Opuscula Math. 36, no. 2 (2016), 253–264
http://dx.doi.org/10.7494/OpMath.2016.36.2.253 Opuscula Mathematica

MULTIPLE SOLUTIONS FOR FOURTH ORDER ELLIPTIC PROBLEMS WITH $p(x)$-BIHARMONIC OPERATORS

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Communicated by Marius Ghergu

Abstract. We study the multiplicity of weak solutions to the following fourth order nonlinear elliptic problem with a $p(x)$-biharmonic operator

$$\begin{cases} \Delta^2_{p(x)} u + a(x)|u|^{p(x)-2}u = \lambda f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $p \in C(\overline{\Omega})$, $\Delta^2_{p(x)} u = \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is the $p(x)$-biharmonic operator, and $\lambda > 0$ is a parameter. We establish sufficient conditions under which there exists a positive number λ^* such that the above problem has at least two nontrivial weak solutions for each $\lambda > \lambda^*$. Our analysis mainly relies on variational arguments based on the mountain pass lemma and some recent theory on the generalized Lebesgue–Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$.

Keywords: critical points, $p(x)$-biharmonic operator, weak solutions, mountain pass lemma.

Mathematics Subject Classification: 35J66, 35J40, 35J92, 47J10.

1. INTRODUCTION

In this paper, we are interested in the existence of multiple weak solutions of the following fourth order nonlinear elliptic problem with a $p(x)$-biharmonic operator

$$\begin{cases} \Delta^2_{p(x)} u + a(x)|u|^{p(x)-2}u = \lambda f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ with $N \geq 1$ is a bounded domain with smooth boundary, $p \in C(\overline{\Omega})$ with $p(x) > N$ on $\overline{\Omega}$, $a \in C(\overline{\Omega})$ is positive, $\lambda > 0$ is a parameter, $f \in C(\overline{\Omega} \times \mathbb{R})$, and

$\Delta^2_{p(x)}u = \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is the so-called $p(x)$-biharmonic operator. It is known that problems with $p(x)$-growth conditions have more complicated nonlinearities than the constant case. For instance, it is not homogeneous, and thus some techniques which can be applied when $p(x)$ is a positive constant, such as the Lagrange Multiplier Theorem, will fail in this new situation. The study of differential equations and variational problems with nonstandard $p(x)$-growth conditions is an interesting and attractive topic and has been the object of considerable attention in recent years. We refer the reader to [1,2,5–7,12–14,18,19] for some recent work on this subject. The reason for such an interest relies on the fact that they have many applications in mathematical physics such as in the modelling of electrorheological fluids and of other phenomena related to image processing, elasticity, and the flow in porous media ([10,20,22]).

Recently, problem (1.1) and its variations have been studied in the literature. For instance, the problem

$$\begin{cases} \Delta^2_{p(x)}u = \lambda|u|^{q(x)-2}u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

has been studied by Ayoujil and El Amrouss in [1] when $p(x) = q(x)$ and in [2] when $p(x) \neq q(x)$. In particular, in [1], by the Ljusternik-Schnirelmann principle on C^1-manifolds, the authors proved among others things the existence of a sequence of eigenvalues and that $\sup \Lambda = \infty$, where Λ is the set of all nonnegative eigenvalues. In [2], using the mountain pass lemma and Ekeland's variational principle, the authors further established several existence criteria for eigenvalues. In [13], the present author studied the existence of at least one weak solution to the problem

$$\begin{cases} \Delta^2_{p(x)}u + a(x)|u|^{p(x)-2}u = \lambda\left(b(x)u^{\gamma(x)-1} - c(x)u^{\beta(x)-1}\right) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases}$$

for $\lambda > 0$ large enough by applying variational arguments. In [14], the existence of at least one weak solution was obtained for problem (1.1) for $\lambda > 0$ sufficiently small.

We point out that when $p(x)$ is a positive constant, a number of variations of problem (1.2) have been investigated in the literature. See, for example, [3,8,9,16,17] and the references therein.

In this paper, by applying variational arguments based on the Ambrosetti and Rabinowitz's mountain pass lemma and the theory of the generalized Lebesgue-Sobolev spaces, we study the existence of at least two nontrivial weak solutions for problem (1.1). More precisely, under appropriate conditions, we show that there exists a positive number λ^* such that problem (1.1) has at least two nontrivial weak solutions for each $\lambda > \lambda^*$.

The rest of this paper is organized as follows. Section 2 contains some preliminary lemmas on the generalized Lebesgue–Sobolev spaces and Section 3 contains the main result of this paper and its proof.

2. PRELIMINARY RESULTS

In this section, we recall some definitions and basic properties of variable spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$, where Ω is given as in problem (1.1). The presentation here can be found in, for example, [1, 4–7, 15, 21].

Throughout this paper, we use the notations

$$h^+ := \max_{x\in\overline{\Omega}} h(x) \text{ and } h^- := \min_{x\in\overline{\Omega}} h(x) \quad \text{for } h \in C(\overline{\Omega}),$$

and

$$C_+(\overline{\Omega}) := \left\{h \ : \ h \in C(\overline{\Omega}) \text{ and } h(x) > 1 \text{ on } \overline{\Omega}\right\}.$$

Let $p \in C_+(\overline{\Omega})$ be fixed. We define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{ u \ : \ u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}.$$

Then, $L^{p(x)}(\Omega)$ is a separable and reflexive Banach space equipped with the so-called Luxemburg norm

$$|u|_{p(x)} = \inf \left\{ \lambda > 0 \ : \ \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

Clearly, when $p(x) = p$, a positive constant, the space $L^{p(x)}(\Omega)$ reduces to the classic Lebesgue space $L^p(\Omega)$ and the norm $|u|_{p(x)}$ reduces to the standard norm $\|u\|_{L^p} = \left(\int_\Omega |u|^p\right)^{1/p}$ in $L^p(\Omega)$.

For any positive integer k, as in the constant exponent case, let

$$W^{k,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) \ : \ D^\alpha u \in L^{p(x)}(\Omega), \ |\alpha| \leq k \right\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index, $|\alpha| = \sum_{i=1}^N \alpha_i$, and $D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial^{\alpha_1} x_1 \ldots \partial^{\alpha_N} x_N}$. Then, $W^{k,p(x)}(\Omega)$ is a separable and reflexive Banach space equipped with the norm

$$\|u\|_{k,p(x)} = \sum_{|\alpha|\leq k} |D^\alpha u|_{p(x)}.$$

We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$.

In the sequel, we let

$$X = W_0^{1,p(x)}(\Omega) \cap W^{2,p(x)}(\Omega).$$

Define a norm $\|\cdot\|_X$ of X by

$$\|u\|_X = \|u\|_{1,p(x)} + \|u\|_{2,p(x)}.$$

Then, endowed with $\|\cdot\|_X$, X is a separable and reflexive Banach space. Moreover, $\|u\|_X$ and $|\Delta u|_{p(x)}$ are two equivalent norms of X by [21, Theorem 4.4].

Let

$$\|u\|_a = \inf\left\{\lambda > 0 \ : \ \int\limits_\Omega \left(\left|\frac{\Delta u(x)}{\lambda}\right|^{p(x)} + a(x)\left|\frac{u(x)}{\lambda}\right|^{p(x)}\right) dx \le 1\right\} \quad \text{for } u \in X.$$

Then, since $a^- > 0$, $\|u\|_a$ is equivalent to the norms $\|u\|_X$ and $|\Delta u|_{p(x)}$ in X. In this paper, for the convenience of discussion, we use the norm $\|u\|_a$ for X.

Proposition 2.1 ([5, Proposition 2.3]). *Let* $\rho_a(u) = \int\limits_\Omega \left(|\Delta u(x)|^{p(x)} + a(x)|u(x)|^{p(x)}\right) dx$ *for* $u \in X$. *Then, we have*

(a) *if* $\|u\|_a \ge 1$, *then* $\|u\|_a^{p^-} \le \rho_a(u) \le \|u\|_a^{p^+}$;
(b) *if* $\|u\|_a \le 1$, *then* $\|u\|_a^{p^+} \le \rho_a(u) \le \|u\|_a^{p^-}$.

Proposition 2.2 ([6, Propositions 2.4 and 2.5] or [15, Theorem 2.1 and Corollary 2.7]). *The conjugate space of* $L^{p(x)}(\Omega)$ *is* $L^{q(x)}(\Omega)$, *where* $1/p(x) + 1/q(x) = 1$. *Moreover, for* $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{q(x)}(\Omega)$, *we have the following inequality of Hölder type*

$$\left|\int\limits_\Omega u(x)v(x)dx\right| \le \left(\frac{1}{p^-} + \frac{1}{q^-}\right)|u|_{p(x)}|v|_{q(x)} \le 2|u|_{p(x)}|v|_{q(x)}.$$

For any $x \in \overline{\Omega}$, let

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-2p(x)} & \text{if } p(x) < \frac{N}{2}, \\ \infty & \text{if } p(x) \ge \frac{N}{2}. \end{cases}$$

Proposition 2.3 ([1, Theorem 3.2]). *Assume that* $q \in C_+(\overline{\Omega})$ *satisfy* $q(x) < p^*(x)$ *on* $\overline{\Omega}$. *Then, there exists a continuous and compact embedding* $X \hookrightarrow L^{q(x)}(\Omega)$.

3. MAIN RESULT

Let

$$F(x,t) = \int\limits_0^t f(x,s)ds \quad \text{for } (x,t) \in \Omega \times \mathbb{R}.$$

We need the following assumptions.

(H1) $\lim_{|t|\to 0} \frac{|F(x,t)|}{|t|^{p(x)}} = 0$ uniformly for $x \in \Omega$;
(H2) $\limsup_{|t|\to\infty} \frac{F(x,t)}{|t|^{p(x)}} \le 0$ uniformly for $x \in \Omega$;
(H3) there exists $w \in X$ such that $\int\limits_\Omega F(x, w(x))dx > 0$.

We say that $u \in X$ is a weak solution of problem (1.1) if

$$\int_\Omega |\Delta u(x)|^{p(x)-2}\Delta u(x)\Delta v(x)dx + \int_\Omega a(x)|u(x)|^{p(x)-2}u(x)v(x)dx \\ -\lambda \int_\Omega f(x,u(x))v(x)dx = 0$$

for all $v \in X$.

We now state our main theorem.

Theorem 3.1. *Assume that* (H1)–(H3) *hold. Then, problem* (1.1) *has at least two nontrivial weak solutions for each* $\lambda > \lambda^*$, *where*

$$\lambda^* = \frac{\int_\Omega \left(|\Delta w(x)|^{p(x)} + a(x)|w(x)|^{p(x)}\right) dx}{p^- \int_\Omega F(x,w(x))dx}. \tag{3.1}$$

The following example illustrate the applicability of Theorem 3.1.

Example 3.2. In problem (1.1), let

$$f(x,t) = \begin{cases} \eta|t|^{\eta-2}t, & |t| \geq 1, \\ \zeta|t|^{\zeta-2}t, & |t| < 1, \end{cases}$$

where $0 < \eta < p^- \leq p^+ < \zeta < \infty$. Then, we claim that, for $\lambda > 0$ large enough, problem (1.1) has at least two nontrivial weak solutions.

Clearly, for f defined above, we have

$$F(x,t) = \begin{cases} |t|^\eta, & |t| \geq 1, \\ |t|^\zeta, & |t| < 1. \end{cases}$$

Thus, (H1) and (H2) trivially hold. Moreover, (H3) also holds for any $w \in X$ with w being nonzero in a subset of Ω of positive measure. Therefore, the claim readily follows from Theorem 3.1.

In the rest of this section, we will prove Theorem 3.1. First, recall that a functional $I \in C^1(X,\mathbb{R})$ is said to satisfy the Palais–Smale (PS) condition if every sequence $\{u_n\} \subset X$, such that $I(u_n)$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence. Here, the sequence $\{u_n\}$ is called a PS sequence of I.

In our proof, we need the following classic mountain pass lemma of Ambrosetti and Rabinowitz (see, for example, [11, Theorem 7.1]). Below, we denote by $B_r(u)$ the open ball centered at $u \in X$ with radius $r > 0$, $\overline{B}_r(u)$ its closure, and $\partial B_r(u)$ its boundary.

Lemma 3.3. *Let $(X, \|\cdot\|)$ be a real Banach space and $I \in C^1(X, \mathbb{R})$. Assume that I satisfies the PS condition and there exist $u_0, u_1 \in X$ and $\rho > 0$ such that*

(A1) $u_1 \notin \overline{B}_\rho(u_0)$;
(A2) $\max\{I(u_0), I(u_1)\} < \inf_{u \in \partial B_\rho(u_0)} I(u)$.

Then, I possesses a critical value which can be characterized as

$$c = \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} I(\gamma(s)) \geq \inf_{u \in \partial B_\rho(u_0)} I(u),$$

where

$$\Gamma = \{\gamma \in C([0,1], X) \ : \ \gamma(0) = u_0, \ \gamma(1) = u_1\}.$$

Now, define functionals $\Phi, \Psi, I_\lambda : X \to R$ by

$$\Phi(u) = \int_\Omega \frac{1}{p(x)} \left(|\Delta u(x)|^{p(x)} + a(x)|u(x)|^{p(x)} \right) dx,$$

$$\Psi(u) = \int_\Omega F(x, u(x)) dx,$$

and

$$I_\lambda(u) = \Phi(u) - \lambda \Psi(u).$$

Lemma 3.4. ([5, Propositions 2.5] and [14, Lemma 2.1]) *We have the following:*

(a) *Φ is weakly lower semicontinuous, $\Phi \in C^1(X, \mathbb{R})$, and*

$$\langle \Phi'(u), v \rangle = \int_\Omega |\Delta u(x)|^{p(x)-2} \Delta u(x) \Delta v(x) dx + \int_\Omega a(x)|u(x)|^{p(x)-2} u(x) v(x) dx$$

for all $u, v \in X$;

(b) *$\Phi'(u) : X \to X^*$ is of type (S_+), i.e., if $u_n \rightharpoonup u$ and $\liminf_{n\to\infty} \langle \Phi'(u_n), u_n - u \rangle \leq 0$, then $u_n \to u$, where X^* is the dual space of X;*

(c) *Ψ is weakly lower semicontinuous, $\Psi \in C^1(X, \mathbb{R})$, and*

$$\langle \Psi'(u), v \rangle = \int_\Omega f(x, u(x)) v(x) dx$$

for all $u, v \in X$.

Remark 3.5. In view of Lemma 3.4 (a) and (c), $I_\lambda \in C^1(X, \mathbb{R})$ and

$$\langle I'_\lambda(u), v \rangle = \int_\Omega |\Delta u(x)|^{p(x)-2} \Delta u(x) \Delta v(x) dx + \int_\Omega a(x)|u(x)^{p(x)-2} u(x) v(x) dx$$
$$- \lambda \int_\Omega f(x, u(x)) v(x) dx$$

for all $v \in X$. Thus, u is a critical point of I_λ if and only if u is a weak solution of problem (1.1).

Lemma 3.6. *Assume that* (H2) *holds. Then, for any $\lambda > 0$, the functional I_λ is coercive and satisfies the PS condition.*

Proof. Let $\lambda > 0$ be fixed. We first show that I_λ is coercive, i.e.,

$$\lim_{\|u\|\to\infty} I_\lambda(u) = \infty \quad \text{for any } u \in X. \tag{3.2}$$

From (H2), there exists $C > 0$ such that

$$F(x,t) \leq \epsilon |t|^{p(x)} \quad \text{for } (x,t) \in \Omega \times \mathbb{R} \text{ with } |t| > C, \tag{3.3}$$

where

$$0 < \epsilon < \frac{a^-}{\lambda p^+}. \tag{3.4}$$

On the other hand, from the continuity of f, there exists $l \in L^1(\Omega, \mathbb{R}^+)$ such that

$$|F(x,t)| \leq l(x) \quad \text{for } (x,t) \in \Omega \times \mathbb{R} \text{ with } |t| \leq C. \tag{3.5}$$

For any $u \in X$ with $\|u\|_a \geq 1$, let $\Omega_1 = \{x \in \Omega \ : \ |u(x)| > C\}$ and $\Omega_2 = \{x \in \Omega \ : \ |u(x)| \leq C\}$. Then, from Proposition 2.1 (a), (3.3), and (3.5), it follows that

$$\begin{aligned}
I_\lambda(u) &\geq \frac{1}{p^+} \int\limits_\Omega \left(|\Delta u(x)|^{p(x)} + a(x)|u(x)|^{p(x)} \right) dx - \lambda \int\limits_{x\in\Omega_1} F(x,u(x))dx \\
&\quad - \lambda \int\limits_{x\in\Omega_2} F(x,u(x))dx \\
&\geq \frac{1}{p^+} \int\limits_\Omega \left(|\Delta u(x)|^{p(x)} + a(x)|u(x)|^{p(x)} \right) dx - \lambda\epsilon \int\limits_{x\in\Omega_1} |u(x)|^{p(x)}dx \\
&\quad - \lambda \int\limits_{x\in\Omega_2} l(x)dx \\
&\geq \frac{1}{p^+} \int\limits_\Omega \left(|\Delta u(x)|^{p(x)} + a(x)|u(x)|^{p(x)} \right) dx - \lambda\epsilon \int\limits_{x\in\Omega} |u(x)|^{p(x)}dx - \lambda\|l\|_{L^1} \\
&= \frac{1}{p^+} \int\limits_\Omega |\Delta u(x)|^{p(x)}dx + \int\limits_\Omega \left(\frac{a(x)}{p^+} - \lambda\epsilon \right) |u(x)|^{p(x)}dx - \lambda\|l\|_{L^1} \\
&\geq \min\left\{ \frac{1}{p^+}, \frac{a^-}{p^+} - \lambda\epsilon \right\} \int\limits_\Omega \left(|\Delta u(x)|^{p(x)} + |u(x)|^{p(x)} \right) dx - \lambda\|l\|_{L^1} \\
&\geq \min\left\{ \frac{1}{p^+}, \frac{a^-}{p^+} - \lambda\epsilon \right\} \min\left\{ 1, \frac{1}{a^+} \right\} \int\limits_\Omega \left(|\Delta u(x)|^{p(x)} + a(x)|u(x)|^{p(x)} \right) dx \\
&\quad - \lambda\|l\|_{L^1} \\
&\geq \min\left\{ \frac{1}{p^+}, \frac{a^-}{p^+} - \lambda\epsilon \right\} \min\left\{ 1, \frac{1}{a^+} \right\} \|u\|_a^{p^-} - \lambda\|l\|_{L^1}.
\end{aligned}$$

Note from (3.4) that $a^-/p^+ - \lambda\epsilon > 0$. Then, $I_\lambda(u) \to \infty$ as $\|u\|_a \to \infty$, i.e., (3.2) holds.

We now show that I_λ satisfies the PS condition. Assume that $\{u_n\} \subset X$ is a PS sequence of I_λ. Then, we have

$$I(u_n) \text{ is bounded and } I'(u_n) \to 0 \text{ as } n \to \infty. \tag{3.6}$$

In view of (3.2) and (3.6), $\{u_n\}$ is bounded in X. Thus, by the reflexivity of X, there exists $u_0 \in X$ such that, up to a subsequence, $u_n \rightharpoonup u_0$ in X. Note that

$$\begin{aligned}|\langle I'_\lambda(u_n), u_n - u_0\rangle| &\le |\langle I'_\lambda(u_n), u_n\rangle| + |\langle I'_\lambda(u_n), u_0\rangle| \\ &\le \|\langle I'_\lambda(u_n)\| \, \|u_n\| + \|\langle I'_\lambda(u_n)\| \, \|u_0\|.\end{aligned}$$

Hence, from (3.6) and the fact that $\{u_n\}$ is bounded in X, it follows that

$$\lim_{n\to\infty} \langle I'_\lambda(u_n), u_n - u_0\rangle = 0. \tag{3.7}$$

Now, we claim that

$$\lim_{n\to\infty} \langle \Psi'(u_n), u_n - u_0\rangle = 0. \tag{3.8}$$

In fact, from Proposition 2.2, we have

$$\begin{aligned}|\langle \Psi'(u_n), u_n - u_0\rangle| &= \left| \int_\Omega f(x, u_n(x))(u_n(x) - u_0(x))dx \right| \\ &\le 2|f(\cdot, u_n(\cdot))|_{\frac{p(x)}{p(x)-1}} |u_n - u_0|_{p(x)}.\end{aligned} \tag{3.9}$$

By [7, Theorem 2.2], for $i = 1, 2$, $W^{i,p(x)}(\Omega)$ can be embedded into $W^{i,p^-}(\Omega)$ continuously. Thus, note that $p^- > N$, we see that there exists a compact embedding of X into $C^0(\overline{\Omega})$, i.e., there exists a constant $\kappa > 0$ such that

$$\sup_{x\in\overline{\Omega}} |u(x)| \le \kappa \|u\|_a \quad \text{for all } u \in X. \tag{3.10}$$

By (3.10) and the fact that $\{u_n\}$ is bounded in X, we have

$$\sup_{x\in\overline{\Omega}} |u_n(x)| < K < \infty \quad \text{for } n \in \mathbb{N} \text{ and some } K > 0.$$

Hence, by (3.9), the continuous and compact embedding of $X \hookrightarrow L^{p(x)}(\Omega)$ (see Proposition 2.3), and the fact that $f \in C(\overline{\Omega} \times \mathbb{R})$, we see that (3.8) holds. Now, from (3.7) and (3.8), we conclude that

$$\lim_{n\to\infty} \langle \Phi'(u_n), u_n - u_0\rangle = 0.$$

Thus, by Lemma 3.4 (b), we have $u_n \to u_0$ in X. Then, I_λ satisfies the PS condition. This completes the proof of the lemma. □

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. We first show that, for each $\lambda > 0$, 0 is a strict local minimizer of I_λ. It is obvious that

$$I_\lambda(0) = \Phi(0) - \lambda\Psi(0) = 0.$$

For ϵ satisfying (3.4), by (H1), there exists $D > 0$ such that

$$|F(x,t)| \leq \epsilon |t|^{p(x)} \quad \text{for } (x,t) \in \Omega \times \mathbb{R} \text{ with } |t| \leq D. \tag{3.11}$$

Note that (3.10) holds. Then, there exists $\varrho > 0$ such that

$$\sup_{x\in\overline{\Omega}} |u| \leq D \quad \text{for all } u \in B_\varrho(0).$$

Hence, for any $u \in B_\varrho(0) \setminus \{0\}$, from Proposition 2.1 (a), (3.4), and (3.11), we have

$$\begin{aligned}
I_\lambda(u) &\geq \frac{1}{p^+}\int_\Omega \left(|\Delta u(x)|^{p(x)} + a(x)|u(x)|^{p(x)}\right)dx - \lambda \int_{x\in\Omega} F(x,u(x))dx \\
&\geq \frac{1}{p^+}\int_\Omega \left(|\Delta u(x)|^{p(x)} + a(x)|u(x)|^{p(x)}\right)dx - \lambda\epsilon \int_{x\in\Omega} |u(x)|^{p(x)}dx \\
&= \frac{1}{p^+}\int_\Omega |\Delta u(x)|^{p(x)}dx + \int_\Omega \left(\frac{a(x)}{p^+} - \lambda\epsilon\right)|u(x)|^{p(x)}dx \\
&\geq \min\left\{\frac{1}{p^+}, \frac{a^-}{p^+} - \lambda\epsilon\right\}\int_\Omega \left(|\Delta u(x)|^{p(x)} + |u(x)|^{p(x)}\right)dx \\
&\geq \min\left\{\frac{1}{p^+}, \frac{a^-}{p^+} - \lambda\epsilon\right\}\min\left\{1, \frac{1}{a^+}\right\}\int_\Omega \left(|\Delta u(x)|^{p(x)} + a(x)|u(x)|^{p(x)}\right)dx \\
&\geq \min\left\{\frac{1}{p^+}, \frac{a^-}{p^+} - \lambda\epsilon\right\}\min\left\{1, \frac{1}{a^+}\right\}\|u\|_a^{p^-} > 0.
\end{aligned}$$

Thus, for each $\lambda > 0$, 0 is a strict local minimizer of I_λ.

Let λ^* be defined by (3.1) and w be given in (H3). Then, it follows that

$$I_\lambda(w) \leq \frac{1}{p^-}\int_\Omega \left(|\Delta w(x)|^{p(x)} + a(x)|w(x)|^{p(x)}\right)dx - \lambda \int_{x\in\Omega} F(x,w(x))dx < 0$$

if $\lambda > \lambda^*$. Thus, 0 is not a global minimizer of I_λ when $\lambda > \lambda^*$.

In what follows, for any λ satisfying $\lambda > \lambda^*$, we show that I_λ has a global minimizer. Let $\mu \in \mathbb{R}$ such that $I_\lambda(w) < \mu < 0$. Define

$$Y = \{u \in X \ : \ I_\lambda(u) \leq \mu\}.$$

Then, $Y \neq \emptyset$ and is bounded since I_λ is coercive by Lemma 3.6. We claim that I_λ is bounded below on Y. Assume, to the contrary, that there exists a sequence $\{u_n\} \subset Y$ such that

$$\lim_{n\to\infty} I_\lambda(u_n) = -\infty. \tag{3.12}$$

Note that $\{u_n\}$ is bounded. Then, by the reflexivity of X, there exists $u_0 \in X$ such that, up to a subsequence, $u_n \rightharpoonup u_0$ in X. As in (3.10), from $p^- > N$, we know that that there exists a compact embedding of X into $C^0(\overline{\Omega})$. Hence, $u_n \to u_0$ in $C^0(\overline{\Omega})$. Therefore, we have

$$\lim_{n\to\infty} I_\lambda(u_n) = \lim_{n\to\infty}(\Phi(u_n) - \lambda\Psi(u_n)) = \Phi(u_0) - \lambda\Psi(u_0) = I_\lambda(u_0) > -\infty.$$

This contradicts (3.12). Thus,

$$0 > \nu := \inf_{u\in Y} I_\lambda(u) = \inf_{u\in X} I_\lambda(u) > -\infty.$$

Let $\{u_n\} \subset Y$ be a sequence such that

$$\lim_{n\to\infty} I_\lambda(u_n) = \nu.$$

Arguing as above, we see that there exists $u_1 \in X$ such that, up to a subsequence, $u_n \to u_1$ in $C^0(\overline{\Omega})$. Hence, we have

$$I_\lambda(u_1) = \nu < 0, \tag{3.13}$$

and so $u_1 \not\equiv 0$. Clearly, u_1 is a critical point of I_λ. Then, by Remark 3.5, u_1 is a nontrivial solution of problem (1.1).

Next, we apply Lemma 3.3 to find a second critical point of I_λ when $\lambda > \lambda^*$. By Lemma 3.6, I_λ satisfies the PS condition. Since 0 is a strict local minimizer of I_λ, there exists $0 < \rho < \|u_1\|_a$ such that

$$r := \inf_{u\in\partial B_\rho(u_0)} I_\lambda(u) > 0.$$

Then, from the fact that $I_\lambda(0) = 0$ and (3.13) holds, we see that all the conditions of Lemma 3.3 are satisfied with $u_0 = 0$ and the above u_1. Hence, Lemma 3.3 implies that there exists a critical point u_2 of I_λ such that

$$I_\lambda(u_2) \geq r > 0. \tag{3.14}$$

By (3.13) and (3.14), we see that $u_1 \neq u_2$ and $u_2 \not\equiv 0$. Thus, by Remark 3.5, u_2 is a second nontrivial solution of problem (1.1). This completes the proof of the theorem. □

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Received: May 14, 2015.
Accepted: August 14, 2015.