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ORIGINAL PAPER

OPTIMISATION OF THE STOCK STRUCTURE OF A SINGLE STOCK ITEM TAKING INTO ACCOUNT STOCK QUANTITY CONSTRAINTS, USING A LAGRANGE MULTIPLIER

Stanisław Krzyżaniak

Poznań School of Logistics, Poznań, **Poland**

ABSTRACT. Background: Optimisation in the area of stock management is most often performed in relation to cycle stock. The classic example here is the Harris-Wilson formula for calculating the economic order quantity. Often these models are not subject to any constraints imposed on the optimised quantities. However, in practice, taking such constraints into account is important. The application of the so-called Lagrange multiplier is helpful here, but the examples of its application usually refer to the multi-position sets of stock items (e.g. the search for the optimum structure of stock of material groups in the case of capital constraints). This paper attempts to optimise the structure of the stock (cycle stock vs. safety stock) for a single stock item.

Methods: To achieve the objective of determining the optimum stock structure for the various conditions under which stock replenishment is implemented, a general model has been built, a component of which is a Lagrange function containing the constraint conditions for the solution. Next, this model has been implemented in the form of an EXCEL spreadsheet application.

Results: The result of solving the optimization task based on the proposed model is a system of equations, the solution of which (with the help of the EXCEL application) allows to determine the optimum value of the Lagrange multiplier, on the basis of which the components of the inventory structure and other related quantities (service level indicators and costs, such as stock replenishment, stock maintenance and stock deficit costs) are calculated. This has been illustrated using a fictitious example, which at the same time made it possible to observe certain general relationships between the adopted constraints and the recorded quantities.

Conclusions: Two types of conclusions can be presented. The first type concerns the approach itself. The possibility of determining the optimum structure of the stock (cycle stock vs. safety stock) depending on various values characterising the adopted stock replenishment system as well as the adopted limitations has been demonstrated. The second type of conclusions results from the presented example of application of the method for the assumed ranges of changes of selected quantities.

Keywords: stock optimization, Lagrange multiplier, service level, stock related costs

INTRODUCTION

Commonly applied models of costs related to stock cover their three groups: stock replenishment, stock maintenance, and stock deficit costs [e.g., Korponai et al., 2017]. It allows optimising the value of parameters controlling stock replenishment [e.g., Samak-Kulkarnia, Rajhansb, 2013]. The objective function adopted in the considerations is the total cost including the above-mentioned cost groups. The cost of stock maintenance and replenishment has been limited to variable costs since all fixed costs, even if included in the model, would disappear after differentiation, as independent of the assumed independent variables. These variables are two quantities that impact both the components considered in the stock of a given stock item: the quantity of order (delivery) quantity q (which affects the cycle part) and the safety factor ω determining the quantity of the safety stock. The optimization of delivery quantity is often applied, and its classic example is the commonly used formula for the so-called economic order size, first presented by F.W.

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Harris [Harris 1913] (despite significant limitations in its application) . The case is different when it comes to optimising the level of service and, more generally, the safety coefficient. Usually these values are determined arbitrarily, on the basis of experience, customers' requirements or by comparison with competitors. Meanwhile, if the costs accompanying the occurrence of stock deficit are known (both related to the occurrence itself and to the deficit quantity), it is possible to determine the optimum value of the safety coefficient, ensuring minimisation of the sum of the expected annual cost of occurrence of stock deficit and the annual cost of maintaining the safety stock.

Figure 1 shows a synthesis of the research problem presented in this paper. For comparison, it shows the problem in the case of no constraints and the situation where some constraints related to the allowable stock size are imposed on the objective function.

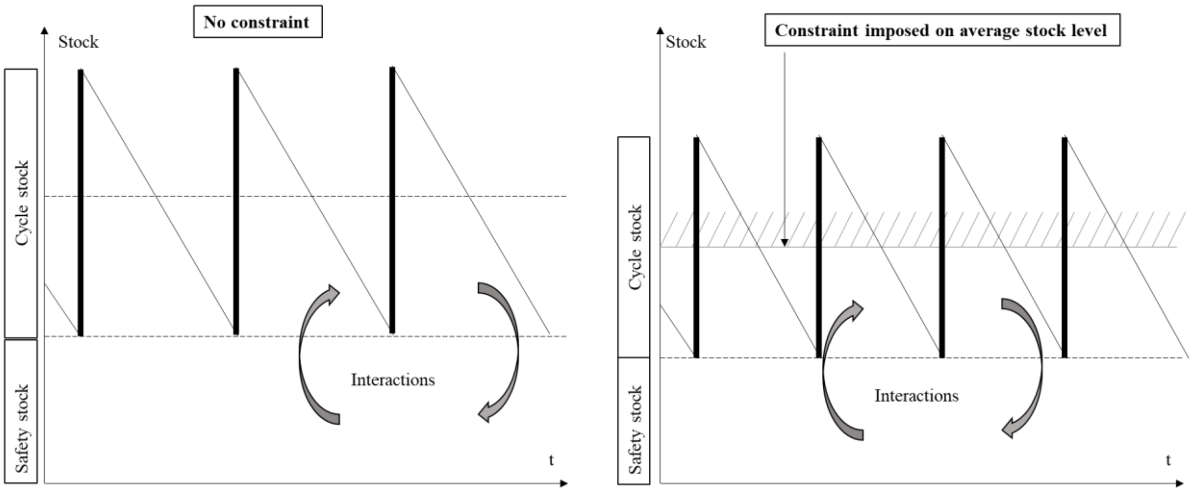

Fig. 1 Synthesis of the research problem - optimization of cycle and safety stock in the absence of constraints and in the situation of imposition of constraints (here - the maximum level of average stock size).

Source: own study.

It is worth noting that there are interactions between the optimal size of cyclic stock (optimization of delivery quantity q) and the optimum level of the safety stock (optimization of safety coefficient ω). For example, decreasing the cycle stock (by decreasing the delivery quantity) results in the need to increase the number of orders, and this increases the expected cost of stock deficit, resulting in the need to increase the safety stock. On the other hand, decreasing the safety stock increases the risk of shortage in the replenishment cycle and forces an increase in the volume of deliveries and consequently in the cycle stock.

TOTAL STOCK COST MODEL WITH CONSIDERATION OF CONSTRAINTS

It has been assumed that the stock is replenished in the Reorder Point system, with a fixed order (delivery) quantity. This system, in the terminology of the European Logistics Association, is denoted as BQ [European Logistics Association 1994].

The model applies the following designations related to demand, costs, and service level:

D – demand in a time unit (e.g., daily/weekly demand),

σ_D – standard deviation of demand in an adopted time unit,

σ_{DLT} – standard deviation of demand in a stock replenishment cycle of mean LT,

D_a – annual total demand,

cc_a – annual stock-carrying cost coefficient,

ω – safety coefficient – directly influencing service level:

αSL – service level (probability of a non-occurrence of stock deficit in its replenishment period, probability to serve demand in a cycle), corresponding to safety coefficient ω treated as an independent variable,

FR (fill rate) - as a percentage realization of demand in a quantitative approach

q – order/delivery quantity – independent variable,

c_r – unit cost of stock replenishment (cost of order, organisation, and execution of a single delivery),

cd_1 cost related to stock deficit occurrence during the stock replenishment cycle,

cd_2 – cost related to stock deficit occurrence in relation to one missing piece of the stock item,

p_u – purchase price (variable production cost) of a unit of the discussed stock item,

nd_a – number of orders (delivery) per year.

The considerations concern stock replenishment on the basis of reorder level, with a fixed delivery quantity. A cost model covering the stock replenishment, stock carrying, and stock deficit of stock will be the starting point.

$$TC = \frac{D_a}{q} \cdot c_r + \frac{1}{2} q \cdot p_u \cdot cc_a + \omega \cdot \sigma_{DLT} \cdot p_u \cdot cc_a + cd_1 \cdot [1 - F(\omega)] \cdot \frac{D_a}{q} + cd_2 \cdot I(\omega) \cdot \sigma_{D,LT} \cdot \frac{D_a}{q} \quad (1)$$

TC	Overall cost
$\frac{D_a}{q} \cdot c_r$	Annual replenishment cost
$\frac{1}{2} q \cdot p_u \cdot cc_a$	Average cost of carrying cycle stock
$\omega \cdot \sigma_{DLT} \cdot p_u \cdot cc_a$	Periodical (e.g., annual) cost of carrying safety stock
$cd_1 \cdot [1 - F(\omega)] \cdot \frac{D_a}{q}$	Stock deficit cost resulting from stock deficit occurrence probability in a replenishment cycle
$cd_2 \cdot I(\omega) \cdot \sigma_{D,LT} \cdot \frac{D_a}{q}$	Stock deficit cost resulting from deficit volume

This formula should now be extended by adding one more component:

$$+\lambda \cdot \left(\frac{1}{2} \cdot q \cdot c + \omega \cdot \sigma_{DLT} \cdot c - C\right) \quad (2)$$

The product of the Lagrange multiplier λ by the assumed constraint.

There are many examples of an application of Lagrange multiplier for stock optimization. Yassa R. I, Ikatrinasari Z.F. (2019) used it for calculations of multi-item Economic Order Quantity. Fergany H. A., Gomaa M. A (2018) applied a Lagrange multiplier based model to analyse how to deduce the optimal order quantity and the optimal reorder point to reach a minimal expected total cost. Lukitosari V. et al. (2019) used the Lagrange multiplier method to solve inventory model for spare parts. Examples of application of Lagrange multiplier for stock optimisation can be also found in a review paper by Hoswari S. et al (2020).

The product (2) is always equal to zero, which follows from the definition of the Lagrange multiplier [e.g., Kowiger 2012]; in the discussed case:

$$\lambda = 0 \quad if \quad \left(\frac{1}{2} \cdot q \cdot c + \omega \cdot \sigma_{DLT} \cdot c - C\right) \neq 0$$

$$\lambda \neq 0 \quad if \quad \left(\frac{1}{2} \cdot q \cdot c + \omega \cdot \sigma_{DLT} \cdot c - C\right) = 0$$

C – constraint

c - the unit quantity of the assumed constraint:

c = p_u (unit price if the stock holding cost is a constraint: C = SHC)

c = v (volume of the stock unit, if the stock volume V is a constraint: C = V)

c = m (mass of the stock unit, if the stock mass M is a constraint: C = M)

c = 1 (when the constraint is the stock quantity in natural units)

After adding the constraint component together with the Lagrange multiplier, the model becomes a Lagrange function and will be further denoted by L.

$$L = \frac{D_a}{q} \cdot c_r + \frac{1}{2} \cdot q \cdot p_u \cdot cc_a + \omega \cdot \sigma_{DLT} \cdot p_u \cdot cc_a +$$
$$+ cd_1 \cdot [1 - F(\omega)] \cdot \frac{D_a}{q} + cd_2 \cdot I(\omega) \cdot \sigma_{DLT} \cdot \frac{D_a}{q}$$
$$+ \lambda \cdot \left(\frac{1}{2} \cdot q \cdot c + \omega \cdot \sigma_{DLT} \cdot c - C\right) \qquad (3)$$

Coefficient ω occurring in the part of formulas (1) and (3), related to the carrying of safety stock, is called here the safety coefficient, and it depends on the adopted service level understood as the probability of serving the entire demand in a replenishment cycle αSL [Tempelmeier, 2000], as well as on the type of demand distribution.

The standard deviation in a stock replenishment cycle σ_{DLT} is generally calculated using the following formula:

$$\sigma_{DLT} = \sqrt{\sigma_D^2 \cdot LT + \sigma_{LT}^2 \cdot D^2} \qquad (2)$$

where:

σ_D – standard deviation of demand in an adopted time unit,

LT – mean stock replenishment cycle time,

σ_{LT} – standard deviation of the replenishment lead time.

Amounts F(ω) and I(ω) present in formula (1) and used to calculate the stock deficit cost are as follows:

F(ω) – distribution function related to the distribution of demand observed in a stock replenishment cycle, equal to service level αSL; thus [1- F(ω)] is a probability (risk) of going out of stock during a replenishment lead time.

I(ω) – standardised number of deficits; expected volume of deficits in a cycle is calculated with the following formula: $I(\omega) \cdot \sigma_{D,LT}$.

The standardized number of deficits can be calculated as follows [e.g. Krzyżaniak 2017]:

$$I(\omega) = f(\omega) - \omega \cdot [1 - F(\omega)] \qquad (3)$$

where f(ω) is the function of the density distribution.

A prerequisite for the existence of a minimum of the L-function is the zeroing of the first derivatives of the L-function with respect to both independent variables: q and ω.

The derivatives with respect to the assumed independent variables are calculated as follows:

$$\frac{\delta L}{\delta q} = -\frac{D_a \cdot c_r}{q^2} + \frac{p_u \cdot cc_a + \lambda \cdot c}{2} - \frac{D_a \cdot \{cd_1 \cdot [1-F(\omega)] + cd_2 \cdot I(\omega) \cdot \sigma_{DLT}\}}{q^2} \quad (4)$$

$$\frac{\delta L}{\delta \omega} = \sigma_{DLT} \cdot p_u \cdot cc_a - cd_1 \cdot \frac{D_a}{q} \cdot \frac{dF(\omega)}{d\omega} + cd_2 \cdot \sigma_{DLT} \cdot \frac{D_a}{q} \cdot \frac{dI(\omega)}{d\omega} + \lambda \cdot \sigma_{DLT} \cdot c \quad (5)$$

It should be noted that $\frac{dF(\omega_i)}{d\omega_i} = f(\omega_i)$, which is the density function of the distribution under consideration.

It can be demonstrated [e.g. Krzyżaniak 2017] that:

$$\frac{dI(\omega)}{d\omega} = \frac{df(\omega)}{d\omega} + \omega \cdot f(\omega) + F(\omega) - 1$$

and in the case of a normal distribution, which is assumed in the solution, there is

$$\frac{dI(\omega)}{d\omega} = F(\omega) - 1$$

In addition, the derivative with respect to the Lagrange multiplier is calculated:

$$\frac{\delta L}{\delta \lambda} = \frac{1}{2} \cdot q \cdot c + \omega \cdot \sigma_{DLT} \cdot c - C = 0 \quad (6)$$

From condition $\frac{\delta L}{\delta q} = 0$ we obtain:

$$q = \sqrt{\frac{2 \cdot D_a \cdot \{c_r + cd_1 \cdot [1-F(\omega)] + cd_2 \cdot I(\omega) \cdot \sigma_{DLT}\}}{p_u \cdot cc_a + \lambda \cdot c}} \quad (7)$$

From formula (5), assuming that $\frac{\delta L}{\delta \omega} = 0$, and considering that $\frac{dF(\omega_i)}{d\omega_i} = f(\omega_i)$ we obtain:

$$q = \frac{D_a \cdot [cd_1 \cdot f(\omega) + \cdot cd_2 \cdot \sigma_{DLT}(1-F(\omega))]}{\sigma_{DLT} \cdot (p_u \cdot cc_a + \lambda \cdot c)} \quad (8)$$

When comparing the delivery quantity q from equations (7) and (8), we obtain the following equation:

$$\sqrt{\frac{2 \cdot D_a \cdot \{c_r + cd_1 \cdot [1-F(\omega)] + cd_2 \cdot [f(\omega) - \omega \cdot [1-F(\omega)]] \cdot \sigma_{DLT}\}}{p_u \cdot cc_a + \lambda \cdot c}} = \frac{D_a \cdot [cd_1 \cdot f(\omega) + \cdot cd_2 \cdot \sigma_{DLT}(1-F(\omega))]}{\sigma_{DLT} \cdot (p_u \cdot cc_a + \lambda \cdot c)} \quad (9)$$

Figure 2 shows a simplified algorithm to determine the optimum value of the Lagrange multiplier λ_{opt}, allowing the calculation of the optimal pair of independent variables $\{q; \omega\}_{opt}$.

TOTAL STOCK COST MODEL - WITHOUT CONSTRAINTS

The solution to the problem of optimizing the order quantity and safety coefficient in the case of no constraints can be found using the above equation, functions and relations, assuming the Lagrange multiplier λ=0 (as indicated above, this applies to the case when $\left(\frac{1}{2} \cdot q \cdot c + \omega \cdot \sigma_{DLT} \cdot c - C\right) \neq 0$, i.e., when no constraint C applies. Equation (3) will then take the following form:

$$TC = \frac{D_a}{q} \cdot c_r + \frac{1}{2} \cdot q \cdot p_u \cdot cc_a + \omega \cdot \sigma_{DLT} \cdot p_u \cdot cc_a + {} + cd_1 \cdot [1-F(\omega)] \cdot \frac{D_a}{q} + cd_2 \cdot I(\omega) \cdot \sigma_{DLT} \cdot \frac{D_a}{q} \quad (14)$$

Formulas (4), (5),)7), (8) and (9) can be transformed in a similar way (that is, by removing the Lagrange multiplier λ from them) and then using the idea of the algorithm shown in Figure 2.

In the case of unconstrained optimization, in order for the pair $\{q, \omega\}_{opt}$ to correspond to the local minimum of the TC function, in addition to the conditions of zeroing of the first derivatives of the TC function with respect to q and ω, the conditions associated with the second derivatives of the TC function must be satisfied:

$$\partial = \left[\frac{\delta^2 TC}{\delta q\ \delta \omega}\right]^2 - \frac{\delta^2 TC}{\delta q^2} \cdot \frac{\delta^2 TC}{\delta \omega^2} < 0 \qquad (10)$$

$$\text{and } \frac{\delta^2 TC}{\delta q^2} > 0 \ and \ \frac{\delta^2 TC}{\delta \omega^2} > 0 \qquad (11)$$

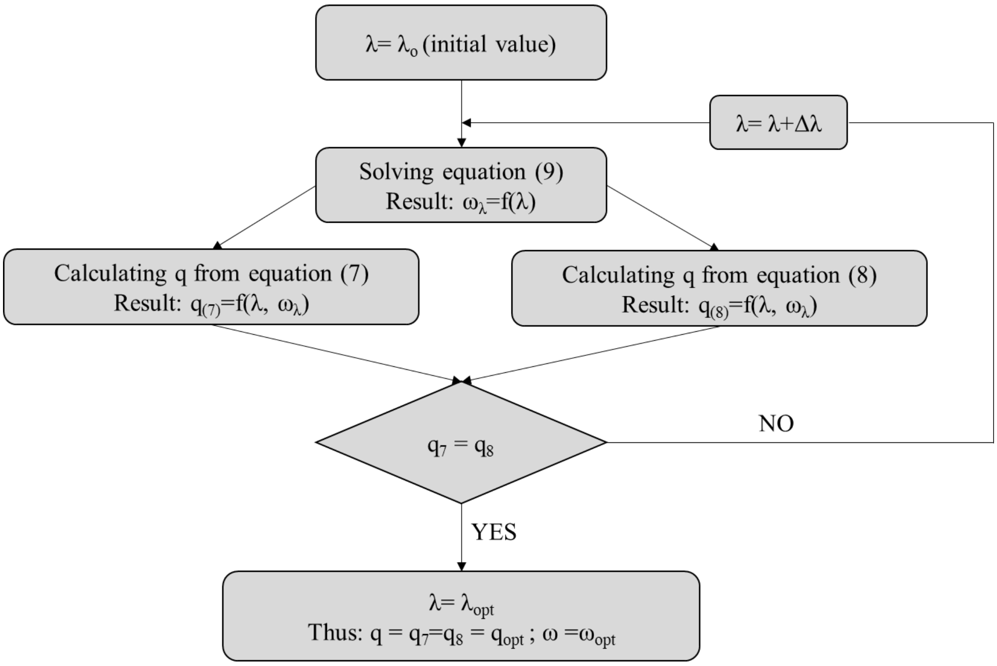

Fig. 2. The algorithm for determining the optimum delivery quantity (q_{opt}) and safety coefficient (ω_{opt})

Source: own study.

Table 1 Data adopted in the presented example

D = 50 units per week
σ_D = 10 unit standard deviation of weekly demand,
LT = 4 weeks (replenishment lead time), with no delays ($\sigma_{DLT} = 0$)
$\boldsymbol{\sigma_{DLT} = \sigma_D \cdot \sqrt{LT} = 20\ units}$
D_a =2,600 units,
cc_a – 0.1,
c_r = 300 €,
cd_1 = 500 €, 1,000 €
cd_2= 50 €
p_u = 500 €
Constraint – C = maximum admissible average stock level: 70, 80, 90, 100, 110, 120 units.

Source: own study.

AN EXAMPLE OF MODEL APPLICATION

The above considerations will be illustrated using the example of hypothetical material item X. The following data have been assumed (according to the previous designations):

The calculation algorithm shown in Figure 2 has been implemented in the EXCEL spreadsheet application. This tool has been used to determine the optimum stock structure and other accompanying indicators for the quantities assumed in the example.

In addition - as an example - the impact of changes in one of the assumed cost quantities (cost related to stock deficit occurrence during the stock replenishment cycle) on the optimal stock structure in the case of optimisation without constraints and for the chosen constraint value (maximum admissible average stock level C=70 units) has been examined. The results are presented in Fig. 4.

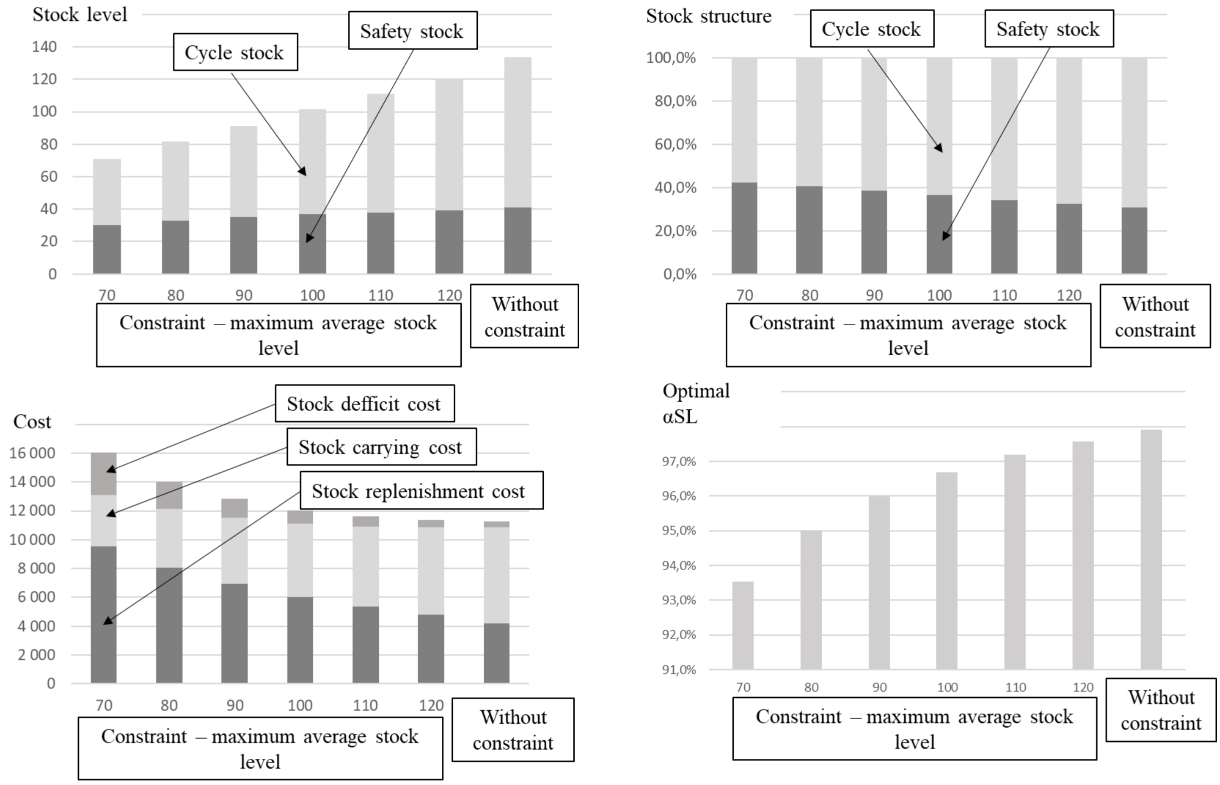

Fig. 3 Results of optimization calculations: stock level, stock structure, stock costs (replenishment, maintenance and deficit) and optimum service level αSL under selected constrain levels in given conditions (maximum acceptable average stock level) and for the case where no constraint has been introduced.
Source: own study.

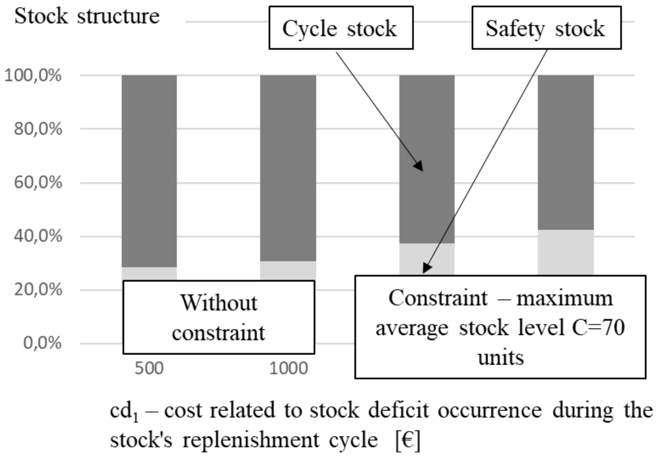

Fig. 4. The impact of changes on one of the assumed cost quantities (cost related to stock deficit occurrence during the stock replenishment cycle) on the optimum stock structure in the case of optimisation without constraints and for the chosen constraint value has been examined.

Source: own study.

CONCLUSIVE REMARKS

The use of the Lagrange multiplier is a well-known way to determine the optimum values of independent variables in the presence of constraints (ties) imposed on the quantities occurring in optimisation models. In the area of inventory management, the application of this method usually concerns sets of stock items, where, e.g. capital constraints (total stock outlays) or spatial constraints (total stock volume) are introduced. The paper demonstrates the possibility of applying the Lagrange multiplier for a single stock item, assuming the total cost of: replenishment, maintenance and deficit to be the objective function. Two types of deficit costs were taken into account: those resulting from the very occurrence and from quantitative shortages. The independent optimised variables have been the order quantity and the so-called safety coefficient, which translates into service level (measured by two indicators). Both of these quantities determine the two main components of the stock structure: the cycle stock and the safety stock. When we impose constraints (related to quantity, value, or space) on the stock, which means that the average total stock cannot exceed a certain level of constraint, we obtain the optimum stock structure under given conditions, i.e. the percentage share of each stock component in the total stock. Optimisation models, taking into account the restrictions imposed on the size of the stock, have been presented under the assumed conditions (stock replenishment in the Reorder Point system, distribution of the demand frequency occurrence in accordance with the normal distribution) and the method of solving this optimisation task has been indicated.

The proposed algorithm using the developed models has been implemented in an EXCEL spreadsheet application and calculations have been carried out for an example data set. Although the purpose of this has been mainly to illustrate the possibility of carrying out optimisation with imposed constraints, the presented results can also be in themselves a source of interesting information about the impact of selected values (mainly the level of constraints) on changes in the optimum stock structure. This example can be an inspiration for more complex research on the impact of various factors on optimisation results.

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Stanisław Krzyżaniak
Poznań School of Logistics, Poznań, **Poland**
e-mail: stanisław.krzyzaniak@wsl.com.pl