

Hence, non-linear vibrations of these beams can be described by the system of nonlinear differential equations for the longitudinal displacements u_0 and the transverse deflection w:

$$\begin{aligned}&\partial[EA(\partial u_0+\tfrac{1}{2}\partial w\partial w)]=0,\\&\mu\ddot{w}-\partial(\vartheta\partial\ddot{w})+c\dot{w}+kw+\partial^2(EJ\partial^2 w)-\partial[EA(\partial u_0+\tfrac{1}{2}\partial w\partial w)\partial w]=q.\end{aligned}\tag{13}$$

The coefficients EA, EJ, μ, ϑ are highly-oscillating, periodic, often non-continuous functions of the x-coordinate. Since equations (13) are not a good tool to analyse vibrations, an approximately equivalent tolerance model is proposed, which describes this problem taking into account the effect of the microstructure size.

3.2. *Tolerance model equations*

Using the introductory concepts and modelling assumptions, e.g. the micro-macro decomposition (5) in the form adopted for basic unknown fields of the beam – u_0, w; substituting these decompositions and averaging resulting equations, after some manipulations we arrive to averaged equations of the tolerance model. Denoting constant coefficients related to the beam properties (h^A, g^K – fluctuation shape functions):

$$\begin{array}{lll}\langle EA\rangle\equiv B, & \langle EA\partial h^A\rangle\equiv lB^A, & \langle\mu\rangle\equiv M,\\ \langle EA\partial g^K\rangle\equiv B^K, & \langle EA\partial h^A\partial g^K\rangle\equiv lB^{AK}, & \langle\mu h^A\rangle\equiv l^2M^A,\\ \langle EA\partial g^K\partial g^L\rangle\equiv B^{KL}, & \langle EA\partial h^A\partial h^B\partial g^K\rangle\equiv l^2B^{ABK}, & \langle\mu h^A h^B\rangle\equiv l^4M^{AB},\\ \langle EJ\rangle\equiv D, & \langle EA\partial h^A\partial h^B\rangle\equiv l^2B^{AB}, & \langle\vartheta\rangle\equiv J,\\ \langle EJ\partial^2 h^A\rangle\equiv D^A, & \langle EA\partial h^A\partial h^B\partial h^C\rangle\equiv l^3B^{ABC}, & \langle\vartheta\partial h^A\rangle\equiv lJ^A,\\ \langle EJ\partial^2 h^B\partial^2 h^A\rangle\equiv D^{AB}, & \langle EA\partial h^A\partial h^B\partial h^C\partial h^D\rangle\equiv l^4B^{ABCD}, & \langle\vartheta\partial h^A\partial h^B\rangle\equiv l^2J^{AB};\end{array}\tag{14}$$

and to the transverse load:

$$\langle q\rangle\equiv Q,\qquad\qquad \langle qh^A\rangle\equiv l^2Q^A,\tag{15}$$

the tolerance model equations take the form, cf. [16]:

$$
\begin{aligned}
&\partial\left(B\partial U + B^{K}T^{K}\right) + \partial\left(\tfrac{1}{2}B\partial W\partial W + lB^{A}V^{A}\partial W + \tfrac{1}{2}l^{2}B^{AB}V^{B}V^{A}\right) = 0,\\
&D\partial^{4}W + D^{A}\partial^{2}V^{A} + M\ddot{W} - J\partial^{2}\ddot{W} + l^{2}M^{A}\ddot{V}^{A} - lJ^{A}\partial\ddot{V}^{A} - Q +\\
&\quad -\partial\left[\left(B\left(\partial U + \tfrac{1}{2}\partial W\partial W\right) + B^{K}T^{K} + lB^{A}V^{A}\partial W + \tfrac{1}{2}l^{2}B^{AB}V^{B}V^{A}\right)\partial W\right] +\\
&\quad -\partial\left[lB^{AK}V^{A}T^{K} + \left(\partial U + \tfrac{1}{2}\partial W\partial W\right)lB^{A}V^{A} +\right.\\
&\quad \left. + l^{2}B^{AB}\partial W V^{B}V^{A} + \tfrac{1}{2}l^{3}B^{ABC}V^{A}V^{B}V^{C}\right] = 0,\\
&\left(D^{AB}\right)V^{B} + l^{2}\left(l^{2}M^{AB} + J^{AB}\right)\ddot{V}^{B} + D^{A}\partial^{2}W + l\left(lM^{A}\ddot{W} + J^{A}\partial\ddot{W}\right) - l^{2}Q^{A} +\\
&\quad + l\left(B^{A}\left(\partial U + \tfrac{1}{2}\partial W\partial W\right) + B^{AK}T^{K}\right)\partial W + l^{2}\left(B^{AB}\partial U + \tfrac{3}{2}B^{AB}\partial W\partial W +\right.\\
&\quad \left. + B^{ABK}T^{K} + \tfrac{3}{2}lB^{ABC}V^{C}\partial W + \tfrac{1}{2}l^{2}B^{ABCD}V^{C}V^{D}\right)V^{B} = 0,\\
&B^{KL}T^{L} + B^{K}\partial U + \tfrac{1}{2}B^{K}\partial W\partial W + lB^{AK}V^{A}\partial W + \tfrac{1}{2}l^{2}B^{ABK}V^{B}V^{A} = 0,
\end{aligned}
\tag{16}
$$

for the macrodisplacements $U(\cdot)$, $W(\cdot)$ and for the fluctuation amplitudes of the axial displacement $T^{K}(\cdot)$ and of the deflection $V^{A}(\cdot)$.

3.3. *Computations results – free linear vibrations*

Let us consider a hinged-hinged beam, which has constant cross section and is provided by a system of periodically distributed system of concentrated masses M_1, M_2 with rotational inertia I_1, I_2, as it is shown in Fig. 4.

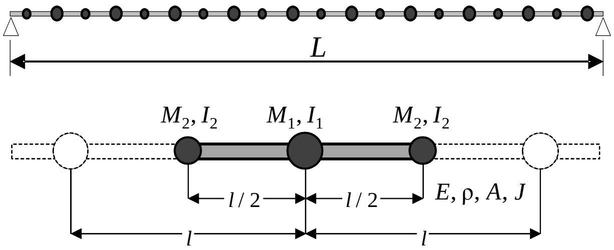

Figure 4. The considered beam and its fragment, cf. [16]

The mass distribution in a periodicity cell is given by

$$
\begin{aligned}
\mu(y) &= \mu_0 + M_1\delta(y) + M_2\delta(y + l/2),\\
\vartheta(y) &= \vartheta_0 + I_1\delta(y) + I_2\delta(y + l/2), \quad y \in \Delta(x).
\end{aligned}
\tag{17}
$$

The solutions of the tolerance model equations can be assumed in the form of truncated Fourier series

$$
\begin{Bmatrix} W(x,t) \\ V^{A}(x,t) \end{Bmatrix} = \sum_{m=1}^{n} \begin{Bmatrix} w_m(t) \\ v_m^{A}(t) \end{Bmatrix} X_m(x).
\tag{18}
$$

For the hinged-hinged boundary conditions the linear natural vibration modes are:

$$
X_m(x) = \sin \xi_m x, \quad \xi_m = m\pi / L.
\tag{19}
$$

The length of the beam is L = 1.0 m, the Young modulus E = 205 GPa, the mass density ρ = 7850 kg/m^3. The other, dimensionless parameters are as follows:

$$\frac{b}{h}=5,\quad \frac{h}{l}=0.1,\quad \frac{l}{L}=0.1,\quad \frac{M_1}{\rho Al}=5.1,\quad \frac{M_2}{M_1}=0.5,\quad \sqrt{\frac{I_1}{M_1 l^2}}=10,\quad \frac{I_2}{I_1}=0.5,\quad q=0. \tag{20}$$

Let us restrict considerations only to the first two (m = 2) terms of the Fourier series (18) and two FSFs (N = 2), so that the model has $m \times (1 + N) = 6$ degrees of freedom.

Table 1. Comparison of the linear eigenfrequencies of considered beam, cf. [16]

mode	finite element	tolerance model			
		approximate *FSF*s		FE-based *FSF*s	
	ω_{FE} [rad/s]	ω_{TA} [rad/s]	Δ [%]	ω_{TA} [rad/s]	Δ [%]
1 w_1	15.861	15.876	0.094	15.872	0.071
2. w_2	32.884	33.050	0.505	33.041	0.477
3. $v^2{}_2$	223.491	239.454	7.143	212.200	5.052
4. $v^2{}_1$	224.244	246.109	9.751	224.088	0.069
5. $v^1{}_2$	14855.007	15129.460	1.848	15034.591	1.209
6. $v^1{}_1$	14989.493	15129.460	0.934	15034.591	0.301

A finite element method procedure for beam dynamics analysis was prepared in Maple to validate the results. The Rayleigh beam elements with Hermitian polynomials and consistent mass matrix were used.

Results of comparative analysis of the calculations obtained for the finite element model (40 elements) and the tolerance model, using the approximate (trigonometric) and refined (finite element based) fluctuation shape functions (*FSF*), are shown in Table 1 (linear eigenfrequencies) and Fig. 5 (linear eigenvectors).

a)

b)

c)

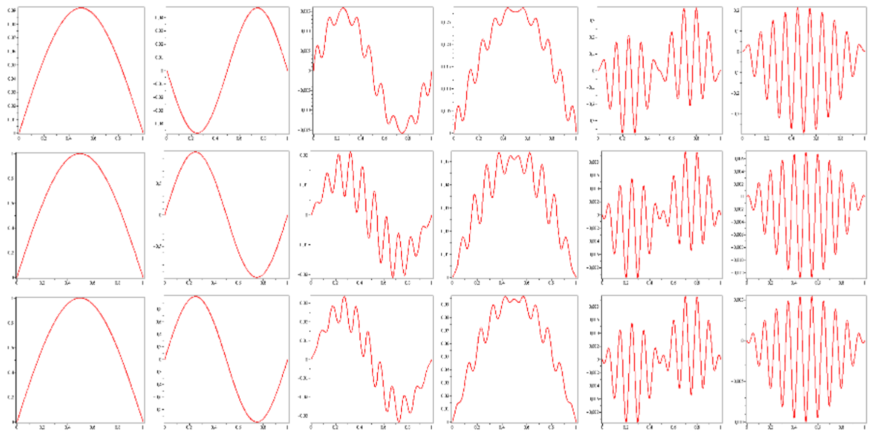

Figure 5. Comparison of the linear eigenforms of considered beam:
a) finite element model, b) tolerance model with trigonometric FSFs,
c) tolerance model with FE-based FSFs, cf. [16]

4. Remarks

In this note the tolerance modelling method is shown, which is applied to description various thermomechanical problems of different microheterogeneous media and structures by the Research Team from Łódź.

This modelling approach allows to replace differential governing equations of the problem, which have highly-oscillating, non-continuous functional coefficients (periodic or tolerance-periodic) by averaged differential governing equations with constant or slowly-varying coefficients.

The following general remarks can be formulated.

- The tolerance modelling method is applied to obtain averaged mathematical models of different thermomechanical problems of periodic or non-periodic microheterogeneous composites and structures.
- Tolerance models allow to analyse various problems, mainly non-stationary, of thermomechanics of solids and structures with taking into account the effect of the microstructure.
- The effect of the microstructure on the overall behaviour of investigated medium can be manifested for instance by:
  - higher order vibrations (e.g. higher frequencies),
  - form of solutions, in which some fluctuations related to heterogeneity can be occurred.

Results of the example can lead to the following remarks:

- The proposed tolerance model makes it possible to investigate the effect of the microstructure size on dynamic problems of periodic beams under consideration, e.g. the "higher order" vibrations related to the beam microstructure.
- The governing equations of *the tolerance model* have a physical sense for unknowns W, U, V^A, $A = 1,...,N$, T^K, $K = 1,...,M$, being slowly-varying functions.

Problems developed but still open there are:

- Geometrically non-linear problems (e.g. large plate deflections).
- Physically non-linear problems (e.g. dependency of heat conduction properties or elastic/inertial properties on temperature).
- Optimisation of distribution of properties in composites and non-periodic structures.
- More detailed analysis of the solutions to the cell problem.

Acknowledgement

This note is partly supported by the National Science Centre, Poland (Grant No. 2014/15/B/ST8/03155).

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