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Free flexural vibration of a sandwich beam on an elastic foundation with variable properties

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Free flexural vibration of a simply supported sandwich beam on an elastic foundation is the main purpose of the presented investigation. An analytical model of multi-layered beam on elastic foundation has been prepared. The authors submitted an original beam-foundation interaction model which based on variable parameters of the foundation and their influence on the beam response. This explanation leads to the possibility of continuous characterization of the beam-foundation interplay. A nonlinear mathematical function for symmetrical properties of the foundation has been adopted. The frequency equation as a function of geometric and mechanical properties of the beam and the parameters of the elastic foundation was derived using the Galerkin method. The analytical investigation has been divided into two parts: the analysis of elastic foundation with constant and variable properties. The unconventional shape function and the function of deflection have been introduced and employed. Moreover, the finite element analysis has been performed. Sample analytical and numerical calculations have been performed, demonstrating a good concurrence between both models. The difference between analytical and numerical values of the fundamental natural frequency did not exceed 0.5%.

Key words: free vibration, elastic foundation, sandwich beam, frequency, Galerkin method.

1. Introduction

Construction-foundation interaction problems are substantial aspects of strength analysis. This is particularly essential in case of roads, railroads, and runways which are loaded not only with static but also dynamic forces. Therefore, the dynamic analysis of constructions on an elastic foundation is a crucial aspect of research. The examples of beams on an elastic foundation are as follows: strip footings, railroad rails, bridges (bridge deck-girder structures resting on elastomeric bearings, bridge abutments, piers), circular concrete tanks, rigid concrete pavements, layered pavements, open foundations for buildings, laterally loaded piles, aqueducts, excavation retaining walls, base slabs of conventional retaining walls, cross support beams, and tunnels in soil foundation. Structures resting on elastic foundations with variable properties can be used for elements placed on variable ground surfaces (foundations for motorways, airports, railroads, sports fields, parking lots, storage capacities, dams, and embankments).

The Winkler model is the simplest description of an elastic foundation behaviour. In this model, it is granted that the displacement of any point on the surface is independent of the displacements of other points. Moreover, the influence of the foundation at a selected point on the surface is proportional to the displacement. Developments of new constructions led to the expansions of new models that allow to determine more detailed influence of the foundation on various constructions: two-parameter (e.g. Filonenko–Borodich, Vlasov–Leontiev, and Pasternak models) and three-parameter models (e.g. Heteneyi [1], Kerr [2], and Reissner models). In addition, the following models can be found in the literature: the elastic-plastic Rhines [3] model and bilinear model [4]. In case of the elastic continuous models, the following can be distinguished: isotropic and anisotropic models, a non-homogeneous model as well as a layered model. The models of the foundation can also be divided due to the physical criterion. According to this assumption, ones can differentiate: linear-elastic, nonlinear-elastic, viscoelastic, elastic-plastic as well as viscoelastic-plastic models [5].

Sandwich beams are the examples of structures resting on elastic foundation. They should be designed in such a way to meet the basic structural criteria (favorable weight to the transferred load ratio and high stiffness). The faces should be thick enough to resist tensile, compressive, and shear stresses. The core, on the other hand, should be characterized by a high strength in order to withstand the shear stresses caused by loads and be thick enough to prevent buckling of the structure. Moreover, the values of the adhesive forces should be sufficient to transfer the shear stresses between the core and the faces.

Research related to the dynamic response of sandwich beams is an essential issue during investigation of constructions. Free vibration is characterized as the vibration of the damped or undamped system of masses with a motion completely influenced by their potential energy. It occurs when there is no externally applied vibration forcing. The free vibration solution is usually approximately sinusoidal. The beam vibration can be defined as the amount and direction of movement that a beam exhibits away from the point of the applied load or support. Vibration factors include type of the material, length of the beam, the value of load applied into the construction or the properties of elastic foundation.

Free vibration of viscoelastic functionally graded sandwich shells with tunable auxetic core has been analyzed by Li and Liu [6]. The equations of motion were derived based on Hamilton's principle. The influence of geometry, temperature, and core parameters were taken into the consideration. It was revealed that the type of shells (cylindrical, spherical, saddle) has a direct influence on the value of natural frequency. Furthermore, the value of frequency decreases with the increase of the viscoelastic loss factor and increases with the increase of the gradient index. Free linear vibration of three-layered sandwich beams with a magnetorheological fluid core has been investigated by Soroor *et al.* [7]. Euler–Bernoulli and Timoshenko theories were adopted to model the faces and the core respectively. The analysis allowed to formulate the following conclusions: the increase of the magnetic flux density increases the value of natural frequency of the beam, the increase of the core thickness decreases the natural frequencies. Moreover, the increase of the gradient index entails the increase of natural frequencies. Free vibration of bidirectional functionally graded sandwich beams has been considered by Le *et al.* [8]. The beam included three layers: an axially functionally graded core and two faces. The properties of the faces varied in thickness and length directions. Disparate boundary conditions were taken into account during the numerical analysis. It was acknowledged that the values of natural frequencies computed based on the Voigt model are higher than those obtained based on the Mori–Tanaka model. Free vibration analysis of functionally graded sandwich curved beams has been carried out by Sayyad and Avhad [9]. The beam being tested was made of functionally graded faces and a homogeneous core. A fifth-order curved beam theory was introduced into the analysis (the effects of transverse shear and normal deformations were included). It was affirmed that the value of non-dimensional fundamental frequency increases with the decrease of the radius of curvature. It was also concluded that the frequency decreases with the increase of the power-law index. Natural frequencies for out-of-plane free vibration of three-layered symmetric sandwich beams have been studied by Gholami *et al.* [10]. The dynamic stiffness matrix has been introduced and employed into the analysis. The exact frequencies and the corresponding vibration modes were attained. Free vibration analysis of symmetric and unsymmetric sandwich beams has been performed by Garg *et al.* [11]. The faces were adopted to be made of functionally graded carbon nanotubes reinforced material and the core was made of balsa wood. The authors conceded that the core thickness and carbon nanotube graduation law play an important role in a mechanical response of beams. Free vibration of sandwich beams with a soft core has been described by Khdeir and Aldraihem [12]. The zig-zag theory was implemented into the analysis. The authors compared the obtained results with those available in the literature (experimental, analytical, and numerical studies). It was stated that the zig-zag theory can be used for predicting the natural frequencies in sandwich beams with a soft core. Free vibration analysis (experimental and numerical) of a sandwich beam with a corrugated core has been formulated by

Xu *et al.* [13]. The graded lattice core was adopted as a continuum solid. The effects of graded parameters, face thickness, core height, and beam length were under consideration. It was concluded that the value of natural frequency can be modified by alternating the graded parameter. Moreover, natural frequencies of the beam increase with the increase of the face thickness and the core height. Nonlinear free vibration analysis of a shear deformable sandwich porous beam has been conducted by Chen *et al.* [14]. The beam was constructed of two faces and a functionally graded core with internal pores. The effects of the porosity coefficient as well as slenderness and thickness ratios were studied to designate the response of the beam. In addition, the influence of transverse shear deformation and rotary inertia were incorporated, based on the Timoshenko beam theory. The analysis allowed to formulate the following conclusions: the nonlinear frequency ratio increases with the increase of the vibration amplitude, the nonlinear frequency ratios decrease with varying amplitudes when the porosity coefficient increases, the linear fundamental frequency is strictly associated with the slenderness ratio [14]. Free vibration analysis of a debonded curved sandwich beam has been presented by Sadeghpour *et al.* [15]. Radial and circumferential rigidities of the core were considered. The displacement in the core was assumed in a form of a quadratic polynomial distribution. The high order theory was introduced and employed by the authors. The analysis revealed that the angle of curvature and boundary conditions are major factors affecting the dynamic response of the beam.

If the reaction force provided by the continuous support is assumed as a function of the displacement of the construction, the support is called the elastic one. A beam resting on an elastic support is called a beam on an elastic foundation. The models of elastic foundations may be considered as an elastic layer of an infinite extent resting on a rigid base and consisted of an infinite sequence of elastic columns. Research related to vibration of beams on an elastic foundation has been conducted by a varying number of authors. Natural vibration of a sandwich beam on the elastic Winkler foundation has been described by Kubenko *et al.* [16]. The authors applied the Bernoulli hypotheses in their work to examine the kinematics of faces. Analytical and numerical analyses were introduced and employed. The analysis revealed that the elastic foundations of medium and high stiffness significantly affect the values of natural frequencies of a sandwich beam. Free vibration of a three-layered sandwich beam has been investigated by Banerjee *et al.* [17]. The material of the face and the core were assumed to be homogeneous and isotropic. In addition, the cross-section of the construction was asymmetric. The dynamic stiffness matrix was developed by the authors. The values of natural frequencies and mode shapes were prepared. Thermomechanical vibration analysis of functionally graded (FG) beams and functionally graded sandwich beams (FGSW) resting on disparate elastic foundations has been studied by Pradhan and Murmu [18]. The Winkler and two-parameter elastic foundations have been taken into the consideration. The evaluation of results acknowledged a good agreement with the values reported in the literature. It was stated for the Winkler and two-parameter foundations that the frequency of beams decreases with the temperature increase. Moreover, the effect of Winkler foundation stiffness on the frequency is higher, compared with two-parameter foundations. Dynamic stability of a functionally graded sandwich beam resting on the Winkler foundation has been considered by Mohanty *et al.* [19]. The finite element method has been used to solve the problem. The beam, hinged at both ends, was subjected to a dynamic axial load. The analysis affirmed that the frequencies and stability of beam increase with the increase of Winkler's modulus. In addition, the stability of beam is diminished for higher values of core thickness. Dynamic stability of smart sandwich beams resting on the Winkler foundation and subjected to harmonic axial loads has been analyzed by Tabassian and Rezaeepazhand [20]. The electro-rheological core was a part of beams. Numerical methods were employed to evaluate the critical dynamic loads. Beam geometry, elastic foundation stiffness, static loads, voltage, and core properties have been taken into the consideration to discuss the behaviour of beams. It was concluded that the occurrence of the electric field in the core increases the values of dynamic critical loads. Furthermore, the increase of critical loads values is also linked to the presence of the elastic foundation. Free vibration analysis of functionally graded sandwich beams with a variable cross-section and resting on the variable Winkler elastic foundation has been conducted by Demir *et al.* [21]. The width altered exponentially along the beam length. The influence of material, geometry, elastic foundation parameters, and the slenderness ratio were taken into account. Analytical and numerical models of the beam were performed. It was conceded that the increase in volume fractions of ceramic leads to the decrease in natural frequencies of the beam. In addition, the frequencies increase with the increase of the elastic foundation index. Moreover, natural frequencies decrease with the increase of the slenderness ratio. Free vibration analysis of a non-symmetric functionally graded sandwich square plate resting on the Winkler–Pasternak foundation has been carried out by Saidi *et al.* [22]. Hamilton's principle has been adopted to derive the equations of equilibrium and boundary conditions. The authors assumed that the transverse shear displacement varied sinusoidal across the thickness. Numerical values were obtained and compared with those available in the literature. The results depict that the values of natural frequencies increase as the ratio of side to plate thickness increases. Moreover, the frequencies increase with the increase of elastic foundation parameters. Free vibration of a three-layered asymmetric sandwich beam resting on the variable Pasternak foundation has been described by Pradhan *et al.* [23]. The beam was subjected to a pulsating axial load. Disparate boundary conditions as well as the parameters of the beam and the elastic foundation were taken into the consideration by the authors. The elastic foundation stiffness was assumed to have parabolic variation of the form. The results indicate that the values of natural frequencies of the beam increase with the increase of elastic foundation parameters. Flexural vibration of functionally graded sandwich plates resting on a two-parameter elastic foundation has been investigated by Tossapanon and Wattanasakulpong [24]. The Chebyshev collocation method has been implemented in this work to acquire the results with several parametric studies. Various factors were taken into account. Good agreement between the obtained numerical results and those available in the literature was observed. Furthermore, it was concluded that the springs constants of the foundation have a meaningful impact on the frequencies values. The higher the values of these parameters, the higher the values of frequencies are attained. Nonlinear dynamic characteristics of a composite orthotropic plate resting on the Winkler–Pasternak elastic foundation have been studied by Gao *et al.* [25]. Natural frequency, linear and nonlinear vibration as well as nonlinear dynamic responses were examined. The research also took into the consideration the influence of temperature on the structure. The plate was subjected to different axial velocities. It was acknowledged that the temperature changes affect the values of vibration frequencies and amplitudes. In addition, the Winkler–Pasternak foundation has an impact on structural dynamic responses of the plate.

Free vibration analysis of functionally graded sandwich Timoshenko beams resting on a nonlocal (size-dependent) elastic foundation has been performed by Zhang *et al.* [26]. The equations of motion as well as boundary conditions were derived with the use of Hamilton's principle. Comparative studies were formulated to validate the numerical model. The beam was assumed to have functionally graded faces and a homogeneous core. It was conceded that the increase of nonlocal parameter values causes the increase of natural frequency. Vibration of a porous nanocomposite sandwich beam resting on the Kerr viscoelastic foundation has been formulated by Keshtegar *et al.* [27]. The beam included two nanocomposite piezoelectric faces. The viscoelastic foundation was composed of two dampers, two springs, and one shear element. It was affirmed that the increase of porous coefficient values decreases the wave velocity and frequency. In addition, the authors conferred that the wave velocity for the Kerr-type viscoelastic foundation would be higher than that obtained for the Pasternak-type and Winkler-type viscoelastic ones. Numerical analysis of forced vibration of a cracked double-beam system connected by a viscoelastic layer has been conducted by Chen *et al.* [28]. The beam was resting on the Winkler–Pasternak elastic foundation and was subjected to harmonic loads. It was indicated that the decrease of natural frequency is associated with the increase of the crack depth ratio. The location of the crack also plays a decisive role in the dynamic characteristic of a double-beam system. Moreover, the stiffness of the connecting layer affects the values of natural frequencies of the construction. Vibration of simply supported sandwich plates with a pyramidal truss core resting on the Winkler–Pasternak elastic foundation has been discussed by Chai *et al.* [29]. The influence of geometric parameters of the plates, material properties as well as the parameters of elastic foundation were taken into the consideration. Two faces and the core were manufactured of the same material. Analytical solution performed by the authors was compared with numerical one. Good accordance between two distinct methods was regarded. The analysis revealed the following conclusions: natural frequencies increase and then decrease with the increase of a core truss radius, the values of frequencies increase with the increase of thickness of the faces. Furthermore, natural frequencies increase with the increase of elastic foundation parameters [29]. Dynamic stability analysis of an asymmetric sandwich beam varying exponentially and resting on the Pasternak elastic foundation has been carried out by Mohanty *et al.* [30]. It was adopted that the elastic foundation stiffness varied linearly with the displacement. The top layer of the beam was made of aluminum, the bottom layer-steel, and the core was composed of glass fiber. In addition, the beam was subjected to an axial pulsating load and one dimensional temperature gradient. The effect of, inter alia, the elastic foundation parameter, shear and taper parameters was taken into account. It was acknowledged that the advantageous dynamic stability of the beam can be attained when the values of d/l (diameter/length), shear parameter, and temperature gradient increase. Opposite situation appears when the values of taper parameters, elastic foundation parameters, and the modulus ratio increase [30]. Free vibration of a sandwich beam with a soft core and resting on the Pasternak foundation has been analyzed by Arefi and Najafitabar [31]. Two functionally graded graphene nanoplatelets reinforced composite faces were incorporated into the construction. The extended high-order sandwich panel theory for free vibration was adopted in the numerical model. It was stated that the natural frequencies decrease with the increase of length to the thickness ratio. Moreover, addition of a slight quantity of nano reinforcement induces the increase of natural frequencies. Research conceded that the application of graphene nanoplatelets as a reinforcement leads to a considerable amelioration of mechanical properties of the beam [31].

Dynamic analysis of functionally graded sandwich plates under multiple moving loads has been carried out by Songsuwan *et al.* [32]. Equations of motion were derived and solved by Ritz and Newmark time integration methods. The effects of, inter alia, boundary conditions, moving load velocity, and layer thickness ratio were taken into the consideration. It was acknowledged that the dynamic deflection alters its values in relation to moving load velocity changes. Moreover, another parameters considered by the authors also play an important role in predicting the dynamic deflection values. A nonlinear transient response of sandwich beams with the functionally graded porous core has been investigated by Songsuwan *et al.* [33]. Beams were made of two isotropic faces, a porous core and were subjected to a moving load. Nonlinear free and forced vibrations were examined with the use of numerical methods. The analysis affirmed that the parameters such as, inter alia, boundary conditions, slenderness ratio, porous coefficient, types of porous distribution, vibration amplitude or velocity of load have a meaningful influence on nonlinear deflection of sandwich beams. The transient or dynamic response of sandwich plates with a functionally graded core and subjected to time-dependent loads has been discussed by Wattanasakulpong and Eiadtrong [34]. The plates were made of two isotropic faces and a functionally graded core with open-cell internal pores. The Ritz method based on Jacobi polynomials was adopted to solve the equations of motion and to study the dynamic behaviour of plates. The authors formulated the following conclusion: increasing the amount of internal pores in the core entails considerable improvement of flexural stiffness. Vibration of porous functionally graded beams resting on the variable Winkler–Pasternak foundation has been presented by Mellal *et al.* [35]. The high-order shear deformation theory has been implemented into the research. Parameters of the Winkler foundation in longitudinal direction had various distribution: linear, parabolic, sinusoidal, cosine, exponential, and uniform. Further analysis conceded that the type of the Winkler foundation has a valid effect on dimensionless fundamental natural frequencies. In addition, the frequencies decrease with the increase of the volume fraction index.

The application of constructions on elastic foundation is a complex issue for engineers. Many instances involve a comparatively rigid structure supported by a more flexible foundation. Such foundations appear in civil engineering where buildings are supported on an elastic medium. However, models describing the elastic foundations comprise some inaccuracies. Literature review, performed by the authors, affirmed one common feature of all elastic substrate models: constant foundation parameters. Due to this assumption, elastic foundation is considered not to play a major role in strength or dynamic analysis of the structure. One could say that it plays an indirect role. The authors submitted the original beam-foundation interaction model which based on variable parameters of the foundation and their influence on the beam response. Thanks to this approach, it is possible to control the behaviour of structure by changing the elastic foundation parameters. The main purpose of the presented study is free flexural vibration of a simply supported sandwich beam resting on elastic foundation. The foundation was depicted by a mathematical function $c(x)$ having changeable parameters. The application of this function allowed for control of foundation parameters and analysis of their impact on free vibration. Analytical and numerical examination were carried out, showing a good agreement between both methods. Stability analysis of a three-layered beam was thoroughly described in previous work of the authors [36]. The presented model in this cited paper [36] is similar to the one considered in this current work. However, the similarity only applies to the model of elastic foundation. The novelty of this work is free vibration investigation of a beam on an elastic foundation (time-dependent issue); governing equations of the considered phenomenon, the solution method and its general nature are different. This work concentrates on the dynamic analysis of a simply supported sandwich beam on an elastic foundation with variable parameters which is a continuation of previous research.

2. Analytical model of a beam

The subject of the paper is a simply supported symmetrical sandwich beam of the length L resting on an elastic foundation (Fig. 1). The width of this beam is b, thicknesses of faces are h_f, the thickness of the core is h_c, consequently, the total depth $h = h_c + 2h_f$. The analytical model of this beam is developed with consideration of the 'broken line' theory (Fig. 2).

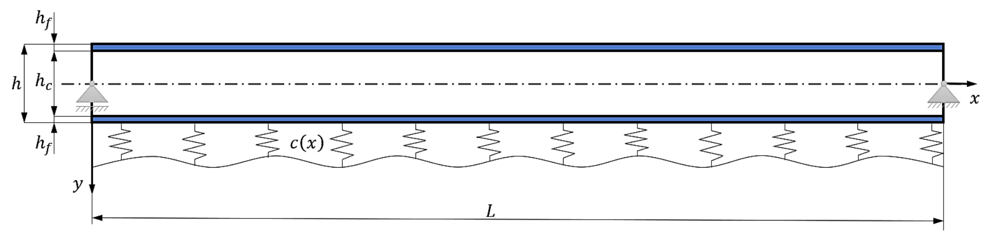

FIG. 1. Scheme of a sandwich beam.

According to that, individual displacement components $u(x,y,t)$ for each layer have the form:

– the upper face: $-\frac{1}{2}h \le y \le -\frac{1}{2}h_c$

$$u(x,y,t) = -\left[y\frac{\partial v}{\partial x} + u_f(x,t)\right] = -h\left[\eta\frac{\partial v}{\partial x} + \psi_f(x,t)\right], \tag{2.1}$$

where $-\frac{1}{2} \le \eta \le -\frac{1}{2}\chi_c$ $\quad \eta = \frac{y}{h}$ – dimensionless coordinate, $\chi_c = \frac{h_c}{h}$ – dimensionless thickness of the core, and $\psi_f(x,t) = \frac{u_f(x,t)}{h}$ – dimensionless function,

– the core: $-\frac{1}{2}h_c \le y \le \frac{1}{2}h_c$

$$u(x,y,t) = -y\frac{\partial v}{\partial x} + 2y\frac{u_f(x,t)}{h_c} = -h\eta\left[\frac{\partial v}{\partial x} - \frac{2}{\chi_c}\psi_f(x,t)\right], \tag{2.2}$$

where $-\frac{1}{2}\chi_c \le \eta \le \frac{1}{2}\chi_c$,

– the lower face: $\frac{1}{2}h_c \leq y \leq \frac{1}{2}h$

$$u(x,y,t) = -y\frac{\partial v}{\partial x} + u_f(x,t) = -h\Big[\eta\frac{\partial v}{\partial x} - \psi_f(x,t)\Big], \tag{2.3}$$

where $\frac{1}{2}\chi_c \leq \eta \leq \frac{1}{2}$.

Then the strains:

– the upper face

$$\varepsilon_x^{uf}(x,\eta,t) = \frac{\partial u}{\partial x} = -h\Big[\eta\frac{\partial^2 v}{\partial x^2} + \frac{\partial \psi_f}{\partial x}\Big], \qquad \gamma_{xy}^{uf}(x,\eta,t) = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0, \tag{2.4}$$

– the core

$$\begin{aligned} \varepsilon_x^{c}(x,\eta,t) &= \frac{\partial u}{\partial x} = -h\eta\Big[\frac{\partial^2 v}{\partial x^2} - \frac{2}{\chi_c}\cdot\frac{\partial \psi_f}{\partial x}\Big], \\ \gamma_{xy}^{c}(x,\eta,t) &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{2}{\chi_c}\psi_f(x,t), \end{aligned} \tag{2.5}$$

– the lower face

$$\varepsilon_x^{lf}(x,\eta,t) = \frac{\partial u}{\partial x} = -h\Big[\eta\frac{\partial^2 v}{\partial x^2} - \frac{\partial \psi_f}{\partial x}\Big], \qquad \gamma_{xy}^{lf}(x,\eta,t) = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0, \tag{2.6}$$

where t – time [s].

Normal and shear stresses are as follows:

– the upper face

$$\sigma_x^{uf}(x,\eta,t) = E_f\cdot\varepsilon_x^{uf}(x,\eta,t), \qquad \tau_{xy}^{uf}(x,\eta,t) = 0, \tag{2.7}$$

where E_f is the Young modulus of the face,

– the core

$$\sigma_x^{c}(x,\eta,t) = E_c\cdot\varepsilon_x^{c}(x,\eta,t), \qquad \tau_{xy}^{c}(x,\eta,t) = \frac{E_c}{2(1+\nu_c)}\cdot\gamma_{xy}^{c}(x,\eta,t), \tag{2.8}$$

where E_c is the Young modulus of the core, and ν_c is the Poisson ratio of the core,

– the lower face

$$\sigma_x^{lf}(x,\eta,t) = E_f\cdot\varepsilon_x^{lf}(x,\eta,t), \qquad \tau_{xy}^{lf}(x,\eta,t) = 0. \tag{2.9}$$

The displacement components $u(x,y,t)$ of a sandwich beam have been determined based on the 'broken line' theory (Fig. 2). Shear stresses, after adopting this theory, are equal to zero in the faces (the upper and the lower one), and other than zero in the core. When the core is characterized by constant proper-

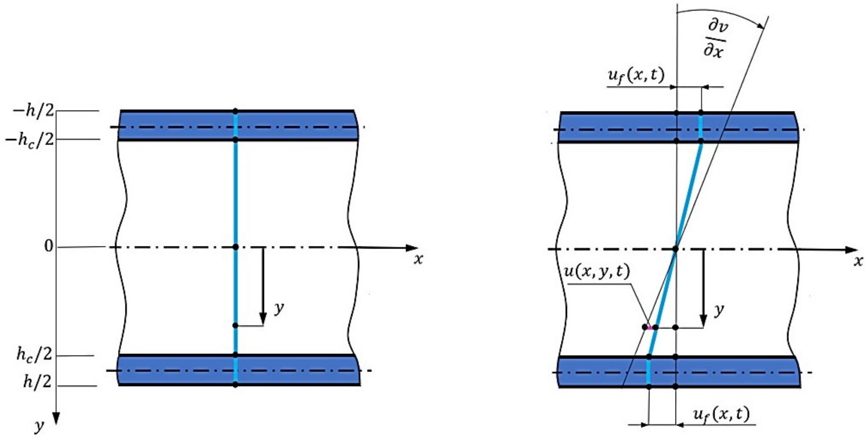

Fig. 2. Scheme of the deformation of the sandwich beam planar cross-section – the 'broken line' theory [37].

ties, the values of shear stresses in the core are also constant. When the core is characterized by various properties, the values of shear stresses in the core are also various. It is also assumed that the deflections of layers are consistent with the deflection of beam axis.

The equations of motion as well as boundary conditions were derived with the use of Hamilton's principle which is assumed in the following form:

$$\delta \int_{t_1}^{t_2} [U_k - (U_\varepsilon - W)]\, dt = 0, \tag{2.10}$$

where U_k is the kinetic energy of the beam, U_ε – the elastic strain energy of the beam, W – the work of load – reaction of the elastic foundation, and t is time [s].

Hamilton's equations of motion correspond to Lagrange's equations, but they are more adequate for the analysis of motion systems since they are of first-order and highly symmetrical.

The kinetic energy of the beam has the form:

$$U_k = \frac{1}{2} bh\rho_b \cdot \int_0^L \left(\frac{\partial v}{\partial t}\right)^2 dx, \tag{2.11}$$

where $\rho_b = (1-\chi_c)\rho_f + \chi_c\rho_c = [1-\chi_c+\sqrt{e}_c\chi_c]\rho_f$ is mass density of the beam, ρ_c, ρ_f – mass density of the core and the faces respectively, $\rho_b = [1-(1-\sqrt{e}_c)\chi_c]\rho_f$, $\sqrt{e}_c = \frac{\rho_c}{\rho_f}$, $e_c = \frac{E_c}{E_f}$. Thus, Eq. (2.11) can be expressed in the following form:

$$U_k = \frac{1}{2} bh\rho_f[1-(1-\sqrt{e}_c)\chi_c] \cdot \int_0^L \left(\frac{\partial v}{\partial t}\right)^2 dx. \tag{2.12}$$

The elastic strain energy of the beam is as follows:

$$U_\varepsilon = \frac{1}{2}bh^3 \int\limits_0^L \Bigg\{ E_f \int\limits_{-1/2}^{-\chi_c/2} \left[\eta^2 \left(\frac{\partial^2 v}{\partial x^2}\right)^2 + 2\eta \frac{\partial^2 v}{\partial x^2} \cdot \frac{\partial \psi_f}{\partial x} + \left(\frac{\partial \psi_f}{\partial x}\right)^2 \right] d\eta$$
$$+ E_f \int\limits_{\chi_c/2}^{1/2} \left[\eta^2 \left(\frac{\partial^2 v}{\partial x^2}\right)^2 - 2\eta \frac{\partial^2 v}{\partial x^2} \cdot \frac{\partial \psi_f}{\partial x} + \left(\frac{\partial \psi_f}{\partial x}\right)^2 \right] d\eta$$
$$+ E_c \left[\left(\frac{\partial^2 v}{\partial x^2}\right)^2 - \frac{4}{\chi_c} \cdot \frac{\partial^2 v}{\partial x^2} \cdot \frac{\partial \psi_f}{\partial x} + \frac{4}{\chi_c^2} \left(\frac{\partial \psi_f}{\partial x}\right)^2 \right] \cdot \int\limits_{-\chi_c/2}^{\chi_c/2} \eta^2 \, d\eta$$
$$+ E_c \frac{2}{1+\nu_c} \cdot \frac{1}{\chi_c} \cdot \frac{\psi_f^2(x,t)}{h^2} \Bigg\} dx.$$

Thus, the above equation is assumed in the following form:

$$U_\varepsilon = \frac{1}{2}bh^3 \int\limits_0^L \Bigg\{ E_f \Bigg[\frac{1}{12}(1-\chi_c^3) \left(\frac{\partial^2 v}{\partial x^2}\right)^2$$
$$- 2 \cdot \frac{1}{4}(1-\chi_c^2) \frac{\partial^2 v}{\partial x^2} \cdot \frac{\partial \psi_f}{\partial x} + (1-\chi_c) \left(\frac{\partial \psi_f}{\partial x}\right)^2 \Bigg]$$
$$+ \frac{1}{12} E_c \cdot \chi_c^3 \left[\left(\frac{\partial^2 v}{\partial x^2}\right)^2 - 2 \cdot \frac{2}{\chi_c} \cdot \frac{\partial^2 v}{\partial x^2} \cdot \frac{\partial \psi_f}{\partial x} + \frac{4}{\chi_c^2} \left(\frac{\partial \psi_f}{\partial x}\right)^2 \right]$$
$$+ E_c \frac{2}{1+\nu_c} \cdot \frac{1}{\chi_c} \cdot \frac{\psi_f^2(x,t)}{h^2} \Bigg\} dx. \tag{2.13}$$

The last part of Eq. (2.10), the work of load, has the form:

$$W = -\frac{1}{2} \int\limits_0^L q_f(x) \cdot v(x) \, dx,$$

where $q_f(x) = c(x) \cdot v(x)$ is the reaction of elastic foundation $\left[\frac{\mathrm{N}}{\mathrm{mm}}\right]$, $c(x)$ – the shape (nonlinear function) of elastic foundation $\left[\frac{\mathrm{N}}{\mathrm{mm}^2}\right]$, $v(x)$ – the deflection of beam.

The shape function (Fig. 3) with variable properties of the foundation is as follows:

$$c(x) = c_{fd}[1 + \alpha_f \sin^n(\pi\xi)], \tag{2.14}$$

where $\xi = x/L$ is dimensionless coordinate and $0 \le \xi \le 1$, n – natural exponent, c_{fd} – the elastic foundation constant $\left[\frac{\mathrm{N}}{\mathrm{mm}^2}\right]$, and α_f – the coefficient of the foundation reaction where $-1 \le \alpha_f \le 1$.

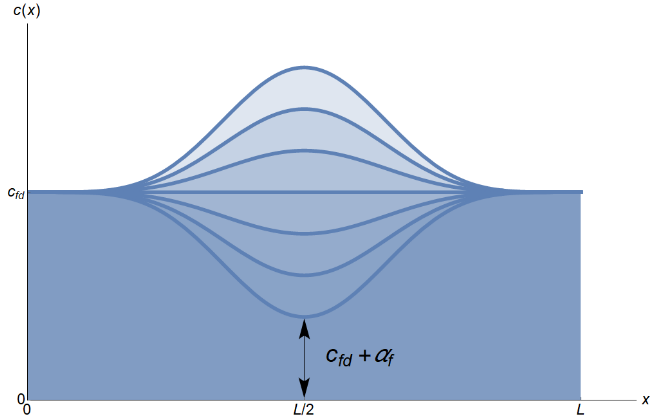

Fig. 3. Shape of nonlinear function $c(x)$ of the foundation [36].

It was adopted that the function (2.14) is symmetrical to both ends of the sandwich beam. However, a nonsymmetrical shape function may also be used for calculations.

The sandwich beam introduced in this article is resting on a variable elastic foundation. The foundation being analyzed is flat but it has a changeable intensity of the foundation reaction. This intensity can be compared to the intensities of soil foundations. Soil foundations have a flat structure but their properties vary with respect to density changes.

Figures 4 and 5 depict the function for variable values of c_{fd}, α_f, and n parameters. The figures reveal that the parameter n is a substantial factor that alters the shape of the function (2.14). The peak in the graph changes its shape, depending on the adopted value of the natural exponent. Furthermore, c_{fd} and α_f parameters also play a significant role in the characteristics of an elastic foundation.

The function of deflection of sandwich beam is assumed in the following form:

$$v(x) = v_a \cdot \sin(m\pi\xi) \cdot \sin^n(\pi\xi), \tag{2.15}$$

where v_a is the amplitude of deflection, whereas m and n are natural numbers.

Therefore, the work of load equation can be expressed in the following form:

$$W = -\frac{1}{2}\int_0^L c(x) \cdot v^2(x)\, dx. \tag{2.16}$$

The variations of kinetic energy of the beam U_k, the elastic strain energy U_ε as well as the work of the load W have the forms:

$$\delta U_k = -bh\rho_f[1-(1-\sqrt{e}_c)\chi_c] \cdot \int_0^L \frac{\partial^2 v}{\partial t^2}\delta v\, dx, \tag{2.17}$$

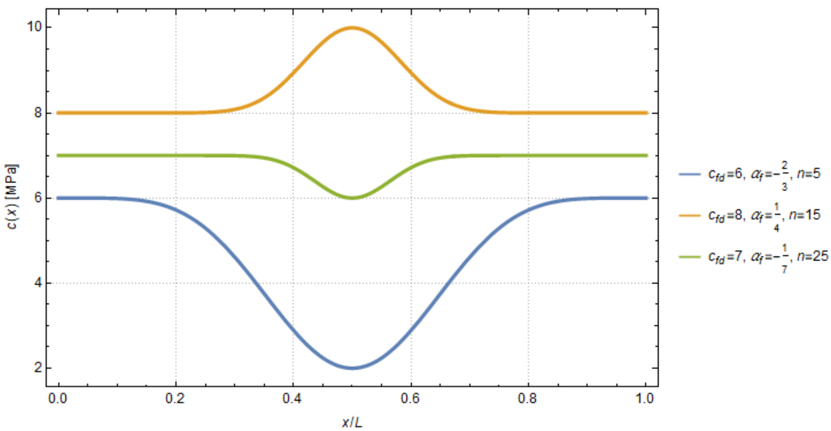

FIG. 4. Shape of $c(x)$ function for variable values of n parameter [36].

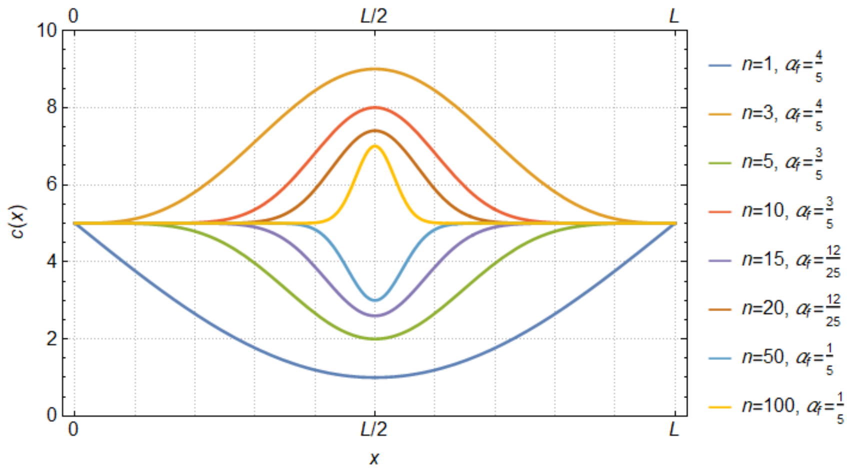

FIG. 5. Examples of shape function $c(x)$ for variable parameters [36].

$$\begin{aligned}
\delta U_{\varepsilon} = bh^3 \int_0^L \Bigg\{ E_f \Bigg[& \frac{1}{12}(1-\chi_c^3)\frac{\partial^4 v}{\partial x^4}\cdot\delta v - \frac{1}{4}(1-\chi_c^2)\frac{\partial^3 \psi_f}{\partial x^3}\cdot\delta v \\
& + \frac{1}{4}(1-\chi_c^2)\frac{\partial^3 v}{\partial x^3}\cdot\delta\psi_f - (1-\chi_c)\frac{\partial^2 \psi_f}{\partial x^2}\cdot\delta\psi_f \Bigg] \\
& + \frac{1}{12}E_c\chi_c^3\Bigg[\frac{\partial^4 v}{\partial x^4}\cdot\delta v - \frac{2}{\chi_c}\cdot\frac{\partial^3 \psi_f}{\partial x^3}\cdot\delta v + \frac{2}{\chi_c}\cdot\frac{\partial^3 v}{\partial x^3}\cdot\delta\psi_f \\
& - \frac{4}{\chi_c^2}\cdot\frac{\partial^2 \psi_f}{\partial x^2}\cdot\delta\psi_f\Bigg] + E_c\frac{2}{1+\nu_c}\cdot\frac{1}{\chi_c}\cdot\frac{\psi_f(x,t)}{h^2}\cdot\delta\psi_f\Bigg\}\,dx,
\end{aligned} \tag{2.18}$$

$$\delta W = -\int_0^L c(x) \cdot v(x) \cdot \delta v \, dx. \tag{2.19}$$

Taking into account Hamilton's principle one obtains:

- the variation of δv:

$$\begin{aligned} & bh\rho_f[1-(1-\sqrt{e}_c)\chi_c] \cdot \frac{\partial^2 v}{\partial t^2} + bh^3\Big\{\frac{1}{12}[E_f(1-\chi_c^3) + E_c \cdot \chi_c^3]\frac{\partial^4 v}{\partial x^4} \\ & \qquad - \Big[\frac{1}{4}E_f(1-\chi_c^2) + \frac{1}{6}E_c\chi_c^2\Big]\frac{\partial^3 \psi_f}{\partial x^3}\Big\} + c(x) \cdot v(x) = 0, \\ & bh\rho_f[1-(1-\sqrt{e}_c)\chi_c] \cdot \frac{\partial^2 v}{\partial t^2} + \frac{1}{12}E_f bh^3\Big\{[1-(1-e_c)\chi_c^3]\frac{\partial^4 v}{\partial x^4} \\ & \qquad - [3-(3-2e_c)\chi_c^2]\frac{\partial^3 \psi_f}{\partial x^3}\Big\} + c(x) \cdot v(x) = 0, \end{aligned} \tag{2.20}$$

- the variation of $\delta\psi_f$:

$$\begin{aligned} & \Big[\frac{1}{4}E_f(1-\chi_c^2) + \frac{1}{6}E_c\chi_c^2\Big]\frac{\partial^3 v}{\partial x^3} - \Big[E_f(1-\chi_c) + \frac{1}{3}E_c\chi_c\Big]\frac{\partial^2 \psi_f}{\partial x^2} \\ & \qquad + 2\frac{E_c}{1+\nu_c} \cdot \frac{1}{\chi_c} \cdot \frac{\psi_f(x,t)}{h^2} = 0, \\ & \frac{1}{12}E_f[3-(3-2e_c)\chi_c^2]\frac{\partial^3 v}{\partial x^3} - \frac{1}{3}E_f[3-(3-e_c)\chi_c]\frac{\partial^2 \psi_f}{\partial x^2} \\ & \qquad + E_f\frac{2}{1+\nu_c} \cdot \frac{e_c}{\chi_c} \cdot \frac{\psi_f(x,t)}{h^2} = 0, \\ & [3-(3-2e_c)\chi_c^2]\frac{\partial^3 v}{\partial x^3} - 4[3-(3-e_c)\chi_c]\frac{\partial^2 \psi_f}{\partial x^2} + \frac{24}{1+\nu_c} \cdot \frac{e_c}{\chi_c} \cdot \frac{\psi_f(x,t)}{h^2} = 0. \end{aligned} \tag{2.21}$$

Therefore, the above equations of motion (2.20) and (2.21) can be presented in the following form:

$$bh\rho_f \cdot c_m \cdot \frac{\partial^2 v}{\partial t^2} + \frac{E_f bh^3}{12}\Big(c_{vv} \cdot \frac{\partial^4 v}{\partial x^4} - c_{v\psi} \cdot \frac{\partial^3 \psi_f}{\partial x^3}\Big) + c(x) \cdot v(x) = 0, \tag{2.22}$$

$$c_{v\psi} \cdot \frac{\partial^3 v}{\partial x^3} - c_{\psi\psi} \cdot \frac{\partial^2 \psi_f}{\partial x^2} + c_\psi \cdot \frac{\psi_f(x,t)}{h^2} = 0, \tag{2.23}$$

where $c_m = 1-(1-\sqrt{e}_c)\chi_c$, $c_{vv} = 1-(1-e_c)\chi_c^3$, $c_{v\psi} = 3-(3-2e_c)\chi_c^2$, $c_{\psi\psi} = 4[3-(3-e_c)\chi_c]$, $c_\psi = \frac{24}{1+\nu_c} \cdot \frac{e_c}{\chi_c}$.

The analytical investigation has been divided into two parts: the analysis of elastic foundation with constant and variable properties. The main objective of the presented work is free vibration analysis of a sandwich beam on an elastic foundation with variable properties. Therefore, the part related to constant properties is only presented for a compliance confirmation of the equations used as well as for a comparison of the results obtained.

2.1. The elastic foundation with constant properties ($\alpha_f = 0$)

During the analysis, the following assumptions have been applied: $c(x) = c_{fd}$, $\xi = x/L$, $\lambda = L/h$, $\bar{v}(\xi,t) = \frac{v(\xi,t)}{L}$, where λ is the ratio of the beam length to its height and $\bar{v}(\xi,t)$ is the dimensionless deflection of the beam.

After making the above assumptions, the equations of motion (2.22) and (2.23) can be transformed into the dimensionless forms:

$$\rho_f \cdot c_m \cdot \frac{\partial^2 \bar{v}}{\partial t^2} + \frac{E_f}{12\lambda^2 L^2}\left(c_{vv} \cdot \frac{\partial^4 \bar{v}}{\partial \xi^4} - c_{v\psi} \cdot \frac{\partial^3 \psi_f}{\partial \xi^3}\right) + \frac{c_{fd}}{bh} \cdot \bar{v}(\xi) = 0, \tag{2.24}$$

$$c_{v\psi} \cdot \frac{\partial^3 \bar{v}}{\partial \xi^3} - c_{\psi\psi} \cdot \frac{\partial^2 \psi_f}{\partial \xi^2} + c_\psi \cdot \lambda^2 \cdot \psi_f(\xi,t) = 0. \tag{2.25}$$

The frequency equation has been obtained for a simply supported beam (pivot ends of the beam). The differential equations of motion (2.24) and (2.25) were approximately solved with the use of two assumed functions:

$$\bar{v}(\xi,t) = \bar{v}_a(t) \cdot \sin(\pi\xi), \tag{2.26}$$

$$\psi_f(\xi,t) = \psi_{fa}(t) \cdot \cos(\pi\xi), \tag{2.27}$$

where $\bar{v}_a$ is the amplitude of flexural vibration and $\bar{v}_a(t) = \bar{v}_a \cdot \sin(\omega t)$.

Substituting (2.26) and (2.27) into Eq. (2.24) once obtained:

$$\rho_f \cdot c_m \cdot \frac{d^2 \bar{v}_a}{dt^2} \cdot \sin(\pi\xi) + \frac{E_f}{12\lambda^2 L^2}(\pi^4 \cdot c_{vv} \cdot \bar{v}_a(t) - \pi^3 \cdot c_{v\psi} \cdot \psi_{fa}(t)) \cdot \sin(\pi\xi) + \frac{c_{fd}}{bh} \cdot \bar{v}_a(t) \cdot \sin(\pi\xi) = 0.$$

Substituting (2.26) and (2.27) into Eq. (2.25) once obtained:

$$-\pi^3 \cdot c_{v\psi} \cdot \bar{v}_a(t) \cdot \cos(\pi\xi) + \pi^2 \cdot c_{\psi\psi} \cdot \psi_{fa}(t) \cdot \cos(\pi\xi) + c_\psi \cdot \lambda^2 \cdot \psi_{fa}(t) \cdot \cos(\pi\xi) = 0,$$

$$(\pi^2 \cdot c_{\psi\psi} + c_\psi \cdot \lambda^2)\psi_{fa}(t) = \pi^3 \cdot c_{v\psi} \cdot \bar{v}_a(t), \tag{2.28}$$

$$\psi_{fa}(t) = \frac{\pi c_{v\psi}}{c_{\psi\psi} + c_\psi\left(\frac{\lambda}{\pi}\right)^2}\bar{v}_a(t).$$

Thus, the first equation of motion (2.24) can be expressed in the following form:

$$\rho_f \cdot c_m \cdot \frac{d^2\bar{v}_a}{dt^2} + \frac{\pi^4 E_f}{12\lambda^2 L^2}\left[c_{vv} - \frac{c_{v\psi}^2}{c_{\psi\psi} + c_\psi\left(\frac{\lambda}{\pi}\right)^2}\right] \cdot \bar{v}_a(t) + \frac{c_{fd}}{bh} \cdot \bar{v}_a(t) = 0. \tag{2.29}$$

Based on the above equations, the frequency equation of a sandwich beam on an elastic foundation has been found:

$$\begin{aligned} \omega^2 &= \frac{\pi^4 E_f}{12\lambda^2 L^2}\left\{c_{vv} - \frac{c_{v\psi}^2}{c_{\psi\psi} + c_\psi\left(\frac{\lambda}{\pi}\right)^2} + \frac{12}{\pi^4}\lambda^3\frac{L}{b} \cdot \frac{c_{fd}}{E_f}\right\}\frac{1}{c_m \cdot \rho_f}, \\ \omega &= \frac{\sqrt{3}\pi^2 \cdot 10^3}{6\lambda L} \cdot \sqrt{\left[c_{vv} - \frac{c_{v\psi}^2}{c_{\psi\psi} + c_\psi\left(\frac{\lambda}{\pi}\right)^2} + \frac{12}{\pi^4}\lambda^3\frac{L}{b} \cdot \frac{c_{fd}}{E_f}\right]\frac{E_f}{c_m \cdot \rho_f}}. \end{aligned} \tag{2.30}$$

Equation (2.30) is solved with the use of $\bar{v}_a(t)$ function and ω [rad/s] or $f_z = \omega/2\pi$ [Hz] is the fundamental natural frequency.

Sample analytical values of natural frequency have been presented for the following data: $E_f = 72$ GPa, $E_c = 3.0$ GPa, $e_c = \frac{1}{24}$, $\rho_f = 2710\,\frac{\text{kg}}{\text{m}^3}$, $\rho_c = 553\,\frac{\text{kg}}{\text{m}^3}$, $\frac{\rho_c}{\rho_f} = \sqrt{e}_c = 0.2041$, $h = 20$ mm, $h_c = 18$ mm, $h_f = 1$ mm, $c_{fd} = 8$ MPa, $\nu_c = 0.3$, $b = 20$ mm, $\lambda = L/h$, and $\lambda_b = L/b$. The results for diverse proportions of beam length have been performed (Table 1). The calculations below confirmed the correctness of the equations used. The values of the natural frequency decrease with the increase of the beam length.

TABLE 1. The values of natural frequency of sandwich beam on elastic foundation with constant properties.

L [m]	0.2	0.3	0.4	0.5
λ	10	15	20	25
$\frac{\omega}{2\pi}$ [Hz]	1383.55	962.77	863.82	833.75

The definition of free vibration is adopted when there is no external load which can induce the motion and the motion is the outcome of initial conditions. In the situation studied, the values of free vibration frequencies are influenced by the change of the beam length. The parameters of an elastic foundation are not taken into the consideration since they are assumed to be constant. During free vibration analysis, energy will remain the same, it is not added or removed from the body. The body keeps vibrating at the same amplitude.

Some examples of free vibration are: oscillations of a simple pendulum, oscillations of an object connected to a horizontal spring, sound produced by a tuning fork in a short distance, notes of musical instruments, an organ pipe, etc.

2.2. The elastic foundation with variable properties ($\alpha_f \neq 0$)

During the analysis, the following assumptions have been applied:

$$c(\xi) = c_{fd}[1 + \alpha_f \sin^n(\pi\xi)], \qquad \bar{v}(\xi,t) = \frac{v(\xi,t)}{L}, \qquad n = 1,2,3,\ldots.$$

After making the above assumptions, the equations of motion (2.22) and (2.23) can be transformed into the dimensionless forms:

$$\rho_f \cdot c_m \cdot \frac{\partial^2 \bar{v}}{\partial t^2} + \frac{E_f}{12\lambda^2 L^2}\Big(c_{vv} \cdot \frac{\partial^4 \bar{v}}{\partial \xi^4} - c_{v\psi} \cdot \frac{\partial^3 \psi_f}{\partial \xi^3}\Big) + \frac{c_{fd}}{bh}[1 + \alpha_f \sin^n(\pi\xi)] \cdot \bar{v}(\xi) = 0, \tag{2.31}$$

$$c_{v\psi} \cdot \frac{\partial^3 \bar{v}}{\partial \xi^3} - c_{\psi\psi} \cdot \frac{\partial^2 \psi_f}{\partial \xi^2} + c_\psi \cdot \lambda^2 \cdot \psi_f(\xi,t) = 0. \tag{2.32}$$

The frequency equation has been obtained for the simply supported beam (pivot ends of the beam). The differential equations of motion (2.31) and (2.32) were approximately solved with the use of two assumed functions:

$$\bar{v}(\xi,t) = [\sin(\pi\xi) + k_v \sin(3\pi\xi)] \cdot \bar{v}_a(t), \tag{2.33}$$

$$\psi_f(\xi,t) = [\cos(\pi\xi) + k_\psi \cos(3\pi\xi)] \cdot \psi_{fa}(t), \tag{2.34}$$

where k_v and k_ψ are the dimensionless coefficients.

Substituting (2.33) and (2.34) into Eq. (2.32) once obtained:

$$k_\psi = 27\frac{\pi^2 c_{\psi\psi} + \lambda^2 c_\psi}{9\pi^2 c_{\psi\psi} + \lambda^2 c_\psi} \cdot k_v, \tag{2.35}$$

$$\psi_{fa}(t) = \frac{\pi c_{v\psi}}{c_{\psi\psi} + \left(\frac{\lambda}{\pi}\right)^2 \cdot c_\psi} \cdot \bar{v}_a(t), \tag{2.36}$$

where

$$\frac{d^4 \bar{v}}{d\xi^4} = \pi^4[\sin(\pi\xi) + 81k_v \sin(3\pi\xi)] \cdot \bar{v}_a(t), \tag{2.37}$$

$$\frac{d^3 \psi_f}{d\xi^3} = \pi^3[\sin(\pi\xi) + 27k_\psi \sin(3\pi\xi)] \cdot \psi_{fa}(t). \tag{2.38}$$

Therefore, after substituting (2.33) and (2.34) into Eq. (2.31) once obtained:

$$\begin{aligned}\rho_f \cdot c_m[\sin(\pi\xi) + k_v \sin(3\pi\xi)]\frac{d^2\bar{v}_a}{dt^2} + \frac{E_f}{12\lambda^2 L^2}\Big\{\pi^4 c_{vv}[\sin(\pi\xi) + 81k_v \sin(3\pi\xi)] \cdot \bar{v}_a(t) \\ - \pi^3 c_{v\psi}[\sin(\pi\xi) + 27k_\psi \sin(3\pi\xi)] \cdot \frac{\pi c_{v\psi}}{c_{\psi\psi} + \left(\frac{\lambda}{\pi}\right)^2 \cdot c_\psi} \cdot \bar{v}_a(t)\Big\} \\ + \frac{c_{fd}}{bh}[1 + \alpha_f \sin^n(\pi\xi)] \cdot [\sin(\pi\xi) + k_v \sin(3\pi\xi)] \cdot \bar{v}_a(t) = 0.\end{aligned}$$

After simple transformations, the first equation of motion (2.31) can be simplified into the following form:

$$
\begin{aligned}
(2.39)\qquad & \rho_f \cdot c_m[\sin(\pi\xi) + k_v \sin(3\pi\xi)]\frac{d^2\bar{v}_a}{dt^2} \\
&+ \frac{\pi^4 E_f}{12\lambda^2 L^2}\{c_{vv}[\sin(\pi\xi) + 81k_v \sin(3\pi\xi)] - c_{se}[\sin(\pi\xi) + 27k_\psi \sin(3\pi\xi)]\} \cdot \bar{v}_a(t) \\
&+ \frac{c_{fd}}{bh}[1 + \alpha_f \sin^n(\pi\xi)] \cdot [\sin(\pi\xi) + k_v \sin(3\pi\xi)] \cdot \bar{v}_a(t) = 0,
\end{aligned}
$$

where

$$
c_{se} = \frac{c_{v\psi}^2}{c_{\psi\psi} + \left(\frac{\lambda}{\pi}\right)^2 \cdot c_\psi}.
$$

The frequency is calculated using the Galerkin method. The main condition of this method is as follows:

$$
(2.40)\qquad \int\limits_0^1 \Omega(\xi,t) \cdot [\sin(\pi\xi) + k_v \sin(3\pi\xi)]\, d\xi = 0.
$$

The general solution can be defined in the following form (integration of the Eq. (2.31)):

$$
\frac{1}{2}\rho_f c_m(1 + k_v^2)\frac{d^2\bar{v}_a}{dt^2} + \frac{\pi^4 E_f}{12\lambda^2 L^2}\left\{\frac{1}{2}c_{vv}(1 + 81k_v^2) - \frac{1}{2}c_{se}(1 + 27k_v k_\psi)\right\} \cdot \bar{v}_a(t) + \frac{c_{fd}}{bh} \cdot J_n \cdot \bar{v}_a(t) = 0.
$$

Therefore, the above equation is as follows:

$$
\begin{aligned}
(2.41)\qquad & \rho_f c_m(1 + k_v^2)\frac{d^2\bar{v}_a}{dt^2} \\
&+ \frac{\pi^4 E_f}{12\lambda^2 L^2}\left\{c_{vv}(1 + 81k_v^2) - c_{se}(1 + 27k_v k_\psi) + \frac{24}{\pi^4}\lambda^3\frac{L}{b}J_n\frac{c_{fd}}{E_f}\right\} \cdot \bar{v}_a(t) = 0.
\end{aligned}
$$

Equation (2.41) is solved with the use of $\bar{v}_a(t) = \bar{v}_a \cdot \sin(\omega t)$ function, where $\bar{v}_a$ – the amplitude of flexural vibration, ω [rad/s] or $f_z = \omega/2\pi$ [Hz] – the fundamental natural frequency.

Thus, substituting the function into Eq. (2.41) one obtains:

$$
\begin{aligned}
(2.42)\qquad & \rho_f c_m(1 + k_v^2) \cdot \omega^2 \\
&= \frac{\pi^4 E_f}{12\lambda^2 L^2}\left\{c_{vv}(1 + 81k_v^2) - c_{se}(1 + 27k_v k_\psi) + \frac{24}{\pi^4}\lambda^3\lambda_b J_n\frac{c_{fd}}{E_f}\right\},
\end{aligned}
$$

where

$$\int_0^L [\sin(\pi\xi) + k_v \sin(3\pi\xi)]^2 \, d\xi = \frac{1}{2}(1 + k_v^2), \tag{2.43}$$

$$\int_0^1 [\sin(\pi\xi) + 81k_v \sin(3\pi\xi)] \cdot [\sin(\pi\xi) + k_v \sin(3\pi\xi)] \, d\xi = \frac{1}{2}(1 + 81k_v^2), \tag{2.44}$$

$$\int_0^1 [\sin(\pi\xi) + 27k_\psi \sin(3\pi\xi)] \cdot [\sin(\pi\xi) + k_v \sin(3\pi\xi)] \, d\xi = \frac{1}{2}(1 + 27k_v k_\psi), \tag{2.45}$$

$$J_n = \int_0^1 [1 + \alpha_f \sin^n(\pi\xi)] \cdot [\sin(\pi\xi) + k_v \sin(3\pi\xi)]^2 \, d\xi. \tag{2.46}$$

Based on the above equations, the frequency equation of the sandwich beam on an elastic foundation has been found:

$$\omega = \min_{k_v} \Bigg\{ \frac{\sqrt{3}\pi^2 \cdot 10^3}{6\lambda L} \cdot \sqrt{\left[c_{vv}(1 + 81k_v^2) - c_{se}(1 + 27k_v k_\psi) + \frac{24}{\pi^4}\lambda^3 \lambda_b J_n \frac{c_{fd}}{E_f}\right] \frac{1}{1 + k_v^2} \cdot \frac{E_f}{c_m \cdot \rho_f}} \Bigg\}. \tag{2.47}$$

The frequency ω is a function of geometric and mechanical properties of the beam, elastic foundation parameters (n and α_f parameters) as well as the k_v coefficient.

Formula (2.47) presents the concept of a minimum frequency which depends on the values of α_f and k_v parameters. In specific elastic foundation conditions, where $\alpha_f = 0$ and $n = 200$, the value of the parameter k_v is equal to 0 and the frequency is equal to f_z $(n \to \infty) = 962.77$ Hz (Table 2).

TABLE 2. Sample values of natural frequency of sandwich beam on elastic foundation with variable properties.

α_f	-0.50	-0.25	0	0.25	0.50
k_v	-0.002580	-0.001294	0	0.001294	0.002580
$\frac{\omega}{2\pi}$ [Hz]	943.34	953.12	962.77	972.30	981.72

3. Results and discussion

3.1. Analytical studies

Sample analytical values of natural frequency have been presented for the following data: $E_f = 72\,\mathrm{GPa}$, $E_c = 3.0\,\mathrm{GPa}$, $e_c = \frac{1}{24}$, $\rho_f = 2710\,\frac{\mathrm{kg}}{\mathrm{m}^3}$, $\rho_c = 553\,\frac{\mathrm{kg}}{\mathrm{m}^3}$, $\frac{\rho_c}{\rho_f} = \sqrt{e}_c = 0.2041$, $h = 20\,\mathrm{mm}$, $h_c = 18\,\mathrm{mm}$, $h_f = 1\,\mathrm{mm}$, $c_{fd} = 8\,\mathrm{MPa}$, $\nu_c = 0.3$, $L = 0.3\,\mathrm{m}$, $b = 20\,\mathrm{mm}$, $\lambda = L/h = 15$, and $\lambda_b = L/b = 15$. The results for various α_f, n, and k_v parameters have been presented below.

TABLE 3. Sample values of k_v of sandwich beam on elastic foundation with variable properties.

	$\alpha_f = -0.50$	$\alpha_f = -0.25$	$\alpha_f = 0$	$\alpha_f = 0.25$	$\alpha_f = 0.50$
$n = 1$	−0.003966	−0.001989	0	0.001996	0.004001
$n = 2$	−0.005832	−0.002925	0	0.002941	0.005901
$n = 3$	−0.006788	−0.003405	0	0.003425	0.006871
$n = 4$	−0.007290	−0.003656	0	0.003678	0.007376
$n = 5$	−0.007546	−0.003784	0	0.003805	0.007630
$n = 10$	−0.007534	−0.003774	0	0.003788	0.007591
$n = 15$	−0.007047	−0.003528	0	0.003538	0.007085
$n = 20$	−0.006565	−0.003286	0	0.003292	0.006591
$n = 30$	−0.005788	−0.002896	0	0.002900	0.005803
$n = 40$	−0.005217	−0.002610	0	0.002612	0.005227
$n = 50$	−0.004183	−0.001763	0	0.003075	0.005495
$n = 100$	−0.002939	−0.001141	0	0.002453	0.004249
$n = 200$	−0.002580	−0.001294	0	0.001294	0.002580

The frequency values are determined by α_f and n parameters. The highest magnitudes can be reached for the highest values of α_f (the coefficient of the foundation reaction). The highest frequency (in the studied area) was equal to $\frac{\omega}{2\pi} = 1098.34\,\mathrm{Hz}$ and has been obtained for $\alpha_f = 0.5$ and $n = 1$, according to Eq. (2.47).

In addition, for negative values of α_f, natural frequency of the beam increases with the increase of n parameter, and decreases for positive values of α_f. This phenomenon is related to the shape of $c(x)$ function and, above all, the peak size on the graph. Narrow peaks do not have much influence on the dynamic response of a sandwich beam. Only wider peaks (lower values of n parameter) change the reaction of analyzed system. Higher values of n tend to stabilize the beam frequency. Therefore, complex analysis of the influence of narrow peaks is not necessary to understand the free vibration phenomenon in sandwich beam. Results of calculations have been introduced in Figs. 6 and 7.

Table 4. Sample values of natural frequency $\frac{\omega}{2\pi}^{A}$ [Hz] of sandwich beam on elastic foundation with variable properties.

	$\alpha_f = -0.50$	$\alpha_f = -0.25$	$\alpha_f = 0$	$\alpha_f = 0.25$	$\alpha_f = 0.50$
$n = 1$	804.40	887.16	962.77	1032.81	1098.34
$n = 2$	824.23	896.25	962.77	1024.87	1083.30
$n = 3$	838.19	902.72	962.77	1019.13	1072.37
$n = 4$	848.69	907.63	962.77	1014.74	1063.99
$n = 5$	856.95	911.51	962.77	1011.24	1057.29
$n = 10$	881.75	923.26	962.77	1000.52	1036.71
$n = 15$	894.75	929.48	962.77	994.77	1025.61
$n = 20$	903.05	933.47	962.77	991.05	1018.40
$n = 30$	913.37	938.46	962.77	986.36	1009.29
$n = 40$	919.73	941.55	962.77	983.44	1003.59
$n = 50$	924.19	943.72	962.77	981.36	999.53
$n = 100$	935.35	949.18	962.77	976.12	989.24
$n = 200$	943.34	953.12	962.77	972.30	981.72

Figure 6 represents the values of natural frequencies of the beam in a function of variable values of the parameter n (natural exponent in Eq. (2.14)). It is affirmed that negative values of α_f tend to increase the natural frequency while for positive values of α_f the situation is opposite (with simultaneous increase of the parameter n). The sing of α_f (the coefficient of the foundation reaction)

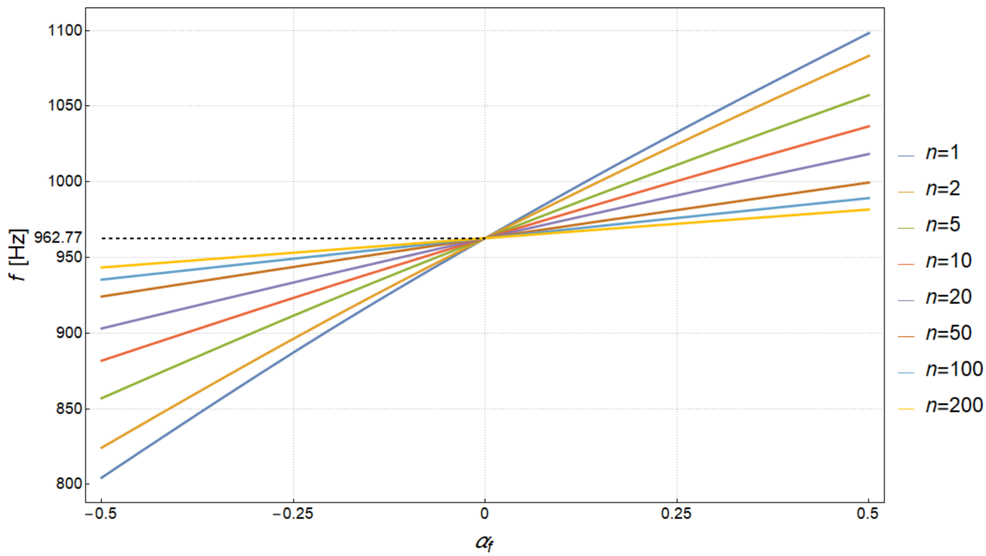

Fig. 6. The influence of n parameter on the values of beam natural frequency.

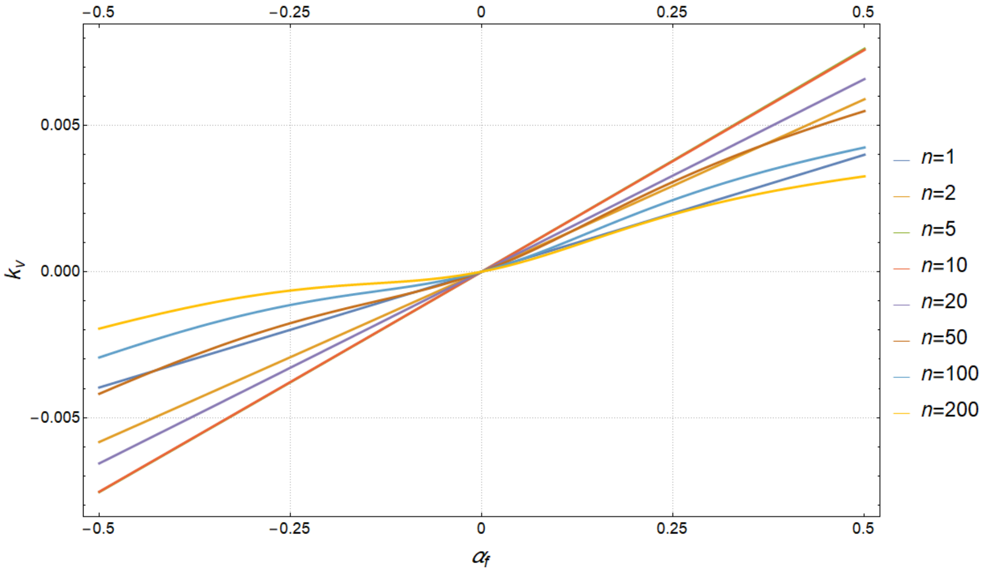

FIG. 7. The influence of n parameter on the values of k_v coefficient k_v.

plays a significant role in calculations. Based on the assumptions from Eq. (2.14), the minimum value of frequency should appear for $\alpha_f = -1$ while the maximum value – for $\alpha_f = 1$ $(n = 1)$.

Figure 7 demonstrates the values of the coefficient k_v for variable values of n parameter. It is apparent from the graph and from the calculations that k_v coefficient is closely dependent on the elastic foundation parameters. Formula (2.47) is a minimum frequency conception of a multi-layered beam. Elastic foundation parameters change the dynamic response of a sandwich beam. In addition, the non-linearity in Fig. 7 is the effect of low values of the coefficient k_v – these values stabilize for higher values of the parameter n.

3.2. Numerical studies

Numerical examination of a sandwich beam has begun with the convergent study and validation in order to verify the accuracy of adopted solutions. The simply supported and symmetrical (symmetry according to the centre line) sandwich beam was considered. The presented numerical results of natural frequencies included the influence of variable parameters of the elastic foundation. Material parameters and dimensions had the same values as those used in the analytical investigation.

Finite element investigation has been accomplished with the use of SolidWorks software. Bonded constraints have been adopted between the core and the faces. The elastic foundation has been substituted by the arrangement of

8 discrete elastic supports. For each elastic support, the value of stiffness has been calculated based on the formula (2.14). Analysis assumptions were as follows:

- number of nodes: 18525,
- number of finite elements: 10298,
- size of the individual element: 5 mm (the tolerance equal to 0.25 mm),
- type of finite elements: SOLID (the second order shape function),
- form of finite elements: tetrahedron (10 nodes),
- type of contact between the core and the faces: bonded with a compatible mesh.

Table below compares the analytical and numerical values of frequency in the sandwich beam on the elastic foundation. Analysis involved the shape function (2.14) where the exponent n was a natural number. The frequency values were dependent on α_f and n parameters. The highest results can be achieved for the highest values of α_f. In addition, for negative values of α_f, natural frequency values of the beam increase with the increase of n parameter, and decrease for positive values of α_f. The difference between analytical and numerical values of the fundamental natural frequency does not exceed 0.5%.

Table 5. Natural frequency $\frac{\omega}{2\pi}^{FE}$ [Hz] of sandwich beam on elastic foundation with variable properties with relative differences δ [%] to analytical calculations (in brackets).

	$\alpha_f = -0.50$	$\alpha_f = -0.25$	$\alpha_f = 0$	$\alpha_f = 0.25$	$\alpha_f = 0.50$
$n = 1$	801.19 (0.40)	883.54 (0.41)	958.75 (0.42)	1028.40 (0.43)	1093.60 (0.43)
$n = 2$	821.06 (0.38)	892.64 (0.40)	958.75 (0.42)	1020.40 (0.44)	1078.50 (0.44)
$n = 3$	835.00 (0.38)	899.11 (0.40)	958.75 (0.42)	1014.70 (0.43)	1067.60 (0.44)
$n = 4$	845.47 (0.38)	904.00 (0.40)	958.75 (0.42)	1010.30 (0.44)	1059.20 (0.45)
$n = 5$	853.70 (0.38)	907.86 (0.40)	958.75 (0.42)	1006.90 (0.43)	1052.60 (0.44)
$n = 10$	878.35 (0.39)	919.55 (0.40)	958.75 (0.42)	996.19 (0.43)	1032.10 (0.44)
$n = 15$	891.25 (0.39)	925.72 (0.40)	958.75 (0.42)	990.48 (0.43)	1021.00 (0.45)
$n = 20$	899.49 (0.39)	929.68 (0.41)	958.75 (0.42)	986.79 (0.43)	1013.90 (0.44)
$n = 30$	909.72 (0.40)	934.63 (0.41)	958.75 (0.42)	982.13 (0.43)	1004.80 (0.44)
$n = 40$	916.04 (0.40)	937.70 (0.41)	958.75 (0.42)	979.23 (0.43)	999.18 (0.44)
$n = 50$	920.42 (0.41)	939.83 (0.41)	958.75 (0.42)	977.20 (0.42)	995.21 (0.43)
$n = 100$	931.53 (0.41)	945.27 (0.41)	958.75 (0.42)	971.99 (0.42)	984.99 (0.43)
$n = 200$	940.21 (0.33)	949.54 (0.38)	958.75 (0.42)	967.84 (0.46)	976.82 (0.50)

Example of the beam vibration mode is shown in figure below (Fig. 8). The system is conservative if the energy is maintained; the algebraic sum of potential and kinetic energy is constant during the motion. It can be acknowledged that all unforced constructions vibrate harmonically. Discrete frequencies which appear at this time are the natural frequencies of a system. Harmonic vibration appears for simple systems, such as the spring – mass system. Notwithstanding, more complex structures, such as the sandwich beam, have more than one natural frequencies so they do not vibrate harmonically. Majority of vibration issues for multi-layered beams have similar equations of motion. Therefore, these equations might be solved once and used to describe alternative and similar approaches.

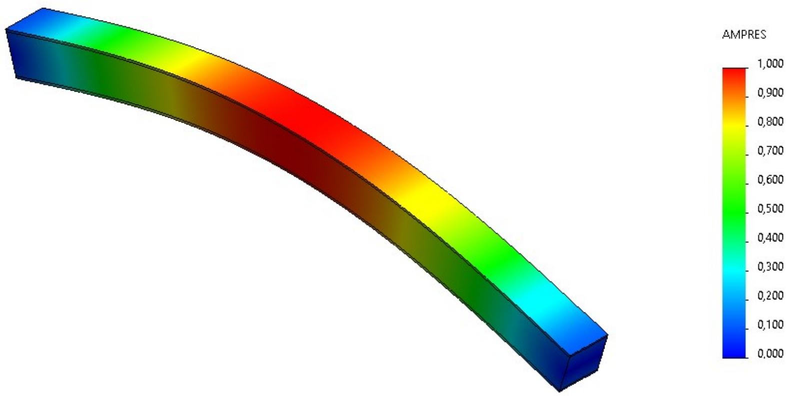

Fig. 8. Example of the beam vibration mode for $\alpha_f = 0.50$ and $n = 1$. Amplitude of vibration is normalized due to maximum displacements.

Natural frequency of the beam is strictly related to the stiffness of the nonlinear elastic foundation. The values being tested increase with the increase of the foundation stiffness. Increase of α_f parameter induces the increase of frequency. The use of the proposed analytical calculations in conjunction with the numerical method allows for a comprehensive analysis of beams on elastic foundations.

4. Conclusions

The main objective of above work was free flexural vibration of a simply supported sandwich beam resting on an elastic foundation. The studies aimed to determine the influence of variable properties of the elastic foundation on the dynamic response of the multi-layered beam. The elastic foundation being tested has been described with the use of an adequate mathematical function (2.14). Alternating parameters of the function have been taken into account: n – natural exponent and α_f – the coefficient of the foundation reaction, which is a novel approach to the analysis of structures on elastic foundations (continuous description of beam-foundation interaction). The investigation confirmed that variable parameters of the elastic foundation have a crucial effect on the dynamic properties of the sandwich beam.

In addition, numerical analysis has been carried out. Sample analytical and numerical calculations have been performed, sharing a good convergence between the results acquired with both models. The disparity between analytical and numerical values of the fundamental natural frequency did not exceed 0.5%.

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