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ALL METRIC BASES AND FAULT-TOLERANT METRIC DIMENSION FOR SQUARE OF GRID

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Abstract. For a simple connected graph $G = (V, E)$ and an ordered subset $W = \{w_1, w_2, \ldots, w_k\}$ of V, the code of a vertex $v \in V$, denoted by code(v), with respect to W is a k-tuple $(d(v, w_1), \ldots, d(v, w_k))$, where $d(v, w_t)$ represents the distance between v and w_t. The set W is called a resolving set of G if code$(u) \neq$ code(v) for every pair of distinct vertices u and v. A metric basis of G is a resolving set with the minimum cardinality. The metric dimension of G is the cardinality of a metric basis and is denoted by $\beta(G)$. A set $F \subset V$ is called fault-tolerant resolving set of G if $F \setminus \{v\}$ is a resolving set of G for every $v \in F$. The fault-tolerant metric dimension of G is the cardinality of a minimal fault-tolerant resolving set. In this article, a complete characterization of metric bases for G^2_{mn} has been given. In addition, we prove that the fault-tolerant metric dimension of G^2_{mn} is 4 if $m + n$ is even. We also show that the fault-tolerant metric dimension of G^2_{mn} is at least 5 and at most 6 when $m + n$ is odd.

Keywords: code, resolving set, metric dimension, fault-tolerant resolving set, fault-tolerant metric dimension.

Mathematics Subject Classification: 05C12, 05C05, 05C90, 05C76.

1. INTRODUCTION

Let $G = (V(G), E(G))$ be a simple connected and undirected graph. A vertex $w \in V$ is said to resolve two vertices u and v if $d(w, u) \neq d(w, v)$. A set $W \subseteq V$ is said to be a *resolving set* for G, if every pair of vertices of G is distinguished by some element of W. A minimum resolving set is called a *metric basis*. The cardinality of a metric basis is called the *metric dimension* of G and it is denoted by $\beta(G)$. For an ordered set $W = \{w_1, w_2, \ldots, w_k\} \subseteq V$, we refer to the k-vector (ordered k-tuple) code(v)= $(d(v, w_1), \ldots, d(v, w_k))$ as the representation of v with respect to W. A set $F \subseteq V$ is called *fault-tolerant resolving* set of G if $F \setminus \{v\}$ is a resolving set of

G for every $v \in F$. The *fault-tolerant metric dimension* of G is the cardinality of a minimal fault-tolerant resolving set. The metric dimension problem was introduced by Slater [21] (independently Harary and Melter [12]) and further studied in [1–3, 5, 7, 8, 11, 13, 14, 16, 19, 20]. Applications of metric basis and resolving sets are in many various platforms such as Robot Navigation [16], Digitization of Image [18], Network Optimization [3], Mastermind game [7] and Chemistry and Drug [5]. Khuller *et al.* [16] studied the metric dimension problem motivated by the robot navigation in a graph space. A resolving set for a graph corresponds to the presence of distinctively labeled (landmark) nodes in the graph. It is assumed that a robot can detect the distance to each node of the landmarks, hence determine uniquely its location in the graphic. Garey and Johnson [9] have shown NP-completeness of the metric dimension problem. Cáceres *et al.* [4] studied the metric dimension of graphs which obtained by the Cartesian product of two or more graphs. Chartrand *et al.* [5] have characterized all graphs of order n having metric dimension 1, $n-2$ or $n-1$. Recently, in [6, 10] and [17], the metric dimension has been widely studied for power of some graphs such as cycles and paths. For two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$, the *Cartesian product* of G with H, denoted by $G \Box H$, is a graph with vertex set

$$V(G \Box H) = V(G) \times V(H) = \{(u, v) : u \in V(G), v \in V(H)\}$$

and two vertices (u_1, v_1) and (u_2, v_2) are adjacent if either $u_1 = u_2$ and $v_1 v_2 \in E(H)$ or $v_1 = v_2$ and $u_1 u_2 \in E(G)$. It is to be noted that $G \Box H$ is isomorphic to $H \Box G$.

The *grid graph* G_{mn} is the Cartesian product of P_m with P_n, i.e.,

$$G_{mn} = P_m \Box P_n,$$

where P_l denotes the path with $l \geq 2$ vertices. A square grid graph, denoted by G^2_{mn}, is obtained from a grid graph G_{mn} by adjoining two vertices which are at distance 2 in G_{mn}. The graph G^2_{mn} has mn vertices and $V(G^2_{mn}) = V(G_{mn})$. A grid network is a computer network consisting of a number of (computer) systems connected in a grid topology. Melter and Tomescu [18] proved that the metric dimension of grid graphs is 2. They have also shown that metric bases correspond to two endpoints of a boundary edge of the grid.

Rest of the paper is organized as follows. Section 2 deals with some preliminary results. In Section 3, we have determined the metric dimension of G^2_{mn} by giving an optimal resolving set with cardinality 3. In Section 4, first we show that the basis elements are not interior points and each basis must contain at least two corner points. After that, we find all metric bases of G^2_{mn}. Section 5 deals with the fault-tolerant resolving set of G^2_{mn}. Here, we prove that the fault-tolerant metric dimension of G^2_{mn} is 4 if $m+n$ is even. We also obtain that if $m+n$ is odd, then the fault-tolerant metric dimension of G^2_{mn} is at least 5 and at most 6.

From here onwards, we denote the vertex set of G^2_{mn} by

$$V(G^2_{mn}) = \{(i, j) : 0 \leq i \leq m-1, 0 \leq j \leq n-1\} = V(G_{mn}).$$

2. PRELIMINARIES

In this section, we give some preliminary results for the graph G^2_{mn} which are to be used in the sequel.

Definition 2.1. A vertex u of G is called a *neighbor* of a vertex $v \in V(G)$ if they are adjacent in G. Throughout the article, $N(u)$ denotes the set of all neighbors of u.

Proposition 2.2. *For any two vertices $u = (i_1, j_1)$ and $v = (i_2, j_2)$ of G^2_{mn}, the following hold:*

(1) $d_{G_{mn}}(u, v) = d_{P_m}(i_1, i_2) + d_{P_m}(j_1, j_2)$,
(2) $d_{G^2_{mn}}(u, v) = \left\lceil \frac{|i_2-i_1|+|j_2-j_i|}{2} \right\rceil$.

Definition 2.3. An *i-th row of G^2_{mn}* is the set

$$\{(i, j) \in V(G^2_{mn}) : 0 \leq j \leq n - 1\}$$

and a *j-th column of G^2_{mn}* is the set

$$\{(i, j) \in V(G^2_{mn}) : 0 \leq i \leq m - 1\}.$$

Definition 2.4. The boundary of G^2_{mn}, denoted by $Bd(G^2_{mn})$, is the set

$$\{(i, j) \in V(G^2_{mn}) : i = 0,\ m - 1,\ 0 \leq j \leq n - 1\}$$
$$\cup \{(i, j) \in V(G^2_{mn}) : j = 0,\ n - 1,\ 0 \leq i \leq m - 1\}.$$

We call a vertex u is an interior vertex if $u \in V(G^2_{mn}) \setminus Bd(G^2_{mn})$. Note that if $m = 2$ or $n = 2$, then $Bd(G^2_{mn}) = V(G^2_{mn})$.

It is to be noted that the graph G^2_{mn} has four boundaries which are given by

(a) top boundary $\{(0, j) : 0 \leq j \leq n - 1\}$, i.e., 0-*th* row,
(b) bottom boundary $\{(m - 1, j) : 0 \leq j \leq n - 1\}$, i.e., $(m - 1)$-*th* row,
(c) left boundary $\{(i, 0) : 0 \leq i \leq m - 1\}$, i.e., 0-*th* column,
(d) right boundary $\{(i, n - 1) : 0 \leq i \leq m - 1\}$, i.e., $(n - 1)$-*th* column.

We call the intersecting points of two boundaries as *corner points*. There are four corner points, namely, $(0, 0)$, $(0, n - 1)$, $(m - 1, 0)$ and $(m - 1, n - 1)$ and we call them as left upper, right upper, left lower and right lower corner points, respectively.

Definition 2.5. For a vertex $(i, j) \in V(G^2_{mn})$, $i + j$ is called the *co-ordinate sum* of the vertex (i, j).

Lemma 2.6. *For any two positive integers m and n the following holds:*

(a) $\left\lceil \frac{n}{2} \right\rceil = \begin{cases} \left\lceil \frac{n+1}{2} \right\rceil, & \textit{if } n \textit{ is odd,} \\ \left\lceil \frac{n-1}{2} \right\rceil, & \textit{if } n \textit{ is even,} \end{cases}$
(b) *if* $\left\lceil \frac{n}{2} \right\rceil = \left\lceil \frac{m}{2} \right\rceil$ *and* $\left\lceil \frac{n-1}{2} \right\rceil = \left\lceil \frac{m-1}{2} \right\rceil$ *then* $n = m$,
(c) *if* $\left\lceil \frac{n}{2} \right\rceil = \left\lceil \frac{m}{2} \right\rceil$ *and* $\left\lceil \frac{n-1}{2} \right\rceil = \left\lceil \frac{m+1}{2} \right\rceil$ *then* $m = n - 1$ *and* m *must be odd,*
(d) $\left\lceil \frac{n}{2} \right\rceil = \left\lceil \frac{m}{2} \right\rceil$ *implies* $n = m$ *and* $n = m - 1$ *according to* $m + n$ *is even or odd.*

Definition 2.7. A *clique* of a graph G is a complete sub-graph of G. From here onward we denote a clique on t vertices by K_t.

Lemma 2.8. *A clique K_3 on three vertices can be resolved by at least two vertices.*

Proof. If possible, let $S = \{x\}$ resolve the clique K_3 and let $V(K_3) = \{u, v, w\}$. Then there exist a vertex, say $u \in V(K_3)$ such that

$$d(x, u) = \min\{d(x, z) : z \in V(K_3)\}.$$

If both $d(x, v)$ and $d(x, w)$ are different from $d(x, u)$, then

$$d(x, v) = d(x, w) = d(x, u) + 1$$

and so S can not resolve v and w. Again if

$$d(x, u) = \min\{d(x, z) : z \in V(K_3)\} = d(x, v),$$

then S can not resolve u and v. □

3. METRIC DIMENSION OF G^2_{mn}

In this section, we determine the exact value of $\beta(G^2_{mn})$. For this, we first show that every resolving set of G^2_{mn} contains at least 3 elements and then we find a resolving set of cardinality 3. The following result is true because every vertex u of G_{mn} lies on a cycle of length four in G_{mn}.

Lemma 3.1. *For every vertex u of G^2_{mn}, there exists a clique K_4 with $u \in V(K_4)$.*

The theorem mentioned below gives the metric dimension of G^2_{mn} and an optimal resolving set for the same.

Theorem 3.2. *For two integers $m \geq 2$ and $n \geq 2$, the metric dimension of G^2_{mn} is* 3.

Proof. First we show that $\beta(G^2_{mn}) \geq 3$. Let $\mathcal{B}$ be an arbitrary resolving set of G^2_{mn} and $u \in \mathcal{B}$. Then applying Lemma 3.1, there exists a clique K_4 with u as a vertex of K_4. Again from Lemma 2.8, to resolve the clique $K_3 = K_4 \setminus \{u\}$ at least two vertices are required. Thus $|\mathcal{B}| \geq 3$ and consequently we have $\beta(G^2_{mn}) \geq 3$. Now we show that there exists a resolving set $\mathcal{B}$ with cardinality 3. If $m = 2$ and $n = 2$, then G^2_{mn} is isomorphic to K_4 and hence any three vertices of G^2_{mn} forms a resolving set. So we assume either $m \geq 3$ or $n \geq 3$. Since G^2_{mn} and G^2_{nm} are isomorphic, we consider the assumption that $n \geq 3$ and $m \geq 2$. Then we take $\mathcal{B} = \{(0, 0), (0, 1), (0, n - 1)\}$ or $\{(0, 0), (1, 0), (0, n - 1)\}$ accordingly as n is odd or even. Our claim $\mathcal{B}$ is a resolving set of G^2_{mn}. We consider the following two cases accordingly as n is odd or even.

Case 1. n is odd. In this case, first we identify the vertices which are not resolved by $(0, 0)$ and $(0, 1)$. If $u = (i_1, 0)$ and $v = (i_2, 0)$ be two vertices on the left boundary, then code$(u) \neq$code(v) for $u \neq v$, i.e., the left boundary is resolved by $\{(0, 0), (0, 1)\}$. Now we consider two vertices $u = (i_1, j_1)$ and $v = (i_2, j_2)$ such that none of them are on the left boundary. Then code(u) =code(v) with respect to $\{(0,0),(0,1)\}$ implies

$$\left\lceil \frac{i_1 + j_1}{2} \right\rceil = \left\lceil \frac{i_2 + j_2}{2} \right\rceil \quad \text{and} \quad \left\lceil \frac{i_1 + j_1 - 1}{2} \right\rceil = \left\lceil \frac{i_2 + j_2 - 1}{2} \right\rceil.$$

Combining these two relations and using Lemma 2.6, we have $i_1 + j_1 = i_2 + j_2$. From this we conclude that if none of u and v are on the left boundary, then they are not resolved by $\{(0,0),(0,1)\}$ only when they have same co-ordinate sum. Again if exactly one of u and v, say v, is on the left boundary, then code(u) =code(v) with respect to $\{(0,0),(0,1)\}$ implies

$$\left\lceil \frac{i_1 + j_1}{2} \right\rceil = \left\lceil \frac{i_2}{2} \right\rceil \quad \text{and} \quad \left\lceil \frac{i_1 + j_1 - 1}{2} \right\rceil = \left\lceil \frac{i_2 + 1}{2} \right\rceil.$$

By using Lemma 2.6 with simple calculation, we get i_2 is odd and $i_2 = i_1 + j_1 - 1$. Thus u and v have same codes with respect to $\{(0,0),(0,1)\}$ only if the difference between their co-ordinate sum is one and the coordinate sum of v is odd. Finally, from the above discussions we can say that two vertices u and v which are not resolved by $\{(0,0),(0,1)\}$ must be in S_k for some $0 \le k \le m + n - 2$, where S_k is given by

$$S_k = \begin{cases} \{(i,j) : i + j = k, j > 0\} \cup \{(k-1,0)\}, & \text{if } 0 \le k \le m - 1 \text{ and } k \text{ is odd,} \\ \{(i,j) : i + j = k, j > 0\}, & \text{otherwise.} \end{cases}$$

Note that if $u \in S_i$ and $v \in S_j$ with $i \neq j$, then they are resolved by $\{(0,0),(0,1)\}$ as they have distinct co-ordinate sum. Now we show that elements of S_k are resolved by $(0, n-1)$. If possible, let (i_1, j_1) and (i_2, j_2) be two vertices in S_k that are not resolved by $(0, n-1)$, where $j_1, j_2 > 0$. Then

$$\left\lceil \frac{i_1 + n - 1 - j_1}{2} \right\rceil = \left\lceil \frac{i_2 + n - 1 - j_2}{2} \right\rceil$$

and hence using Lemma 2.6, we have $i_1 - j_1 = i_2 - j_2$. Again $(i_1, j_1), (i_2, j_2) \in S_k$ gives $i_1 + j_1 = i_2 + j_2$. Solving these two relations we have $i_1 = i_2$ and $j_1 = j_2$. Therefore, $(i_1, j_1) = (i_2, j_2)$, which is a contradiction. Again for two vertices (i, j) and $(k-1, 0)$ of S_k with $i + j = k$, an odd integer, the equality

$$d((0, n-1)(i,j)) = d((0, n-1)(k-1, 0))$$

gives

$$\left\lceil \frac{i + n - 1 - j}{2} \right\rceil = \left\lceil \frac{n - k - 2}{2} \right\rceil.$$

Using Lemma 2.6, we have $i - j = k$. Now solving $i - j = k$ and $i + j = k$, we have $j = 0$ which contradicts the fact that $j > 0$. Therefore, $\{(0,0),(0,1),(0,n-1)\}$ forms a resolving set.

Case 2. n is even. Here first we determine the vertices which are not resolved by $(0,0)$ and $(1,0)$. By similar argument as in Case 1, two vertices u and v which are not resolved by $\{(0,0),(0,1)\}$ must be in S_k for some $0 \le k \le m+n-2$, where S_k is given by

$$S_k = \begin{cases} \{(i,j) : i+j = k, i > 0\} \cup \{(0,k-1)\}, & 0 \le k \le m-1 \text{ and } k \text{ is odd}, \\ \{(i,j) : i+j = k, i > 0\} & \text{otherwise.} \end{cases}$$

Note that if $u \in S_i, v \in S_j$ (with $i \neq j$), then they are resolved by $\{(0,0),(0,1)\}$ as they have distinct co-ordinate sum. Now we show that elements of S_k are resolved by $(0, n-1)$. If possible, let two distinct vertices (i_1, j_1) and (i_2, j_2) of S_k have same distance from $(0, n-1)$, where $i_1, i_2 > 0$. This generates an equation:

$$\left\lceil \frac{i_1 + n - 1 - j_1}{2} \right\rceil = \left\lceil \frac{i_2 + n - 1 - j_2}{2} \right\rceil.$$

Then using Lemma 2.6, we have $i_1 - j_1 = i_2 - j_2$. Since $(i_1, j_1), (i_2, j_2) \in S_k$, it gives $i_1 + j_1 = i_2 + j_2$. Solving these two equations, we get $i_1 = i_2, j_1 = j_2$. Therefore, $(i_1, j_1) = (i_2, j_2)$, which is a contradiction. Again for two vertices (i,j) and $(0, k-1)$ of S_k with $i + j = k$ is odd, the equality $d((0,n-1)(i,j)) = d((0,n-1)(0,k-1))$ gives

$$\left\lceil \frac{i + n - 1 - j}{2} \right\rceil = \left\lceil \frac{n-k}{2} \right\rceil.$$

This implies $i+n-1-j = n-k$ or $n-k+1$. By solving the equations $i+n-1-j = n-k$ or $n-k+1$ and $i+j = k$, we get $j = \frac{2k-1}{2}$ or $k+1$, which contradicts $j \in \mathbb{N}$ and $i + j = k$. Therefore, $\{(0,0),(1,0),(m-1,0)\}$ forms a resolving set. This completes the proof of the theorem. □

4. CHARACTERIZATION OF AN OPTIMAL RESOLVING SET

In this section, we determine the positions of elements of a metric basis for G^2_{mn}. In Section 3, we have shown that any metric basis contains exactly three elements. Here we show that all three elements of a metric basis are on the boundary and at least two of them are corner points. In this section, we also find all metric bases for G^2_{mn}.

Theorem 4.1. *For any metric basis $\mathcal{B}$ of G^2_{mn}, $\mathcal{B} \subset Bd(G^2_{mn})$.*

Proof. Since $Bd(G^2_{mn}) = V(G^2_{mn})$ when $m = 2$ or $n = 2$, so the result is true when at least one of m and n is 2. Now we assume $m \ge 3$ and $n \ge 3$. If possible, let there be a metric basis $\mathcal{B} = \{(i_1,j_1),(i_2,j_2),(i_3,j_3)\}$ of G^2_{mn} containing an interior element, say (i_1, j_1). Let

$$\begin{aligned} P = \{&(i_1-1,j_1),(i_1,j_1+1),(i_1+1,j_1),(i_1,j_1-1), \\ &(i_1-1,j_1-1),(i_1-1,j_1+1),(i_1+1,j_1+1),(i_1+1,j_1-1)\} \end{aligned}$$

be a subset of $V(G^2_{mn})$. Note that all the elements of P are the neighbors of (i_1, j_1). If $\mathcal{B} \cap P$ is non-empty and $u \in \mathcal{B} \cap P$, then there exists a clique K_3 such that $V(K_3) \subset P$ and $d(u,v) = d(u,w)$ for all $v, w \in V(K_3)$. For example, if $u = (i_1 - 1, j_1)$, then

$$V(K_3) = \{(i_1+1,j_1),(i_1,j_1-1),(i_1,j_1+1)\}$$

and if $u = (i_1 - 1, j_1 - 1)$, then

$$V(K_3) = \{(i_1 - 1, j_1 + 1), (i_1, j_1 - 1), (i_1 - 1, j_1)\}.$$

Thus applying Lemma 2.8, we have $|\mathcal{B}| \geq 2 + 2 = 4$ when $\mathcal{B} \cap P \neq \emptyset$. Now we take $\mathcal{B} \cap P = \emptyset$. We divide $V(G^2_{mn}) \setminus \{(i_1, j_1)\}$ into six sets which are given below

$$\begin{aligned} A &= \{(i,j) : i < i_1, j < j_1\} \cup \{(i,j) : i > i_1, j > j_1\} = A_1 \cup A_2, \\ B &= \{(i,j) : i < i_1, j > j_1\} \cup \{(i,j) : i > i_1, j < j_1\} = B_1 \cup B_2, \\ C &= \{(i_1, j) : 0 \leq j < j_1\}, \\ D &= \{(i_1, j) : j_1 < j \leq n - 1\}, \\ E &= \{(i, j_1) : 0 \leq i < i_1\}, \\ F &= \{(i, j_1) : i_1 < i \leq m - 1\}. \end{aligned} \tag{4.1}$$

It is clear that the set

$$S = \{(i_1 - 1, j_1), (i_1, j_1 + 1), (i_1 + 1, j_1), (i_1, j_1 - 1)\} = \{x, y, z, w\}$$

forms a clique K_4 and each element of S is adjacent to (i_1, j_1). In Table 1, we calculate the distances of x, y, z and w from the vertex $(i,j) \in C \cup D \cup E \cup F$. In this table, a, b, c and d denote the integers $\left\lceil \frac{j_1 - j - 1}{2} \right\rceil$, $\left\lceil \frac{j - j_1 - 1}{2} \right\rceil$, $\left\lceil \frac{i_1 - i - 1}{2} \right\rceil$ and $\left\lceil \frac{i - i_1 - 1}{2} \right\rceil$, respectively.

From Table 1, if both (i_2, j_2) and (i_3, j_3) are in $C \cup D \cup E \cup F$, then there are at least two elements of S whose codes are same. This contradicts that $\mathcal{B}$ is a metric basis. Again if one of (i_2, j_2),(i_3, j_3), say (i_2, j_2), belongs to $C \cup D \cup E \cup F$, then from Table 1, it is clear that there are three vertices of S which are at same distance from (i_2, j_2). Since S forms a clique K_4, so there is a clique K_3 whose vertices are same distances from the both (i_1, j_1) and (i_2, j_2). Then Lemma 2.8 implies that it can not be resolved by only one vertex $(i,j) \in V(G^2_{mn}) \setminus \{(i_1, j_1), (i_2, j_2)\}$. So $(i_2, j_2), (i_3, j_3) \notin C \cup D \cup E \cup F$ and hence $(i_2, j_2), (i_3, j_3) \in A \cup B$. Now we calculate distances among elements of S and $A \cup B$ in Table 2, where a, b, c, d denote the integers $\left\lceil \frac{i_1 - i + j_1 - j - 1}{2} \right\rceil$, $\left\lceil \frac{i - i - 1 + j - j_1 - 1}{2} \right\rceil$, $\left\lceil \frac{i_1 - i + j - j_1 - 1}{2} \right\rceil$ and $\left\lceil \frac{i - i_1 + j - j_1 - 1}{2} \right\rceil$, respectively.

Table 1. Distances between elements of S and $C \cup D \cup E \cup F$

	$(i,j) \in C$	$(i,j) \in D$	$(i,j) \in E$	$(i,j) \in F$
$x = (i_1 - 1, j_1)$	$a + 1$	$b + 1$	c	$d + 1$
$y = (i_1, j_1 + 1)$	$a + 1$	$b + 1$	$c + 1$	d
$z = (i_1 + 1, j_1)$	$a + 1$	b	$c + 1$	$d + 1$
$w = (i_1, j_1 - 1)$	a	$b + 1$	$c + 1$	$d + 1$

Table 2. Distances between the elements of S and $A \cup B$

	$(i,j) \in A_1$	$(i,j) \in A_2$	$(i,j) \in B_1$	$(i,j) \in B_2$
x	a	$b + 1$	c	$d + 1$
y	$a + 1$	b	c	$d + 1$
z	$a + 1$	b	$c + 1$	d
w	a	$b + 1$	$c + 1$	d

From Table 2, if both $(i_2, j_2), (i_3, j_3)$ are in A or in B, then there are two vertices having same codes. Thus one of $(i_2, j_2), (i_3, j_3)$ is in A and the other one must be in B. Without loss of generality, we may assume $u = (i_2, j_2) \in A$. We show that there exists a clique K_3 such that $V(K_3) \subset N(u)$ and $d(u, v) = d(u, w)$ for all $v, w \in V(K_3)$. If $u \in A_1$ with $i_1 + j_1 - (i_2 + j_2)$ is even or $u \in A_2$ with $i_2 + j_2 - (i_1 + j_1)$ is odd, then the clique K_3 with

$$V(K_3) = \{(i_1, j_1 + 1), (i_1 + 1, j_1), (i_1 + 1, j_1 + 1)\}$$

has the above property. Again if $u \in A_1$ with $i_1 + j_1 - (i_2 + j_2)$ is odd or $u \in A_2$ with $i_2 + j_2 - (i_1 + j_1)$ is even, then we consider the clique K_3 with

$$V(K_3) = \{(i_1, j_1 - 1), (i_1 - 1, j_1), (i_1 - 1, j_1 - 1)\}.$$

Thus in any case, applying Lemma 2.8, we have $|\mathcal{B}| \geq 2+2 = 4$, which is a contradiction. Which completes the proof of the theorem. □

Theorem 4.1 tells us that every basis must be on the boundary. The theorem below gives an idea about the number of basis elements from each boundary.

Theorem 4.2. *Every metric basis of G^2_{mn} must contain at least two points from same boundary.*

Proof. If possible, let there be a metric basis $\mathcal{B}$ with elements from three distinct boundaries. Since every metric basis consists three elements, without loss of generality, we may assume $\mathcal{B} = \{(0, i), (j, 0), (m - 1, k)\}$.

Now two cases may arise.

Case 1. $k \neq i$. Then from Table 3, we get:

(a) the codes of $(0, i - 1)$ and $(1, i)$ are same when $k < i$,
(b) the codes of $(m - 1, k - 1)$ and $(m - 2, k)$ are same when $k > i$.

Case 2. $k = i$. First we consider $i + j$ is even. Then from Table 4, we have the following:

(a) the codes of $(1, i + 1)$ and $(1, i)$ are same if m is an odd integer,
(b) the codes of $(0, i - 1)$ and $(1, i)$ are same if m is an even integer.

Also if $i + j$ is odd integer, then Table 4 shows the following:

(a) the codes of $(1, i - 1)$ and $(1, i)$ are same if m is an odd integer,
(b) the codes of $(2, i)$ and $(1, i)$ are same if m is an even integer.

From Case 1 and Case 2, finally we get the result. □

Table 3. Two pairs of vertices which are not resolve by $\mathcal{B}$

	$(0, i)$	$(j, 0)$	$(m-1, k)$
$(0, i-1)$	1	$\lceil \frac{j+i-1}{2} \rceil$	$\lceil \frac{m-2+\lvert i-k \rvert}{2} \rceil$
$(1, i)$	1	$\lceil \frac{j+i-1}{2} \rceil$	$\lceil \frac{m-2+\lvert i-k \rvert}{2} \rceil$
$(m-1, k-1)$	$\lceil \frac{m-2+\lvert k-i \rvert}{2} \rceil$	$\lceil \frac{m-2-j+k}{2} \rceil$	1
$(m-2, k)$	$\lceil \frac{m-2+\lvert k-i \rvert}{2} \rceil$	$\lceil \frac{m-2-j+k}{2} \rceil$	1

Table 4. Three pairs of vertices which are not resolve by $\mathcal{B}$

	$(0, i)$	$(j, 0)$	$(m-1, i)$
$(0, i-1)$	1	$\lceil \frac{j+i-1}{2} \rceil$	$\lceil \frac{m}{2} \rceil$
$(1, i)$	1	$\lceil \frac{j+i-1}{2} \rceil$	$\lceil \frac{m-2}{2} \rceil$
$(1, i+1)$	1	$\lceil \frac{j+i}{2} \rceil$	$\lceil \frac{m-1}{2} \rceil$
$(2, i)$	1	$\lceil \frac{j+i-2}{2} \rceil$	$\lceil \frac{m-3}{2} \rceil$
$(1, i)$	1	$\lceil \frac{j+i-1}{2} \rceil$	$\lceil \frac{m-2}{2} \rceil$
$(1, i-1)$	1	$\lceil \frac{j+i-2}{2} \rceil$	$\lceil \frac{m-1}{2} \rceil$

Theorem 4.3. *Every metric basis $\mathcal{B}$ of G^2_{mn} must contain at least two corner points.*

Proof. For $m = 2 = n$, the graph G^2_{mn} is isomorphic to K_4 and hence any metric basis must contain three corner points. Thus the result holds true when $m = 2 = n$. So we assume either $m \geq 3$ or $n \geq 3$. Without loss of generality, we take $n \geq 3$. First we show that $\mathcal{B}$ contains at least one corner point. If possible, let $\mathcal{B}$ does not contain any corner point. Theorem 4.2 gives that at least two elements, say x and y, of $\mathcal{B}$ must be on same boundary. If $m = 2$ and both x, y are either on the left boundary or on the right boundary, then $\mathcal{B}$ contains at least one corner point. So, we assume both x and y are either on the top boundary or on the bottom boundary. Then without loss of generality, we assume $\mathcal{B} = \{(0, i), (0, j), (k, 0)\}$, where $i < j$ and $(0, i), (0, j), (k, 0) \notin \{(0, 0), (0, n-1), (m-1, 0), (m-1, n-1)\}$, i.e., none of $(0, i), (0, j), (k, 0)$ are the corner points. Then from Table 5, codes of $(0, i-1)$ and $(1, i)$ are same with respect to $\mathcal{B}$ and hence $\mathcal{B}$ must contains at least one corner point.

Table 5. Distances of the vertices $(0, i-1)$ and $(1, i)$ from all vertices of $\mathcal{B}$

	$(0, i)$	$(0, j)$	$(k, 0)$
$(0, i-1)$	1	$\lceil \frac{j-i+1}{2} \rceil$	$\lceil \frac{k+i-1}{2} \rceil$
$(1, i)$	1	$\lceil \frac{j-i+1}{2} \rceil$	$\lceil \frac{k+i-1}{2} \rceil$

Now we show that $\mathcal{B}$ contains at least two corner points. If possible, let $\mathcal{B}$ contains exactly one corner point. Without loss of generality, we may assume that $(0, 0) \in B$. Applying Theorem 4.2, we may assume $\mathcal{B} = \{(0, 0), (0, s), (t, 0)\}$, where $(0, s), (t, 0) \notin \{(0, n-1), (m-1, 0), (m-1, n-1)\}$.

Then from Table 6, codes of $(t+1, s)$ and $(t, s+1)$ are same with respect to $\mathcal{B}$ and hence $\mathcal{B}$ must contains at least two corner points. $\square$

Table 6. Distances of the vertices $(t+1, s)$ and $(t, s+1)$ from all vertices of $\mathcal{B}$

	$(0,0)$	$(0,s)$	$(t,0)$
$(t+1,s)$	$\lceil \frac{t+s+1}{2} \rceil$	$\lceil \frac{t+1}{2} \rceil$	$\lceil \frac{s+1}{2} \rceil$
$(t,s+1)$	$\lceil \frac{t+s+1}{2} \rceil$	$\lceil \frac{t+1}{2} \rceil$	$\lceil \frac{s+1}{2} \rceil$

In the lemma below, we give conditions for the resolvability of two consecutive vertices in a column by a vertex u lying on the upper boundary or the lower boundary.

Lemma 4.4. *Let j be an integer such that $0 \le j \le n-1$. Then any two consecutive vertices (i,j) and $(i+1,j)$ on j-th column are resolved by*

(a) $(0,l)$ *only if $i+j-l$ is even,*
(b) $(m-1,l)$ *only if $m+l-i-j$ is even.*

Proof. Let $u=(i,j)$ and $v=(i+1,j)$. The distances of u and v from $w=(0,l)$ are given by

$$d(w,u) = \begin{cases} \left\lceil \frac{i+j-l}{2} \right\rceil & l \le j, \\ \left\lceil \frac{i+l-j}{2} \right\rceil & l > j. \end{cases}$$

and

$$d(w,v) = \begin{cases} \left\lceil \frac{i+1+j-l}{2} \right\rceil & l \le j, \\ \left\lceil \frac{i+1+l-j}{2} \right\rceil & l > j. \end{cases}$$

Therefore $d(w,v)$ is equal to $d(w,u)$ or $d(w,u)+1$ accordingly as $i+j-l$ is odd or even, i.e., $w=(0,l)$ resolves u and v only if $i+j-l$ is even. Again the distances of u and v from the vertex $z=(m-1,l)$ on bottom boundary are given by

$$d(z,u) = \begin{cases} \left\lceil \frac{m-1-i+j-l}{2} \right\rceil & l \le j, \\ \left\lceil \frac{m-1-i+l-j}{2} \right\rceil & l > j. \end{cases}$$

and

$$d(w,v) = \begin{cases} \left\lceil \frac{m-i-2+j-l}{2} \right\rceil & l \le j, \\ \left\lceil \frac{m-i-2+l-j}{2} \right\rceil & l > j. \end{cases}$$

Therefore $d(z,v)$ is equal to $d(z,u)$ or $d(z,u)-1$ accordingly as $m+l-i-j$ is odd or even, i.e., z resolves u and v only if $m+l-i-j$ is even. $\square$

Remark 4.5. A vertex u on the left or on the right boundary may not resolve two consecutive vertices of j-*th* column. For example, if $(k, 0)$ is a vertex on the left boundary, then it can not resolve (k, j) and $(k+1, j)$ when j is odd. Similarly, the vertex $(k, n-1)$ on the right boundary can not resolve the vertices (k, j) and $(k+1, j)$ when $n - j$ is even.

Any two non-consecutive vertices in i-*th* row are resolved by a vertex u on the left or the right boundary. In the following lemma we give the conditions for which a vertex u on the left or the right boundary resolves two consecutive vertices on i-*th* row for every i with $0 \leq i \leq m-1$.

Lemma 4.6. *Let i be an integer such that $0 \leq i \leq m-1$. Then any two consecutive vertices (i, j) and $(i, j+1)$ on i-th row are resolved by*

(a) $(k, 0)$ *only if $i + j - k$ is even,*
(b) $(k, n-1)$ *only if $n + k - i - j$ is even.*

Proof. Let $u = (i, j)$ and $v = (i, j+1)$. The distances u and v from a vertex $w = (k, 0)$ on the left boundary are given by

$$d(w, u) = \begin{cases} \left\lceil \frac{i-k+j}{2} \right\rceil & k \leq i, \\ \left\lceil \frac{k-i+j}{2} \right\rceil & k > i. \end{cases}$$

and

$$d(w, v) = \begin{cases} \left\lceil \frac{i-k+j+1}{2} \right\rceil & k \leq i, \\ \left\lceil \frac{k-i+j+1}{2} \right\rceil & k > i. \end{cases}$$

From above equations, w resolves u and v only if $i + j - k$ is even. Using similar arguments we may show that $(k, n-1)$ resolve u and v only if $n + k - i - j$ is even. $\square$

Remark 4.7. A vertex u on the top or the bottom boundary may not resolve two consecutive vertices in i-*th* row. For example, if $(0, l)$ is a vertex on the top boundary, then it can not resolve the vertices (i, l) and $(i, l+1)$ when i is odd. Similarly, the vertex $(m-1, l)$ on the bottom boundary can not resolve the vertices (i, l) and $(i, l+1)$ when $m - i$ is even.

In Theorem 3.2, we have given a metric basis for G^2_{mn} containing the upper corner points $(0, 0)$ and $(0, n-1)$. In the theorem below we construct all metric bases of G^2_{mn} containing these two corner points.

Theorem 4.8. *Let $\mathcal{B}$ be a metric basis of G^2_{mn} containing $(0, 0)$ and $(0, n-1)$. Then $\mathcal{B}$ is any one of the following:*

(a) $\{(0, 0), (0, n-1), (0, j)\}$ *if both n and j are odd,*
(b) $\{(0, 0), (0, n-1), (m-1, j)\}$ *if both n and $m + j$ are odd,*
(c) $\{(0, 0), (0, n-1), (m-1, 0)\}$ *if $m + n$ is even,*
(d) $\{(0, 0), (0, n-1), (m-1, n-1)\}$ *if $m + n$ is even,*
(e) $\{(0, 0), (0, n-1), (i, 0)\}$ *if i is odd and n is even,*
(f) $\{(0, 0), (0, n-1), (i, n-1)\}$ *if $n + i$ is odd and n is even.*

Proof. Let $\mathcal{B} = \{(0,0),(0,n-1),(s,t)\}$ be an arbitrary basis containing two elements $(0,0)$ and $(0,n-1)$. Theorem 4.1 tells that $(s,t) \in Bd(G^2_{mn})$. Therefore (s,t) is one of $(0,l),(m-1,l),(k,0),(k,n-1)$ for some $0 \le l \le m-1$, $0 \le k \le n-1$. We prove the results by taking two cases according to n is odd or even.

Case 1. n is odd. In this case, first we determine the vertices which are not resolved by $\{(0,0),(0,n-1)\}$. Assume (i_1,j_1) and (i_2,j_2) have same codes with respect to $\{(0,0),(0,n-1)\}$. Then

$$\left\lceil \frac{i_1+j_1}{2} \right\rceil = \left\lceil \frac{i_2+j_2}{2} \right\rceil \quad \text{and} \quad \left\lceil \frac{i_1+n-j_1-1}{2} \right\rceil = \left\lceil \frac{i_2+n-j_2-1}{2} \right\rceil .$$

Combining these two equations, we have $j_1 = j_2$, $i_2 = i_1+1$ or $j_1 = j_2$, $i_2 = i_1 - 1$ according to $i_1 + j_1$ is odd or even. Thus two vertices u and v which are not resolved by $\{(0,0),(0,n-1)\}$ are of the form $u = (i,j)$, $v = (i+1,j)$ with $i+j$ is odd. Thus (s,t) must resolve such u and v. Applying Lemma 4.4, all possible (s,t) are listed in below:

(a) $(0,l)$ on top boundary only if $i+j-l$ is even, i.e., l is odd,
(b) $(m-1,l)$ on bottom boundary only if $m+l-i-j$ is even, i.e., $m+l$ is odd integer,
(c) $(m-1,0)$ only if $m+0-i-j$ is even, i.e., m is odd,
(d) $(m-1,n-1)$ only if $m+0-i-j$ is even, i.e., m is odd.

Again from Remark 4.5, it follows that (s,t) can not be on the left or the right boundary.

Case 2. n is even. As in Case 1, here two vertices u and v which are not resolved by $\{(0,0),(0,n-1)\}$ are of the form $u = (i,j)$ and $v = (i,j+1)$, where $i+j$ is odd. Thus u and v must be resolved by (s,t). Applying Lemma 4.6, all possible (s,t) are listed in below:

(a) $(k,0)$ on the left boundary only if $i+j-k$ is even, i.e., k is odd,
(b) $(k,n-1)$ on the right boundary only if $n+k-i-j$ is even, i.e., k is odd integer,
(c) $(m-1,0)$ only if $m-1$ is odd, i.e., m is even,
(d) $(m-1,n-1)$ only if $m-1$ is odd, i.e., m is even.

From Remark 4.5, it follows that (s,t) can not be on top or bottom boundary. On account of Case 1 and Case 2, we obtain the results. □

From the symmetricity of G^2_{mn}, we have the following theorem that gives all metric bases containing two lower corner points $(m-1,0)$ and $(m-1,n-1)$.

Theorem 4.9. *Let $\mathcal{B}$ be a metric basis of G^2_{mn} containing $(m-1,0)$ and $(m-1,n-1)$. Then $\mathcal{B}$ is any one of the following:*

(a) $\{(m-1,0),(m-1,n-1),(0,j)\}$ *if both n and j are odd,*
(b) $\{(m-1,0),(m-1,n-1),(m-1,j)\}$ *if both $m+j$ and n are odd,*
(c) $\{(0,0),(m-1,0),(m-1,n-1)\}$ *if $m+n$ is even,*
(d) $\{(0,n-1),(m-1,0),(m-1,n-1)\}$ *if $m+n$ is even,*
(e) $\{(m-1,0),(m-1,n-1),(i,0)\}$ *if i is odd and n is even,*
(f) $\{(m-1,0),(m-1,n-1),(i,n-1)\}$ *if $i+n$ is odd and n is even.*

Now, in the theorem below we give all metric bases of G^2_{mn} containing the left corner points $(0,0)$ and $(m-1,0)$.

Theorem 4.10. *Let $\mathcal{B}$ be a metric basis of G^2_{mn} containing $(0,0)$ and $(m-1,0)$. Then $\mathcal{B}$ is any one of the following:*

(a) $\{(0,0),(m-1,0),(i,0)\}$ *if both m and i are odd,*
(b) $\{(0,0),(m-1,0),(i,n-1)\}$ *if both m and $n+i$ are odd,*
(c) $\{(0,0),(m-1,0),(m-1,n-1)\}$ *if $m+n$ is even,*
(d) $\{(0,0),(m-1,0),(0,n-1)\}$ *if $m+n$ is even,*
(e) $\{(0,0),(m-1,0),(0,j)\}$ *if j is odd and m is even,*
(f) $\{(0,0),(m-1,0),(m-1,j)\}$ *if $m+j$ is odd and m is even.*

Proof. Let $\mathcal{B}=\{(0,0),(m-1,0),(s,t)\}$ be a basis. Here we have to determine all possible pair (s,t). From Theorem 4.1, we have $(s,t)\in Bd(G^2_{mn})$, i.e., (s,t) is one of $(0,l),(m-1,l),(k,0),(k,n-1)$ for some $0\le k\le m-1$ and $0\le l\le n-1$. Now we consider the following two cases according as m is odd or even.

Case 1. m is odd. First we determine the vertices which are not resolved by $\{(0,0),(m-1,0)\}$. Let $(i_1,j_1),(i_2,j_2)$ be two distinct vertices of G^2_{mn} which are not resolved by $\{(0,0),(m-1,0)\}$. Then

$$\left\lceil\frac{i_1+j_1}{2}\right\rceil=\left\lceil\frac{i_2+j_2}{2}\right\rceil \quad\text{and}\quad \left\lceil\frac{m-1-i_1+j_1}{2}\right\rceil=\left\lceil\frac{m-1-i_2+j_2}{2}\right\rceil.$$

Solving these two equations, we have $i_2=i_1$, $j_2=j_1+1$ or $i_2=i_1$, $j_2=j_1-1$ according to i_1+j_1 is odd or even. Therefore, the vertices u and v which are not resolved by $\{(0,0),(0,n-1)\}$ are of the form $u=(i,j)$ and $v=(i,j+1)$ with $i+j$ is odd. Thus, u and v must be resolved by (s,t). Applying Lemma 4.6, all possible (s,t) are listed in below:

(a) $(s,t)=(k,0)$ on the left boundary only if $i+j-k$ is even, i.e., k is odd,
(b) $(s,t)=(k,n-1)$ on the right boundary only if $n+k-i-j$ is even, i.e., k is odd integer,
(c) $(s,t)=(m-1,0)$ only if $m-1$ is odd, i.e., m is even,
(d) $(s,t)=(m-1,n-1)$ only if $m-1$ is odd, i.e., m is even.

Again from Remark 4.5, it follows that (s,t) can not be on the top or the bottom boundary.

Case 2. m is even. As in Case 1, here two vertices u and v which are not resolved by $\{(0,0),(0,n-1)\}$ are of the form $u=(i,j)$ and $v=(i+1,j)$, where $i+j$ is odd. Thus, such type u and v must be resolved by (s,t). Applying Lemma 4.4, all possible (s,t) are listed in below:

(a) $(s,t)=(0,l)$ on top boundary only if $i+j-l$ is even, i.e., l is odd,
(b) $(s,t)=(m-1,l)$ on bottom boundary only if $m+l-i-j$ is even, i.e., $m+l$ is odd integer,
(c) $(s,t)=(m-1,0)$ only if $m+0-i-j$ is even, i.e., m is odd,
(d) $(s,t)=(m-1,n-1)$ only if $m+0-i-j$ is even, i.e., m is odd.

From Remark 4.5, it follows that (s,t) can not be on the left or the right boundary. From Case 1 and Case 2, we obtain the results. □

From the symmetricity of G^2_{mn}, we have the following result that gives all metric bases containing the right corner points $(0,n-1)$ and $(m-1,n-1)$.

Theorem 4.11. *Let $\mathcal{B}$ be a metric basis of G^2_{mn} containing $(0,n-1)$ and $(m-1,n-1)$. Then $\mathcal{B}$ is any one of the following:*

(a) $\{(0,n-1),(m-1,n-1),(i,0)\}$ *if both m and i are odd,*
(b) $\{(0,n-1),(m-1,n-1),(i,n-1)\}$ *if both m and $n+i$ are odd,*
(c) $\{(0,0),(0,n-1),(m-1,n-1)\}$ *if $m+n$ is even,*
(d) $\{(m-1,0),(0,n-1),(m-1,n-1)\}$ *if $m+n$ is even,*
(e) $\{(0,n-1),(m-1,n-1),(0,j)\}$ *if j is odd and m is even,*
(f) $\{(0,n-1),(m-1,n-1),(m-1,j)\}$ *if $m+j$ is odd and m is even.*

Now our remaining task is to determine all metric bases containing the corner points which are diagonally placed. In the theorem below we settle these bases.

Theorem 4.12. *Let $\mathcal{B}$ be a metric basis of G^2_{mn} containing $(0,0)$ and $(m-1,n-1)$. If $m+n$ is even, then $\mathcal{B}$ has one of the following forms:*

(a) $\{(0,0),(0,n-1),(m-1,n-1)\}$,
(b) $\{(0,0),(m-1,0),(m-1,n-1)\}$.

Moreover, if $m+n$ is odd then there is no metric basis containing $(0,0)$ and $(m-1,n-1)$.

Proof. Let $\mathcal{B}=\{(0,0),(m-1,n-1),(s,t)\}$ be an arbitrary metric basis. Now we determine the vertices which are not resolved by $\{(0,0),(m-1,n-1)\}$. Let $(i_1,j_1),(i_2,j_2)$ be two arbitrary distinct vertices of G^2_{mn} having same code with respect to $\{(0,0),(m-1,n-1)\}$. Then we have

$$\left\lceil \frac{i_1+j_1}{2} \right\rceil = \left\lceil \frac{i_2+j_2}{2} \right\rceil$$

and

$$\left\lceil \frac{m-1-i_1+n-1-j_1}{2} \right\rceil = \left\lceil \frac{m-1-i_2+n-1-j_2}{2} \right\rceil.$$

Solving these two equations, we get $i_1+j_1=i_2+j_2$. Thus, the vertices having same codes with respect to $\{(0,0),(m-1,n-1)\}$ are the only vertices whose co-ordinate sums are equal. Now we consider the following cases.

Case 1. (s,t) is on the top boundary, i.e., $(s,t)=(0,j)$ with $0<j\le n-1$. If $j\ne n-1$, then the vertices $u=(0,j+1)$ and $v=(1,j)$ are adjacent to $(0,j)$, so these are not resolved by $(0,j)$. Again since co-ordinate sum of u and v are same, these are also not resolved by $\{(0,0),(m-1,n-1)\}$. Thus, $\{(0,0),(m-1,n-1),(0,j)\}$ with $j\ne n-1$ is not a metric basis. Now we show that $\{(0,0),(m-1,n-1),(0,n-1)\}$ forms a metric

basis, i.e., we have to show that two vertices $u = (i_1, j_1)$ and $v = (i_2, j_2)$ having same co-ordinate sum must be resolved by $(0, n-1)$. The equality

$$d((0, n-1), (i_1, j_1)) = d((0, n-1), (i_2, j_2))$$

implies that

$$\left\lceil \frac{i_1 + n - 1 - j_1}{2} \right\rceil = \left\lceil \frac{i_2 + n - 1 - j_2}{2} \right\rceil.$$

Also, we have $i_1 + j_1 = i_2 + j_2$ as u and v have same co-ordinate sum. These two relations give $i_1 = i_2$ and $j_1 = j_2$ and this implies that $(0, n-1)$ resolves the vertices u and v.

Case 2. (s,t) is on the left boundary, i.e., $(s,t) = (i,0)$ with $0 < i \leq m-1$. If $i \neq m-1$, then the vertices $u = (i, 1)$ and $v = (i+1, 0)$ are adjacent to $(i, 0)$, so these are not resolved by $(i, 0)$. Again since both u and v have same co-ordinate sum, so they are not resolved by $\{(0,0), (m-1, n-1)\}$. Thus, $\{(0,0), (m-1, n-1), (i, 0)\}$ with $i \neq n-1$ does not form a metric basis. By similar argument as in Case 1, we can show that two vertices $u = (i_1, j_1)$ and $v = (i_2, j_2)$ having same co-ordinate sum must be resolved by $(m-1, 0)$.

Case 3. (s,t) is on the right boundary, i.e., $(s,t) = (i, n-1)$ with $0 \leq i < m-1$. If $i \neq 0$, then the vertices $u = (i-1, n-1)$ and $v = (i, n-2)$ are adjacent, so these two vertices are not resolved by $(i, n-1)$. Again since co-ordinate sums of u and v are same, these are also not resolved by $\{(0,0), (m-1, n-1)\}$. Thus, $\{(0,0), (m-1, n-1), (i, n-1)\}$ with $i \neq 0$ does not form a basis. If $i = 0$, then as in Case 1, $\{(0,0), (m-1, n-1), (0, n-1)\}$ forms a metric basis.

Case 4. (s,t) is on the bottom boundary, i.e., $(s,t) = (m-1, j)$ with $0 \leq j < n-1$. If $j \neq 0$, then the vertices $u = (m-1, j-1)$ and $v = (m-2, j)$ are adjacent, so these two vertices are not resolved by $(m-1, j)$. Again since co-ordinate sum of u and v are same, these are also not resolved by $\{(0,0), (m-1, n-1)\}$. Thus, $\{(0,0), (m-1, n-1), (0, j)\}$ with $j \neq 0$ does not form a basis. If $j = 0$, then as in Case 2, $\{(0,0), (m-1, n-1), (m-1, 0)\}$ forms a metric basis. Now we show that there does not exist any metric basis containing two corner points $(0,0)$ and $(m-1, n-1)$ when $m+n$ is odd. If possible, let $\mathcal{B} = \{(0,0), (m-1, n-1), (s,t)\}$ be a basis containing $(0,0)$ and $(m-1, n-1)$ where $m+n$ is odd. Let

$$T = \{(m-1, n-2), (m-2, n-2), (m-2, n-1)\}.$$

Then u and $(m-1, n-1)$ are adjacent for each $u \in T$.

Table 7 shows that there is no vertex in any boundary which can resolves these three vertices. Therefore, there is no basis containing $(0,0)$ and $(m-1, n-1)$. □

Table 7. Distances among the vertices of T and the boundary vertices. Here a stands for $m+n-3$

Distance	$(0,j)$	$(m-1,j)$	$(i,0)$	$(i,n-1)$
$(m-1,n-2)$	$\lceil\frac{a-j}{2}\rceil$	$\lceil\frac{n-2-j}{2}\rceil$	$\lceil\frac{a-i}{2}\rceil$	$\lceil\frac{m-i}{2}\rceil$
$(m-2,n-2)$	$\lceil\frac{a-1-j}{2}\rceil$	$\lceil\frac{n-1-j}{2}\rceil$	$\lceil\frac{a-1-i}{2}\rceil$	$\lceil\frac{m-1-i}{2}\rceil$
$(m-2,n-1)$	$\lceil\frac{a-j}{2}\rceil$	$\lceil\frac{n-j}{2}\rceil$	$\lceil\frac{a-i}{2}\rceil$	$\lceil\frac{m-2-i}{2}\rceil$

From the symmetricity of G^2_{mn}, we have the following result that gives all metric bases containing the corner points $(0,n-1)$ and $(m-1,0)$.

Theorem 4.13. *Let $\mathcal{B}$ be a metric basis of G^2_{mn} containing $(m-1,0)$ and $(0,n-1)$. If $m+n$ is an even integer, then $\mathcal{B}$ is of one of the following forms:*

(a) $\{(0,0),(0,n-1),(m-1,0)\}$,
(b) $\{(m-1,0),(0,n-1),(m-1,n-1)\}$.

Moreover, if $m+n$ is odd then there is no basis containing $(m-1,0)$ and $(0,n-1)$.

5. FAULT-TOLERANT RESOLVING SET AND FAULT-TOLERANT METRIC DIMENSION

In this section, we give an optimal fault-tolerant resolving set of G^2_{mn} when $m+n$ is an even integer. For the remaining case when $m+n$ is odd, we give a lower and an upper bound of fault-tolerant metric dimension with difference one. In Section 3, it has been shown that metric dimension of G^2_{mn} is 3, so we have the following result.

Lemma 5.1. *For any fault-tolerant resolving set F of G^2_{mn}, $|F|\geq 4$.*

Theorem 5.2. *The fault-tolerant metric dimension of G^2_{mn} is 4 when $m+n$ is even*

Proof. From Lemma 5.1, the fault-tolerant metric dimension of G^2_{mn} is at least 4. Now we consider the set $F=\{(0,0),(0,n-1),(m-1,0),(m-1,n-1)\}$. Then from Theorem 4.12 and Theorem 4.13, we get that $F\setminus\{v\}$ forms a resolving set for every $v\in F$. Hence, F is a fault-tolerant resolving set of G^2_{mn}. □

Lemma 5.3. *The fault-tolerant metric dimension of G^2_{mn} is at least 5 when $m+n$ is odd.*

Proof. If possible, let F be a fault-tolerant resolving set of G^2_{mn} with cardinality 4. From Theorem 4.3 and Theorem 4.1, it is clear that every metric basis $\mathcal{B}$ contains at least two corner points and $\mathcal{B}\subset Bd(G^2_{mn})$. So F contains at least three corner points, say u,v,w. Out of these three corner points two corner points, say, u and v, must be diagonally placed in G^2_{mn}. Since $F\subset Bd(G^2_{mn})$ and $u,v,w\in F$, $F=\{u,v,w,z\}$ for some $z\in Bd(G^2_{mn})\setminus\{u,v,w\}$. Then applying Theorem 4.12 and Theorem 4.13, it follows that $F\setminus\{w\}$ is not a resolving set of G^2_{mn}, which contradicts that F is a fault-tolerant resolving set and hence the fault-tolerant metric dimension of G^2_{mn} is at least 5 when $m+n$ is odd. □

Theorem 5.4. *The set*

$$F = \{(0,0),(0,1),(1,0),(0,n-1),(m-1,0),(m-1,n-1)\}$$

forms a fault-tolerant resolving set of G^2_{mn} when $m+n$ is an odd integer.

Proof. First we show that $F \setminus \{(0,0)\}$ forms a resolving set. To show this we determine the vertices which are not resolved by $\{(0,1),(1,0),(m-1,n-1)\}$. If $u=(i_1,0)$ and $v=(i_2,0)$ be any two vertices from the left boundary, then codes of u and v with respect to $\{(0,1),(m-1,n-1)\}$ are distinct. Thus, the vertices on the left boundary are resolved by $\{(0,1),(m-1,n-1)\}$. Similarly, the vertices on the top boundary are resolved by $\{(1,0),(m-1,n-1)\}$. Now we consider $u=(i_1,j_1)$ and $v=(i_2,j_2)$ be two vertices neither from the left nor from the top boundary. Then $\mathrm{code}(u)=\mathrm{code}(v)$ implies

$$\left\lceil \frac{i_1+j_1-1}{2} \right\rceil = \left\lceil \frac{i_2+j_2-1}{2} \right\rceil$$

and

$$\left\lceil \frac{m+n-2-i_1-j_1}{2} \right\rceil = \left\lceil \frac{m+n-2-i_2-j_2}{2} \right\rceil.$$

These two relations give $i_2+j_2=i_1+j_1$. But then

$$\left\lceil \frac{m-1-i_1+j_1}{2} \right\rceil \neq \left\lceil \frac{m-1-i_2+j_2}{2} \right\rceil,$$

i.e., $(m-1,0)$ resolves u and v. Therefore, $\{(0,1),(1,0),(m-1,n-1),(m-1,0)\}$ forms a resolving set. Similarly, $F \setminus \{(m-1,n-1)\}$ also forms a resolving set. Again applying Theorems 4.8 and 4.9, we have $F \setminus \{x\}$ is a resolving set for every $x \in \{(0,1),(1,0),(0,n-1),(m-1,0)\}$. Therefore, F is a fault-tolerant resolving set of G^2_{mn} when $m+n$ is an odd integer. □

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