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Analysis of Dynamic Characteristics of Selected Pneumatic Systems with Fractional Calculus. Simulation and Laboratory Research

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ABSTRACT: The section of the paper on simulation studies presents the application of fractional calculus to describe the dynamics of pneumatic systems. In the construction of mathematical models of the analysed dynamic systems, the Riemann-Liouville definition of differ-integral of non-integer order was used. For the analysed model, transfer function of integer and non-integer order was determined. Functions describing characteristics in time and frequency domains were determined, whereas the characteristics of the analysed systems were obtained by means of computer simulation. MATLAB were used for the simulation research. The section of the paper on laboratory research presents the results of the laboratory tests of the injection system of the internal combustion engine with special attention to the verification of simulated tests of selected pneumatic systems described with the use of fractional calculus.

1 INTRODUCTION

The problems of differential and integral calculus of non-integer orders – commonly known as fractional calculus – have been known since the days of famous mathematicians Gottfried Wilhelm Leibniz (1646-1716) and Guillaume François Antoine de l'Hospital (1661-1704) [1–3, 10, 14–16]. However, until today, a description of the dynamic properties of an object by means of fractional calculus has not been used due to the barriers resulting from the lack of appropriate calculation methods and possibilities of verifying them (related, among others, to the limited calculating potential of earlier computers). Nowadays, technical and calculating possibilities cause that former limitations have disappeared and the said problems can now be solved [8, 11]. There are more and more publications dealing with the issue of differential equations of non-integer orders. Majority of them, however, deal with theoretical aspects of the problem.

The dynamic development of research in recent years on the use of fractional calculus for the analysis of dynamic systems has prompted the authors to use it in the analysis and modelling of pneumatic systems that have been described so far with "classical" mathematical analysis [4–13]. The authors of the paper have developed a method for describing the dynamic properties of pneumatic systems, based on fractional calculus which allows to analyse the properties of a wide range of pneumatic systems of any order.

The simulation tests of the membrane pressure transducer, presented in the paper [13], were performed with the use of classical differential calculus and fractional calculus. In the construction of the mathematical model of the analysed dynamical system, the Riemann-Liouville definition of the differ-integral of non-integer order was used.

For simulation studies MATLAB were used [5, 6, 8, 9]. In the laboratory tests, which were the verification of the simulation tests of the membrane pressure transducer, the following assumptions were made: the analysed pneumatic systems were modelled as a linear system; the pneumatic system was described with a transfer function characterizing the dynamics of this system and the components contained therein, assuming constant physical parameters and omission of aging of its components; an assessment was accepted of the dynamic properties of the pneumatic systems in terms of amplitude and phase; pneumatic systems with a pressure of up to 1MPa were analysed while operating in the frequency range up to 500Hz; with the variability of the thermodynamic parameters of the air as a working medium, it can be treated as an ideal gas; in the analysis of the pneumatic systems, an adiabatic process was assumed whereas the pressure distribution in the whole volume of the measuring chamber was homogeneous.

2 MEMBRANE PRESSURE TRANSDUCER

Simulation tests of the membrane pressure transducer were performed using a classical and fractional differential calculus. The tested transducer was made from a pressure chamber and an inlet pipe, which supplied a working medium (air). To determine how the connection of the intake pipe to the transducer chamber affected its dynamic properties, the acoustic system shown schematically in Figure 1 is considered.

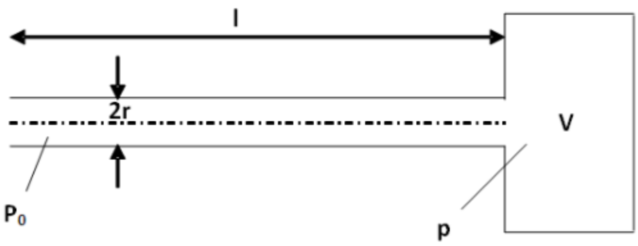

Figure 1. Pressure chamber with inlet pipe: r, l - tube dimensions, p_0 - input pressure (force), p - pressure in transducer chamber

The relationship that binds the output signal p(t) (pressure inside the chamber) to the signal $p_0(t)$ (pressure at the open end) and referring to the RLC electrical circuits can be represented as:

$$\frac{d^2 p(t)}{dt^2} + \frac{R_p}{L_p}\frac{dp(t)}{dt} + \frac{1}{L_p C_p} p(t) = \frac{1}{L_p C_p} p_0(t) \tag{1}$$

Constants occurring in expression (1) can be represented as:

$$C_p = \frac{V}{\rho c^2} \left[Ns^2 m^{-5}\right] \tag{2a}$$

$$L_p = \frac{4l\rho}{3\pi r^2} \left[m^5 N^{-1}\right] \tag{2b}$$

$$R_p = \frac{8\eta l}{\pi r^2} \left[Nsm^{-5}\right] \tag{2c}$$

where:
ϱ [kgm^{-3}] - gas density;
η [$kgm^{-1}s^{-1}$] - dynamic viscosity;
C_p [Ns^2m^{-5}] - pneumatic capacity (gas compressibility);
p(t) [Pa] - pressure in transducer chamber;
$p_0(t)$ [Pa] - input pressure;
V [m^3] - transducer chamber volume;
L_p [m^3N^{-1}] - pneumatic induction (gas inertia);
R_p [Nsm^{-5}] - flow resistance;
c [ms^{-1}] - speed of sound in the gas;
r, l [m] - dimensions of the inlet pipe.

By specifying the frequency ω0 and damping ratio ξ as:

$$\omega_0 = \frac{1}{\sqrt{L_p C_p}} = \sqrt{\frac{3\pi^2 c^2}{4lV}} \tag{3a}$$

$$\xi = \frac{R_p C_p \omega_0}{2} = \frac{R_p}{2}\sqrt{\frac{C_p}{L_p}} = 2\frac{\eta\sqrt{\frac{3lV}{\pi}}}{r\rho c} = 2\sqrt{\frac{3\eta^2 lV}{\pi r^2 \rho^2 c^2}} \tag{3b}$$

where: ξ <1.

Expression (1) finally assumes the form:

$$\frac{d^2 p(t)}{dt^2} + 2\xi\omega_0 \frac{dp(t)}{dt} + \omega_0^2 p(t) = \omega_0^2 p_0(t) \tag{4}$$

Equation (4) is a mathematical description of the dynamics of the analysed pneumatic system, using classical differential calculus (of integer orders). The impulse response of the analysed pneumatic system is given by:

$$g(t) = \frac{\omega_0}{\sqrt{1-\xi^2}} e^{-\xi\omega_0 t} \sin \omega_0 \sqrt{1-\xi^2}\, t \tag{5}$$

The step response of the system is expressed by:

$$h(t) = 1 - \frac{e^{-\xi\omega_0 t}}{\sqrt{1-\xi^2}} \sin\left(\omega_0 \sqrt{1-\xi^2}\, t + \varphi\right) \tag{6}$$

where:

$$\varphi = arctg \frac{\sqrt{1-\xi^2}}{\xi} \tag{7}$$

Given that the derivatives of integer orders in the fractional calculus are a special case of fractional derivatives, equation (4) can be written as:

$${}_0^{RL}D_t^{2v} p(t) + 2\xi\omega_0\, {}_0^{RL}D_t^{v} p(t) + \omega_0^2 p(t) = \omega_0^2 p_0(t) \tag{8}$$

where v>0.

In order to determine the pressure in the transducer chamber, the definition of Riemann-Liouville differ-integral of non-integer order was used, defined by a following formula (9):

$$ {}_{a}^{RL}D_t^{\alpha} f(t) = \frac{1}{\Gamma(k-\alpha)} \frac{d^k}{dt^k} \int_a^t (t-\tau)^{k-\alpha-1} f(\tau)\, d\tau \tag{9} $$

where:
α – the order of integration within bounds (a, t) of the function f(t), k-1⩽a⩽k and $\alpha \in R+$ and Γ - Euler's gamma function.

The Laplace transform for a fractional derivative defined by Riemann-Liouville takes the form:

$$ L\left[{}_{0}^{R-L}D_t^{\alpha} f(t)\right] = s^{\alpha} F(s) - \sum_{k=0}^{j-1} s^k \, {}_{0}^{R-L}D_t^{\alpha-k-1} f(0) \tag{10} $$

where $j-1 \leqslant \alpha \leqslant j \in N$.

The practical application of the formula determining the Laplace transform of a Riemann-Liouville fractional derivative faces some difficulties related to the lack of physical interpretation of the initial values of successive derivatives of fractional orders. Assuming zero initial conditions, the difficulties associated with their physical interpretation will be eliminated.

Using the Laplace transform to equation (8), for zero initial conditions, we obtain:

$$ s^{2v} p(s) + 2\xi\omega_0 s^v p(s) + \omega_0^2 p(s) = \omega_0^2 p_0(s) \tag{11} $$

Thus the transfer function of non-integer order of the analysed pneumatic system is obtained:

$$ G^{(v)}(s) = \frac{\omega_0^2}{s^{2v} + 2\xi\omega_0 s^v + \omega_0^2} \tag{12} $$

Transfer function denominator of non-integer order has two complex roots as the system damping is $\xi < 1$.

3 IMPULSE RESPONSE TO THE MEMBRANE PRESSURE TRANSDUCER

By transforming expression (12), we obtain $G^{(v)}(s)=p(s)/p_0(s)$ such that [12, 13]:

$$ G^{(v)}(s) = \frac{\omega_0^2}{s^{2v} + 2\xi\omega_0 s^v} \cdot \frac{1}{1 + \dfrac{\omega_0^2}{s^{2v} + 2\xi\omega_0 s^v}} \tag{13} $$

Using the properties of the geometric series, we obtain:

$$ G^{(v)}(s) = \frac{\omega_0^2}{s^{2v} + 2\xi\omega_0 s^v} \cdot \sum_{k=0}^{\infty} \frac{(-\omega_0^2)^k}{(s^{2v} + 2\xi\omega_0 s^v)^k} \tag{14} $$

Conducting elementary transformations, we obtain:

$$ G^{(v)}(s) = \sum_{k=0}^{\infty} \frac{(-\omega_0^2)^k}{k!} \frac{k!\,\omega_0^2 s^{v-(2v+vk)}}{\left(s^v - (-2\xi\omega_0)\right)^{k+1}} \tag{15} $$

Using the formula:

$$ L\left\{t^{\alpha m+\beta-1} E_{\alpha,\beta}^{(m)}(at^{\alpha})\right\} = \frac{m!\, s^{\alpha-\beta}}{(s^{\alpha}-a)^{m+1}} \tag{16} $$

we obtain:

$$ g^{(v)}(t) = L^{-1}\left(G^{(v)}(s)\right) = = \sum_{k=0}^{\infty} \frac{(-\omega_0^2)^k \omega_0^2}{k!} t^{vk+2v+vk-1} E_{v,2v+vk}^{k}\left(-2\xi\omega_0 t^v\right) \tag{17} $$

where:

$$ E_{\alpha,\beta}^{(k)} = \sum_{n=0}^{\infty} \frac{(n+k)!}{n!} \frac{t^n}{\Gamma(\alpha n + \alpha k + \beta)} \tag{18} $$

The simulation of the pressure impulse response in the transducer chamber required a program to be written in the MATLAB environment. The program for the given parameters and derivative orders calculates the function values and draws out their impulse response. We present, for comparison, the graphs of the function obtained for the classic solution (v=1) and for several fractional orders.

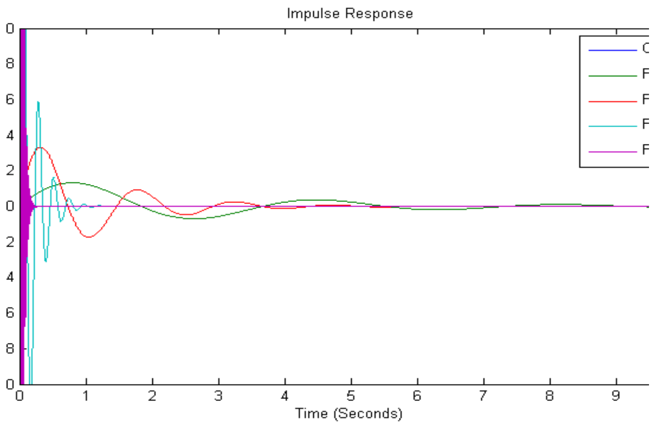

Figure 2. Impulse response of a pneumatic transducer described with an integer and non-integer order: $F_{0,5}$ for v=0.5, $F_{0,7}$ for v=0.7, $F_{0,9}$ for v=0.9, $F_{1,0}$ for v=1, C_2 - classical model (integer order).

In Figure 2, the impulse response of the analysed pneumatic transducer was determined by simulating equation (17) for the selected parameter values: $F_{0,5}$ for v=0.5, $F_{0,7}$ for v=0.7, $F_{0,9}$ for v=0.9 and $F_{1,0}$ for v=1.

The impulse response (characteristics C_2 in the above figure) was also presented, by simulating a computer equation (5) which is a mathematical model of the analysed pneumatic system, with the use of a classical differential calculus (described by ordinary differential equation). It is worth noting that while reducing the row, it reduces the response time, which is desirable in measuring transducers.

4 STEP RESPONSE OF THE MEMBRANE PRESSURE TRANSDUCER

The step response of the tested transducer is defined by the relationship:

$$H^{(v)}(s)=\frac{\omega_0^2}{s\left(s^{2v}+2\xi\omega_0 s^v+\omega_0^2\right)}=$$
$$=\frac{\omega_0^2 s^{-1}}{s^{2v}+2\xi\omega_0 s^v}\cdot\frac{1}{1+\dfrac{\omega_0^2}{s^{2v}+2\xi\omega_0 s^v}} \tag{19}$$

Using the properties of geometric series, we obtain

$$H^{(v)}(s)=\frac{\omega_0^2 s^{-1}}{s^{2v}+2\xi\omega_0 s^v}\cdot\sum_{k=0}^{\infty}\frac{\left(-\omega_0^2\right)^k}{\left(s^{2v}+2\xi\omega_0 s^v\right)^k} \tag{20}$$

Conducting elementary transformations, we obtain:

$$H^{(v)}(s)=\sum_{k=0}^{\infty}\frac{\left(-\omega_0^2\right)^k}{k!}\frac{k!\,\omega_0^2 s^{v-(2v+vk+1)}}{\left(s^v-(-2\xi\omega)\right)^{k+1}} \tag{21}$$

Using the formula (16), we obtain:

$$h^{(v)}(s)=\sum_{k=0}^{\infty}\frac{\left(-\omega_0^2\right)^k\omega_0^2}{k!}t^{vk+2v+vk+1-1}E^k_{x,2v+vk+1}\left(-2\xi\omega_0 t^v\right) \tag{22}$$

in which $E^{(k)}_{\alpha,\beta}$ is the Mittag-Leffler function defined by the equation (18).

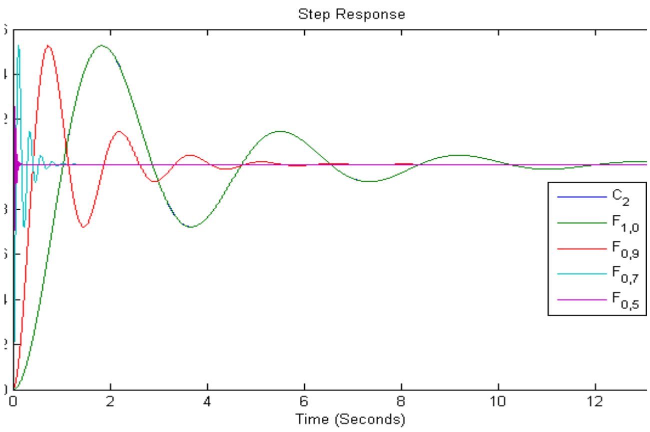

Figure 3. The step response of the pneumatic system: $F_{0,5}$ for v=0.5, $F_{0,7}$ for v=0.7, $F_{0,9}$ for v=0.9, $F_{1,0}$ for v=1, C_2 - classical model (integer order).

Running a simulation of a pneumatic transducer, a unit step signal was applied and the received step response is shown in Figure 3.

The model described by equation (17) and (22) correctly reproduces the amplitude of the input signal as the classical model - the graphs coincide (graph $F_{1,0}$ - the parameter v=1 coincides with C_2 – the classical model). This confirms the correctness of the method and that the differential calculus with derivatives of integer orders is a special case of fractional calculus. The step response (Figure 3) shows that regardless of the differential order, the amplitude of the signal is constant. The smaller the order of the derivative leads to the faster the reaction of the system to the unit step.

5 FREQUENCY RESPONSE OF THE MEMBRANE PRESSURE TRANSDUCER

In order to determine the relationships describing the frequency response, the spectral transfer function of the tested transducer was determined [6, 9]. Substituting (23):

$$s=j\omega=\omega e^{j\frac{\pi}{2}}=\omega\left[\cos\left(\frac{\pi}{2}\right)+j\sin\left(\frac{\pi}{2}\right)\right] \tag{23}$$

to the formula (12), the spectral transfer function of the transducer is obtained:

$$G^{(v)}(j\omega)=\frac{\omega_0^2}{(j\omega)^{2v}+2\xi\omega_0(j\omega)^v+\omega_0^2} \tag{24a}$$

$$G^{(v)}(j\omega)=$$
$$=\frac{\omega_0^2}{\omega^{2v}\left[\cos(v\pi)+j\sin(v\pi)\right]+2\xi\omega_0\omega^v\left[\cos\left(v\frac{\pi}{2}\right)+j\sin\left(v\frac{\pi}{2}\right)\right]+\omega_0^2} \tag{24b}$$

By making elementary transformations, the real and imaginary part of the spectral transfer function was calculated:

$$G^{(v)}(j\omega)=P^{(v)}(\omega)+jQ^{(v)}(\omega) \tag{25}$$

where:

$$P^{(v)}(\omega)=$$
$$=\frac{\omega_0^2\omega^{2v}\cos(v\pi)+2\xi\omega_0^3\omega^v\cos\left(\frac{v\pi}{2}\right)+\omega_0^4}{\left[\omega^{2v}\cos(v\pi)+2\xi\omega_0\omega^v\cos\left(\frac{v\pi}{2}\right)+\omega_0^2\right]^2+\left[\omega^{2v}\sin(v\pi)+2\xi\omega_0\omega^v\sin\left(\frac{v\pi}{2}\right)\right]^2} \tag{26a}$$

$$Q^{(v)}(\omega)=$$
$$=\frac{\omega_0^2\omega^{2v}\sin(v\pi)+2\xi\omega_0^3\omega^v\sin\left(\frac{v\pi}{2}\right)}{\left[\omega^{2v}\cos(v\pi)+2\xi\omega_0\omega^v\cos\left(\frac{v\pi}{2}\right)+\omega_0^2\right]^2+\left[\omega^{2v}\sin(v\pi)+2\xi\omega^v\omega_0\sin\left(\frac{v\pi}{2}\right)\right]^2} \tag{26b}$$

Knowing the real and imaginary part of the spectral transfer function of the transducer, one can determine the equation describing the logarithmic amplitude characteristic:

$$L^{(v)}(\omega)=20\log\sqrt{\left[P^{(v)}(\omega)\right]^2+\left[Q^{(v)}(\omega)\right]^2} \tag{27}$$

and the equation describing the logarithmic phase step:

$$\varphi^{(v)}(\omega) = arctg\left[\frac{Q^{(v)}(\omega)}{P^{(v)}(\omega)}\right] = = -arctg\left[\frac{\omega^{2v}\sin(v\pi) + 2\xi\omega_0\omega^v \sin\left(\frac{v\pi}{2}\right)}{\omega^{2v}\cos(v\pi) + 2\xi\omega_0\omega^v \cos\left(\frac{v\pi}{2}\right) + \omega_0^2}\right] \tag{28}$$

In order to verify the relationships describing logarithmic steps of amplitude (27) and phase (28) of the tested transducer, the pneumatic pressure transducer was modelled in the MATLAB environment, described by means of ordinary differential equation and differential equation with derivatives of non-integer order. Describing the transducer with the use of a differential equation of non-integer orders, the parameter v=1 was assumed and the obtained logarithmic amplitude and phase steps were compared to the logarithmic amplitude and phase steps obtained from the transducer description by means of the ordinary differential equation.

The transfer function of the pneumatic pressure transducer, calculated from the ordinary differential equation, has the form:

$$G(s) = \frac{p(s)}{p_0(s)} = \frac{\omega_0^2}{s^2 + 2\xi\omega_0 s + \omega_0^2} \tag{29}$$

By performing the simulation of equation (29) which presents the dynamics of the phenomena occurring in the analysed pneumatic system, in the MATLAB programming environment, the frequency response presented in Figure 4 was obtained:

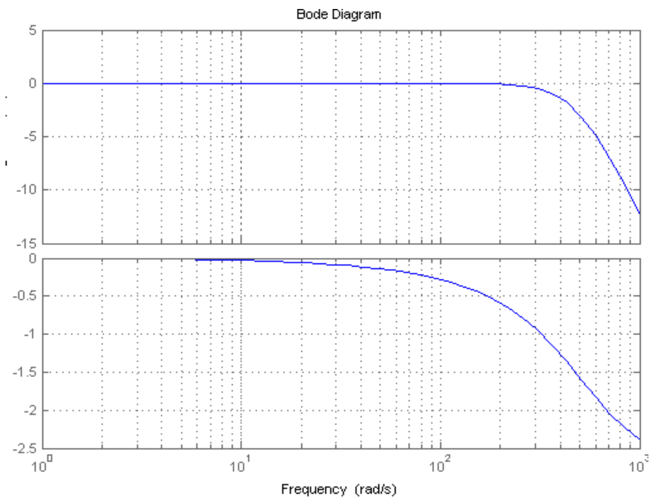

Figure 4. Logarithmic frequency response of the transducer described by the ordinary differential equation

When simulating equations (27) and (28) in the MATLAB environment which describe a pneumatic pressure transmitter by means of a differential equation of non-integer order, assuming a coefficient v=1 for damping ξ=0.8, the response shown in Figure 5 was obtained:

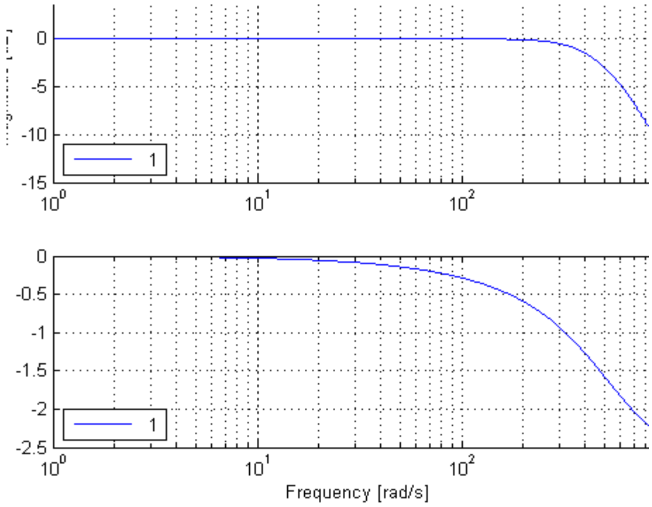

Figure 5. Logarithmic frequency response of a pneumatic transducer described by means of a differential equation with non-integer order for v=1 (equation 27 and 28)

Logarithmic frequency response of amplitude and phase presented by the simulation of ordinary differential equation (Figure 4), coincide with frequency response obtained by the simulation of the equations describing logarithmic response of amplitude (27) and phase (28), obtained from the equation of the transducer described with the help of non-integer orders (figure 5) for the parameter v=1.

In order to obtain a Bode plot, the equations (27) and (28) were simulated by writing an appropriate program in the MATLAB environment. Written in the MATLAB environment, the program allows analysing the transducer for different orders of derivatives, with any step, because the order was given as a parameter. The simulation results for the selected values of parameter v are shown in Figure 6 and Figure 7.

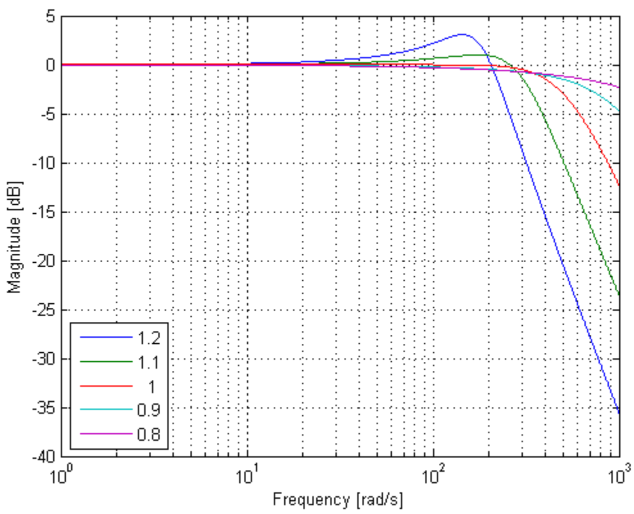

Figure 6. Logarithmic amplitude response of a pneumatic transducer described by means of differential equation with fractional derivatives of non-integer orders in the range (0.8-1.2)

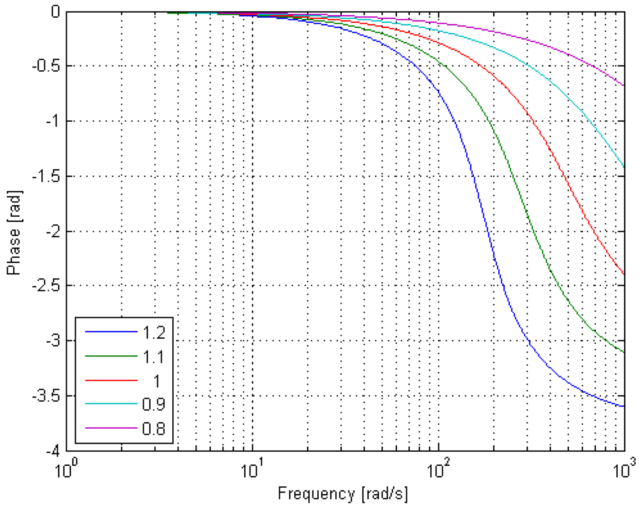

Figure 7. Logarithmic phase response of a pneumatic transducer described by means of differential equation with fractional derivatives of non-integer orders in the range (0.8-1.2)

The analysis of the responses shows that for the parameter v<1, the logarithmic amplitude responses (Figure 6) are monotonically decreasing functions. For the parameter v>1, the logarithmic amplitude responses have a maximum depending on the order of the differential. The maximum is achieved with resonant frequency ω_R=110rad/s.

Increasing the order of derivative, the frequency responses acquire the character of a second-order oscillatory element, and while decreasing the order of the derivative, the responses acquire the character of the first order inertial element.

Decreasing the order of the derivative causes the transducer to become more linear, which allows the scope of work to be increased.

Increasing the parameter v above one results in resonance, although it should not be visible in the response, because the simulation was carried out for the damping ξ=0.8. The model then does not reflect the real system.

6 LABORATORY TESTS OF THE PRESSURE TRANSDUCER

In order to identify the dynamics of the pressure transducer, the measuring system was constructed as it is shown in Figure 8.

The AVL single-cylinder automatic ignition engine was used for testing [6, 13]. It is an internal combustion engine with a capacity of 511cm^3, cylinder diameter 85.01mm and a piston stroke of 90mm. The measurements were made in the inlet air system to the engine. The air supply was provided by an additional system consisting of a rotary screw compressor. Thanks to this system it was possible to regulate the air pressure in the intake system.

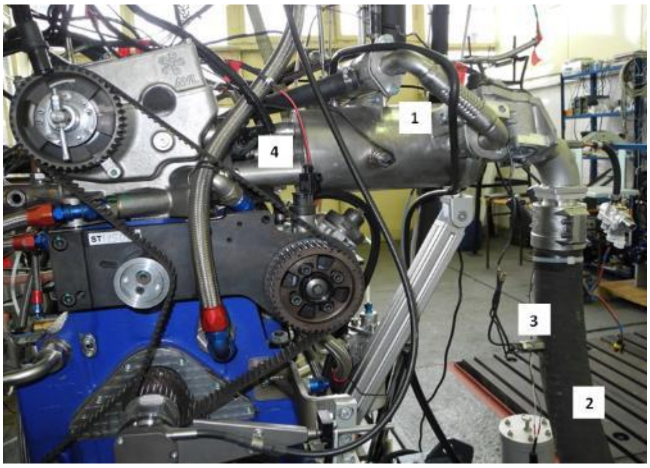

Figure 8. View of measuring station: 1 - measuring chamber, 2 - intake manifold, 3-input pressure transducer, 4 - output pressure transducer

In the air intake system leading into the measuring chamber, the first pressure transducer was installed. The second transducer was installed inside the measuring chamber, at the outlet of the air into the combustion chamber. Kulite pressure transducers, type ETL-189- 190M-10 BARA were used. Two identical pressure transducers were used in the system.

The tests were performed with Concerto and Puma software, whose interfaces are shown in Figure 9.

The presented measuring system allows studying the dynamic properties of the pressure transducer. The studies refer to the time and frequency analysis of the investigated pressure transducer described with integer and non-integer order.

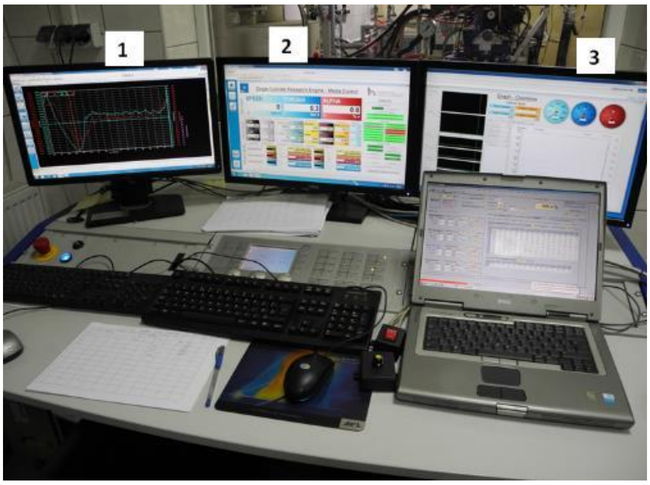

Figure 9. Interfaces of the programs for motor control and recording fast-changing parameters: 1 - Concerto program window for fast variable parameters recording (monitor 1), 2 - Puma program window for controlling and archiving engine parameters (monitor 2), 3 - Puma program window for controlling and archiving engine parameters (monitor 3)

In the measuring system, computers with Concerto and Puma software were used. The Concerto program allows recording fast-changing system parameters and recording them in time and numeric format. The Puma program was used to control the engine. The air supplied into the measuring system is provided by a rotary screw compressor. The engine draws air into the combustion chamber from the measuring system by opening the valve located at the exit of the measuring chamber. The valve opens and closes every two turns of the engine crankshaft. Cyclical opening of the valve caused the same effect as supplying the system with a pneumatic rectangular signal generator, which allowed experimental evaluation of the step response of the measuring system. The step response of the system was determined using the Concerto program, and its graphic representation is shown in Figure 10.

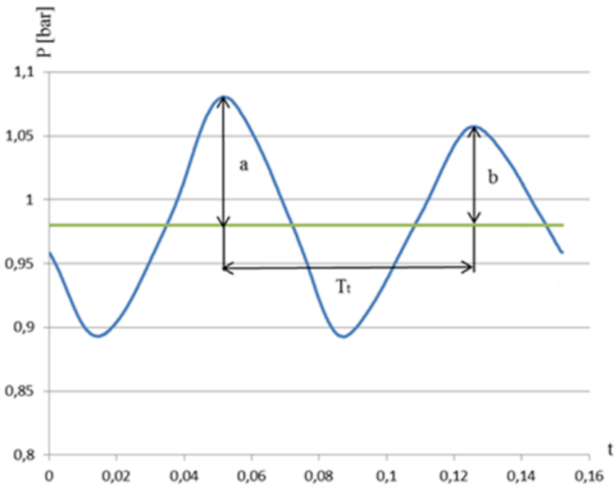

Figure 10. The step response of the measuring system in the measuring chamber and inlet tube obtained experimentally.

The obtained graph is a step response of a typical oscillating element with frequency ω_0 and damping ratio ξ. In order to identify the dynamic properties of the tested pneumatic system it is convenient to determine its transfer function.

Knowing the dependency (30):

$$G(s)=\frac{p(s)}{p_0(s)}=\frac{\omega_0^2}{s^2+2\xi\omega_0 s+\omega_0^2} \qquad (30)$$

and the parameters ω_0 and ξ (31a 31b) obtained from Figure 10:

$$\xi=0.042 \qquad (31a)$$

$$\omega_0=84.684\left[\frac{rad}{s}\right] \qquad (32b)$$

allows determining the transfer function of non-integer order of the analysed pneumatic system. The pulsatance and the damping ratio of the tested system can be determined directly from the obtained step response. The transfer function of non-integer order of the tested system presents the following dependence (32):

$$G^{(v)}(s)=\frac{7171.38}{s^{2v}+7.1135s^{v}+7171.38} \qquad (32)$$

Using the dependence (33):

$$h^{(v)}(s)= =\sum_{k=0}^{\infty}\frac{\left(-\omega_0^2\right)^k\omega_0^2}{k!}t^{vk+2v+vk+1-1}E^{k}_{v,2v+vk+1}\left(-2\xi\omega_0 t^{v}\right) \qquad (33)$$

we obtain a relationship describing the step response of non-integer order of the analysed pneumatic system:

$$h^{(v)}(s)= =\sum_{k=0}^{\infty}\frac{(-7171.38)^k\,7171.38}{k!}t^{2vk+2v}E^{k}_{v,2v+vk+1}\left(-7.1135t^{v}\right) \qquad (34)$$

The equation (34) was simulated with the MATLAB software and as a result the step response of the non-integer order of the tested pneumatic system inside the measuring chamber was obtained, the graph of which is shown in Figure 11.

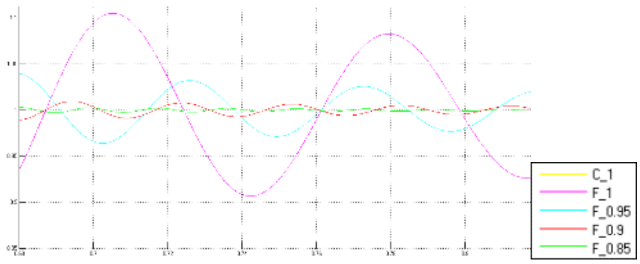

Figure 11. The step response of the measuring system in the transducer measuring chamber

The step response shown in Figure 11, resulting from the simulation of equation (35), for the parameter v=1 (F_1) coincides with the graph determined by means of the differential equation of integer order (C_1) and the graph of the step response obtained experimentally. This means that the model has been correctly designated. Decreasing the order of the derivative causes a decrease in the amplitude of the step response.

$$G^{(v)}(j\omega)=P^{(v)}(\omega)+jQ^{(v)}(\omega) \qquad (35)$$

The spectral transfer function of non-integer order of the tested transducer was obtained by using the experimentally determined values (31a) and (31b) in equations (36a) and (36b), which is the real and imaginary part of the spectral transfer function (35).

$$P^{(v)}(\omega)= =\frac{\omega_0^2\omega^{2v}\cos(v\pi)+2\xi\omega_0^3\omega^{v}\cos\left(\frac{v\pi}{2}\right)+\omega_0^4}{\left[\omega^{2v}\cos(v\pi)+2\xi\omega_0\omega^{v}\cos\left(\frac{v\pi}{2}\right)+\omega_0^2\right]^2+\left[\omega^{2v}\sin(v\pi)+2\xi\omega_0\omega^{v}\sin\left(\frac{v\pi}{2}\right)\right]^2} \qquad (36a)$$

$$Q^{(v)}(\omega)= =\frac{\omega_0^2\omega^{2v}\sin(v\pi)+2\xi\omega_0^3\omega^{v}\sin\left(\frac{v\pi}{2}\right)}{\left[\omega^{2v}\cos(v\pi)+2\xi\omega_0\omega^{v}\cos\left(\frac{v\pi}{2}\right)+\omega_0^2\right]^2+\left[\omega^{2v}\sin(v\pi)+2\xi\omega^{v}\omega_0\sin\left(\frac{v\pi}{2}\right)\right]^2} \qquad (36b)$$

The equation describing the logarithmic amplitude characteristic can be determined from the equation (27):

$$L^{(v)}(\omega) = 20\log\sqrt{\left[P^{(v)}(\omega)\right]^2 + \left[Q^{(v)}(\omega)\right]^2} \quad (37)$$

The equation describing the logarithmic phase characteristic is given by the dependence (38):

$$\varphi^{(v)}(\omega) = -arctg\left[\frac{7171.38\omega^{2v}\sin(v\pi) + 51013.295\omega^{v}\sin\left(\frac{v\pi}{2}\right)}{7171.38\omega^{2v}\cos(v\pi) + 51013.295\omega^{v}\cos\left(\frac{v\pi}{2}\right) + 51428689.039}\right] \quad (38)$$

After simulating equation (37), with the use of equation (36a), a logarithmic amplitude response of the non-integer order of the tested measurement system was obtained for different values of the parameter v, which is shown in Figure 12. Logarithmic phase response (Figure 12) was obtained by simulating equation (36b). Against the response obtained by means of simulation of the mathematical model in which the actual parameters of the tested transducer were applied, the transducer response obtained experimentally was presented.

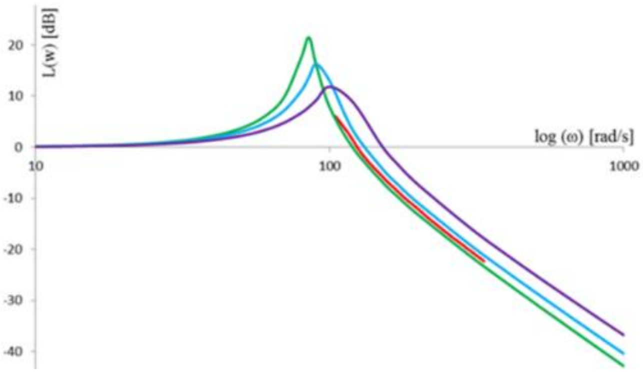

Figure 12. Logarithmic amplitude response determined experimentally and theoretically for different values of the parameter v: red – laboratory results.

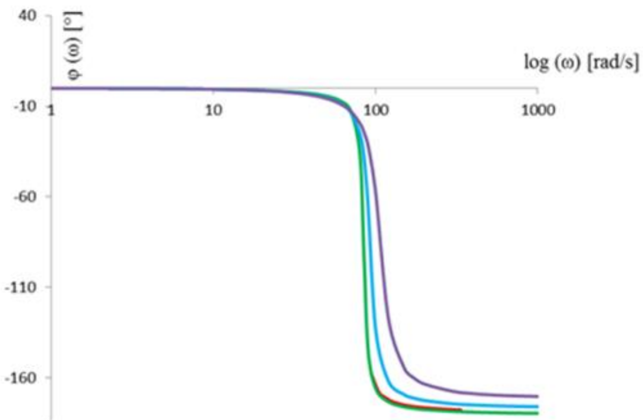

Figure 13. Logarithmic phase response determined experimentally and for different values of the parameter v: red – laboratory results.

The test stand facilitates changing the rotation speed of the rotor and positioning it so as to take measurements. The speed change allows setting the opening frequency of the valve and thus changing the air pressure in the measuring chamber. The regulation of the opening speed of the valve gives the same effect as feeding the system with a pneumatic generator with adjustable frequency, which facilitates obtaining logarithmic frequency response.

Changes in the pressure values in the measuring chamber were recorded for different rotor speed values in the range (100rpm-3200rpm), which corresponds to the frequency ω(104.717[rad/s]-335.093[rad/s]).

The input signal is the pressure in the intake pipe p0, the output signal is the pressure in the transducer chamber (Figure 1). Logarithmic phase response was determined by measuring the phase shift between the output and input signal for each set motor shaft speed.

The minimum rotation speed of the internal combustion engine used in the tests is 1000rpm. This limitation made it impossible to obtain experimentally the full frequency response shown in Figures 12 and 13. The obtained responses were made from frequency ω=104.717[rad/s] to ω=335.093 [rad/s].

By comparing the obtained responses with the responses of non-integer order for the parameter v=1, it can be stated that the tested transducer is of a slightly smaller order than the second order oscillating element. Experientially determined frequency responses are included between the simulated frequency response for parameter v=0.98 and v=1. This means that the transducer should be modelled with the equations of non-integer order. It can therefore be concluded that the description with the classical method would be inaccurate.

The presented simulation studies were performed in the MATLAB development environment (manufacturer: The MathWorks). The authors of the paper declare that, using the above mentioned trademark, they did so only with reference to this publication and with such an intention that it would be for the benefit of the trademark holders but without the intention of infringing the trademark.

7 CONCLUSIONS

The pneumatic system, analysed in the paper in the part on simulation, represents the second order damping oscillator with damping ratio ξ<1, which means that the characteristic equation of the model does not have real solutions. Therefore, the authors were required to develop the original method for determining the relationships describing the time and frequency responses for dynamic systems described with fractional calculus. In the construction of the mathematical model of the analyzed dynamic system, the definition of Riemann-Liouville differ-integral (of fractional order) was used.

The paper presents the results of the laboratory tests of the pressure transducer, which was described with a mathematical model. The analysis of the dynamic properties of the model in terms of time and frequency was conducted. The parameters of the tested pressure transducer were determined experimentally - the damping ratio ξ and the frequency ω_0, which were used to determine the transfer function of integer and non-integer order. Having the knowledge of transfer function, the step response was determined as well as logarithmic amplitude response and phase response of integer and non-integer order. It was found out that the time response obtained experimentally, coincides with the response determined by the developed model of non-integer order of the transducer for the parameter v=1. This confirms the correctness of the designated model.

Frequency responses obtained experimentally differ slightly from the responses obtained in computer simulation. The logarithmic amplitude response and phase response obtained experimentally are of the order of the oscillating element for the parameter v of the interval $0.98<v<1$.

In the real system shown in the paper, the pressure in the transducer chamber was measured at the outlet of the air into the combustion chamber, where, at the moment of aspiration of air through the engine, the air reaches the speed of sound. Such conditions may account for a slight decrease in the order of the oscillating element under testing.

The analysis of the logarithmic amplitude response of the models presented in the paper shows that the local maximum present in these characteristics is dependent on the order of the derivative and the bigger the amplitude, the higher the order of the derivative. For the parameter v=1 (classical model) the amplitude reaches the maximum at the resonant frequency for damping $\xi<1$. With the decreasing order of the derivative, the increase in the resonant frequency of the circuit can be observed.

Fractional calculus is particularly useful in building dynamic models of mathematical systems working in conditions that cannot be described with differential equations of integer orders. This can be deduced by analyzing systems such as the long electric line of infinitely large length or the supercapacitor of a few thousand Farads, which are now also described with fractional calculus.

REFERENCES

1. Caponetto, R., Dongola, G., Fortuna, L., Petráš, I.: Fractional Order Systems. World Scientific (2010). https://doi.org/10.1142/7709.
2. Diethelm, K.: The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type. Springer-Verlag, Berlin Heidelberg (2010). https://doi.org/10.1007/978-3-642-14574-2.
3. Kaczorek, T.: Selected Problems of Fractional Systems Theory. Springer-Verlag, Berlin Heidelberg (2011). https://doi.org/10.1007/978-3-642-20502-6.
4. Luft, M., Łukasik, Z., Pietruszczak, D.: Analysis of dynamic characteristics of selected pneumatic systems with fractional calculus used in telematics. Archives of Transport System Telematics. 11, 4, 37–44 (2018).
5. Luft, M., Nowocień, A., Pietruszczak, D.: Analiza właściwości dynamicznych wybranych układów pneumatycznych za pomocą rachunku różniczkowego niecałkowitych rzędów. Cz. 1, Badania symulacyjne. atest. 18, 12, 1050–1055 (2017).
6. Luft, M., Nowocień, A., Pietruszczak, D.: Analiza właściwości dynamicznych wybranych układów pneumatycznych za pomocą rachunku różniczkowego niecałkowitych rzędów. Cz. 2, Badania laboratoryjne. atest. 18, 12, 1056–1060 (2017).
7. Luft, M., Nowocień, A., Pietruszczak, D.: Modele matematyczne kaskady pneumatycznej oraz membranowego siłownika pneumatycznego opisane rachunkiem różniczkowym niecałkowitych rzędów. atest. 19, 12, 526–531 (2018). https://doi.org/10.24136/atest.2018.444.
8. Luft, M., Nowocień, A., Pietruszczak, D.: Właściwości dynamiczne wybranych podstawowych członów automatyki niecałkowitych rzędów. atest. 19, 12, 1056–1060 (2018). https://doi.org/10.24136/atest.2018.443.
9. Luft, M., Nowocień, A., Pietruszczak, D., Lisak, A., Chamela, W.: Frequency response of the pressure transducer model described by the fractional order differential equation. TTS. 12, 128–132 (2016).
10. Ostalczyk, P.: Zarys rachunku różniczkowo-całkowego ułamkowych rzędów : teoria i zastosowania w automatyce. Wydawnictwo Politechnika Łódzka (2008).
11. Pietruszczak, D.: Laboratory investigations of dynamic properties of accelerometers with fractional orders for application in telematic equipment. Archives of Transport System Telematics. 5, 2, 26–31 (2012).
12. Pietruszczak, D., Luft, M., Lesiak, P.: Some applications of fractional calculus in modelling of accelerometer and pressure transducer. Zeszyty Naukowe Wydziału Elektrotechniki i Automatyki Politechniki Gdańskiej. 47, (2015).
13. Pietruszczak, D., Nowocień, A.: Description of selected pneumatic elements and systems using fractional calculus. jaeee. 1, 1, 43–49 (2019). https://doi.org/10.24136/jaeee.2019.006.
14. Podlubny, I.: Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, Some Methods of Their Solution and Some of Their Applications. Academic Press (1998).
15. Sabatier, J., Agrawal, O.P., Machado, J.A.T. eds: Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering. Springer Netherlands (2007). https://doi.org/10.1007/978-1-4020-6042-7.
16. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives: Theory and Applications. CRC Press (1993).