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RELIABILITY ANALYSIS AND OPTIMIZATION OF EQUAL LOAD-SHARING *K*-OUT-OF-*N* PHASED-MISSION SYSTEMS

ANALIZA NIEZAWODNOŚCI ORAZ OPTYMALIZACJA SYSTEMÓW FAZOWYCH TYPU „*K* Z *N*" O RÓWNYM PODZIALE OBCIĄŻENIA ELEMENTÓW SKŁADOWYCH

There are many studies on k-out-of-n systems, load-sharing systems (LSS) and phased-mission systems (PMS); however, little attention has been given to load-sharing k-out-of-n systems with phased-mission requirements. This paper considers equal load-sharing k-out-of-n phased-mission systems with identical components. A method is proposed for the phased-mission reliability analysis of the studied systems based on the applicable failure path (AFP). A modified universal generating function (UGF) is used in the AFP-searching algorithm because of its efficiency. The tampered failure rate load-sharing model for the exactly k-out-of-n: F system is introduced and integrated into the method. With the TFR model, the systems with arbitrary load-dependent component failure distributions can be analyzed. According to the time and space complexity analysis, this method is particularly suitable for systems with small k-values. Two applications of the method are introduced in this paper. 1) A genetic algorithm (GA) based on the method is presented to solve the operational scheduling problem of systems with independent submissions. Two theorems are provided to solve the problem under some special conditions. 2) The method is used to select the optimal number of components to make the system reliable and robust.

Keywords: *Applicable Failure Path (AFP), Genetic Algorithm (GA), Phased-mission System (PMS), Tampered Failure Rate (TFR) Model, Universal Generating Function (UGF).*

Istnieje wiele badań na temat systemów typu „k z n", systemów z podziałem obciążenia (load-sharing systems, LSS) oraz systemów fazowych (tj. systemów o zadaniach okresowych) (phased-missionsystems, PMS); jak dotąd mało uwagi poświęcono jednak systemom typu „k z n" z podziałem obciążenia wymagającym realizacji różnych zadań w różnych przedziałach czasowych. Niniejszy artykuł omawia systemy fazowe typu „k z n" o równym podziale obciążenia przypadającego na identyczne elementy składowe. Zaproponowano metodę analizy niezawodności badanych systemów w poszczególnych fazach ich eksploatacji opartą na pojęciu właściwej ścieżki uszkodzeń (applicablefailurepath, AFP). W algorytmie wyszukującym AFP zastosowano zmodyfikowaną uniwersalną funkcję tworzącą (universal generating function, UGF), która cechuje się dużą wydajnością. Wprowadzono model manipulowanej intensywności uszkodzeń (tamperedfailurerate, TFR) elementów o równym podziale obciążenia dla systemu, w którym liczba uszkodzeń wynosi dokładnie k z n. Model ten włączono do proponowanej metody analizy niezawodności. Przy pomocy modelu TFR można analizować systemy o dowolnych rozkładach uszkodzeń części składowych, gdzie uszkodzenia są zależne od obciążenia. Zgodnie z analizą złożoności czasowej i przestrzennej, metoda ta jest szczególnie przydatna do modelowania układów o małych wartościach k. W pracy przedstawiono dwa zastosowania metody. 1) oparty o omawianą metodę algorytm genetyczny (GA) do rozwiązywania problemu harmonogramowania prac w systemach z niezależnymi podzadaniami. Sformułowano dwa twierdzenia pozwalające na rozwiązanie problemu w pewnych szczególnych warunkach. 2) Wybór optymalnej liczby elementów składowych pozwalającej na zachowanie niezawodności i odporności systemu.

Słowa kluczowe: *Właściwa ścieżka uszkodzeń (AFP), algorytm genetyczny (GA), system fazowy (o zadaniach okresowych) (PMS), Model manipulowanej intensywności uszkodzeń (TFR), Uniwersalna funkcja tworząca (UGF).*

Acronyms

AFP	applicable failure path
GA	genetic algorithm
i.i.d.	independent identically distributed
LSS	load-sharing system
PMS	phased-mission system
TFR	tampered failure rate
UGF	universal generating function

Nomenclature

M	Number of phases
N	Number of components
k_i	k-value of the i_{th} phase
$\bar{\boldsymbol{K}}$	A set of k-values
K_s	An execution sequence vector
L_i	Load of the i_{th} phase
τ_i	Duration of the i_{th} phase
μ	Component capacity

k_{mi}	Minimum value of $\{k_i, k_{i+1},\ldots, k_M\}$, i.e., the maximum allowed number of failed components in the i_{th} phase
k_{mavr}	Average value of $\{k_{m1}, k_{m2},\ldots, k_{mM}\}$
K_m	Vector $[k_{m1}, k_{m2},\ldots, k_{mM}]$, the maximum allowed number of failed components in each phase
Δk_i	Number of newly failed components in the i_{th} phase
ΔK	A failure path
ΔK^{sub}	A failure subpath
ΔK^a	A applicable failure path
ΔK_l^a	The l_{th} applicable failure path in set V
V	Set of all applicable failure paths
$V(K_m)$	A set of all applicable failure paths related to a specified K_m
C	Cardinal number of AFP set V
$U(z)$	Universal generating function
$U_i(z)$	Universal generating function of the i_{th} phase
$\Omega_{\varphi,\phi},\varphi,\phi$	Composition operators for the UGFs of two different phases
$h(t)$	Failure rate or hazard rate
α_i	Failure rate of the system when (i-1) components have failed
$\delta(L)$	Tampered factor at load L
R or $R(V)$	Phased-mission reliability of the studied system
$R(V(K_m))$	Phased-mission reliability of studied system related to a specified K_m
$R_e(k,n,t)$	Reliability of exactly k-out-n: F system
$R_e(m_i,n_i,\tau_i,L_i)$	Reliability of exactly m_i -out-n_i: F system under duration τ_i and load L_i
τ_{ei}	Effective age of first (i-1) phases
$F(\bar{\boldsymbol{K}})$	A set of all execution sequences
$R_a(k,n,t)$	Reliability of the load-sharing k-out-of-n: F system

1. Introduction

The operation life of phased-mission systems (PMSs) consists of consecutive and non-overlapping time periods (phases). The environmental or operational conditions, stresses and reliability requirements may vary with the phases. Practical phased-mission systems in the real world involve aerospace, nuclear power, airborne weapon systems and distributed computing systems. A classic example is an aircraft with the phases of taxi, takeoff, ascent, level-flight, descent, and landing [36]. Compared with single-phased systems, PMSs have much more complicated characteristics, such as dynamic behavior and dependence across phases. This dynamic behavior may require a distinct model for each phase, and the dependence across phases indicates that the state of a given component at the beginning of a new phase may be identical to the state at the end of the previous phase [34]. Various approaches, e.g., BDDs (binary decision diagrams) [38], recursive algorithms [20] and numerical methods with Markov chain [33], have been suggested for analyzing the reliability of PMSs. Xing *et al*. [35] analyzed a generalized phased mission system (GPMS) with combinatorial phase requirements (CPR), which have flexible failure criteria and outcomes. The latest review of different PMSs and analysis techniques was presented by Xing and Amari [34]. However, because of the complicated characteristics of PMSs and the computational complexity of the existing methods, only some small-scale PMS problems can be accurately solved [35].

An important characteristic of systems with high reliability is fault tolerance, which is often achieved by redundancy. A common realization of redundancy is the k-out-of-n system, which was introduced by Birnbaum *et al*. [8]. A system with n components that properly operates only when no fewer than k components are functioning is called a k-out-of-n: G system. Analogously, a system that fails if and only if at least k out of n components fail is named a k-out-of-n: F system. Hence, a k-out-of-n: G system is equivalent to a (n-k+1)-out-of-n: F system and vice versa. For example, a four-engined plane in which at least two engines must work for the plane to fly normally can be considered a two-out-of-four system [26]. Both parallel and series systems are special cases of the k-out-of-n system. The k-out-of-n redundancy widely applies in many military and industrial systems, e.g., space systems, airborne weapon systems, and distributed computing systems. However, these systems are also PMSs [35]. Load-sharing k-out-of-n systems are an important type of systems with k-out-of-n redundancy. The term load-sharing indicates that the remaining surviving components share a load that is imposed on the system; in other words, a failed component may increase the load on other surviving components. Subsequently, it may increase the failure rate of any surviving component; it is natural to assume that a heavier load indicates a higher failure rate. A typical load-sharing system (LSS) is the Daniels system [16], which has n fibers that are subject to a load process. Because the performance of both load-sharing systems and k-out-of-n systems depends on the number of functioning components, these two types of systems can be integrated together as a load-sharing k-out-of-n system in practical applications. An example is a long belt conveyor, which was studied by Yun *et al.* [37]. Because of the loading policy [30] and dependence between components, the reliability analysis of a load-sharing k-out-of-n system is more complex than that of a k-out-of-n system. Moreover, the performance of some load-sharing k-out-of-n systems is affected by load [9].

Most research on load-sharing k-out-of-n systems is single-phased. Little attention is paid to phased-mission requirements. Mohammad *et al*. [24] studied phased-mission systems with load-sharing components, but subsystems should be independent. Amari and Xing [4] proposed an efficient method to precisely evaluate the reliability of k-out-of-n systems with identical components that are subject to phased-mission requirements. Later, Xing *et al*. [35] used a similar method for the reliability analysis of phased-mission systems with imperfect fault coverage. However, the component failures are s-independent in the studies of Amari [4] and Xing *et al*. [35]. Hence, the method [35] is unsuitable for systems with load-sharing components.

In this paper, we propose a method to precisely evaluate the reliability of equal load-sharing k-out-of-n phased-mission systems with identical components. The method is based on the applicable failure path (AFP). A modified universal generating function (UGF [25]) is used in the AFP-searching procedure. The tampered failure rate (TFR) model is introduced to describe the load-dependent failure rate of each component. With the TFR load-sharing model for exactly k-out-of-n systems, the method can analyze systems with arbitrary load-dependent component failure distributions. Two different applications are introduced in this paper: 1) a genetic algorithm (GA) based on the method is offered to solve the operational scheduling problem of phased-mission systems with independent submissions and 2) the method is used to select the optimal number of components for systems.

The remainder of the paper is organized as follows. Section 2 provides an overview of the problem to be solved, including a system description, assumptions, and problem inputs. Section 3 describes the AFP-based method for the reliability evaluation of the studied phased-mission system. Section 4 presents the applications of the method with a genetic algorithm (GA) in the operational scheduling of phased-mission systems with independent submissions. The effectiveness of the method is verified using three examples in section 5. Finally, section 6 concludes the paper.

1. Problem statement

This paper considers the reliability evaluation of equal load-sharing k-out-of-n phased-mission systems and optimizes the operational scheduling of systems with independent submissions. The assumptions and inputs for the problem are listed in the following subsections.

2.1. System description and assumptions

1. The system mission consists of M consecutive, non-overlapping phases.
2. The system has N identical components. Each component has two states: failed and operational.
3. The system is a load-sharing k-out-of-n: F phased-mission system. The load and k-value may differ by phase.
4. The components share the total system load equally. The load is constant in each phase.
5. The failure rate of each component is affected by the load imposed on it.
6. The system is non-repairable during each phase, but at the end of each phase, all functional components are maintained. A maintained component is as reliable as a new component.
7. If the system fails in any phase, the overall mission is considered failed.

2.2. Input parameters

All required input parameters to solve the problem are listed as follows:

1. M: Number of phases.
2. L_i: Total system load in each phase.
3. τ_i: Duration of each phase.
4. μ: Component capacity.
5. n: Number of components.
6. k_i: Failure criterion for each phase.
7. Baseline failure time distribution of components and tampered factor of the TFR model.

This paper focuses on the system-level reliability evaluation with the above input parameters.

3. Phased-Mission Reliability Analysis

3.1. An AFP-based method for the reliability evaluation

Suppose that a system has n components and requires at least n-k_i+1 components under normal operation in the i_{th} phase with load L_i. It should be noted that the k-values of different phases are not independent. For example, because the system is non-repairable, if $k_i > k_j$ $(i < j)$ and $(k_i - 1)$ components fail in the i_{th} phase, the system will certainly fail in the j_{th} phase (because $k_i - 1 \geq k_j$). Therefore, if we want the system to be operational at the beginning of the j_{th} phase, the maximum allowed number of failed components in the i_{th} phase is $\min(k_i,k_j) - 1 = k_j - 1$ $(i < j)$. Thus, we have

$$k_{mi}=\min(k_i,k_{i+1},\ldots,k_M) \tag{1}$$

where k_{mi} is the actual k-value instead of k_i.

Define $K_m=[k_{m1}, k_{m2},\ldots, k_{mM}]$, where k_{mi} is non-decreasing w.r.t. i. To guarantee the normal operation in the i_{th} phase, the total number of failed components must be less than k_{mi}, which indicates:

$$\sum_{j=1}^{i} \Delta k_j < k_{mi} \tag{2}$$

where Δk_j is the number of newly failed components in the j_{th} phase.

Definition 3.1.1: A Failure Path is a sequence of the numbers of newly failed components in all phases:

$$\Delta K = [\Delta k_1, \Delta k_2, \ldots, \Delta k_M] \tag{3}$$

Definition 3.1.2: A Failure Subpath is a part of a failure path:

$$\Delta K^{sub} = [\Delta k_{d_1}, \Delta k_{d_1+1}, \ldots, \Delta k_{d_2}], 1 \leq d_1 < d_2 \leq M \tag{4}$$

Definition 3.1.3: An Applicable Failure Path (AFP) is a failure path that satisfies (2) for any $i \leq M$:

$$\Delta K^a = [\Delta k_1, \Delta k_2, \ldots, \Delta k_M], \forall i \leq M, \sum_{j=1}^{i} \Delta k_j < k_{mi} \tag{5}$$

Theorem 3.1.1: The necessary and sufficient condition for the system under normal operation is that the numbers of newly failed components in each phase can comprise an AFP.

Theorem 3.1.1 clearly holds for the condition of AFP. Define a set of AFPs:

$$V = \{\Delta K_l^a, l = 1, 2, \ldots, C\} \tag{6}$$

where C is the cardinal number of the set V; in other words, C is the total number of AFPs. The phased-mission reliability can be evaluated as:

$$R = R(V) = \sum_{\Delta K_l^a \in V} R(\Delta K_l^a) = \sum_{\Delta K_l^a \in V} \prod_{i=1}^{M} R_e(\Delta k_{li}, N - \sum_{j=0}^{i-1} \Delta k_{lj}, \tau_i) \tag{7}$$

where $\Delta k_{l0}=0$, τ_i is the duration time of the i_{th} phase and $R_e(k,n,t)$ is the reliability of exactly k-out-n: F systems (or exactly $(n$-k+1)-out-of-n: G systems [19]). **An exactly k-out-n: F system is a system with exactly k failed components at time t**. The last equation in (7) is deduced from assumption 6 in section 2. Furthermore, according to assumption 5, $R_e(k,n,t)$ depends on the load in each phase.

$$R = \sum_{\Delta K_l^a \in V} \prod_{i=1}^{M} R_e(\Delta k_{li}, N - \sum_{j=0}^{i-1} \Delta k_{li}, \tau_i, L_i) \tag{8}$$

To compute the exact phased-mission reliability of a system using (8), we must obtain the AFP set V and the closed-form analytical solution of $R_e(k,n,t)$.

3.2. An AFP-Searching Algorithm Based on a Modified UGF

The AFP set V of a sequence of specified k-values is obtained using a modified UGF method. The universal generating function (UGF), which is a simple and rapid technique, was first introduced by Ushakov [29]. It is widely adopted in reliability analyses because of its efficiency. It is particularly useful in the reliability analysis of systems with multi-states and large numbers of components [12, 18, 19]. In the traditional UGF method, the UGF of a component i over state space G^i is written as a z-transformation polynomial:

$$U_i(z) = \sum_{g_j^i \in G^i} p_j^i z^{g_j^i} \tag{9}$$

where g_j^i is the j_{th} state of the i_{th} component, and p_j^i is the probability of the i_{th} component in state g_j^i. Composition operators are used to obtain the UGF of a subset that is composed of a certain number of components. For example, the resulting UGF of component $i1$ and $i2$ is:

$$U(z)=\Omega_\phi(U_{i1}(z),U_{i2}(z))=\sum_{g_r^{i1}\in G^{i1}}\sum_{g_s^{i2}\in G^{i2}} p_r^{i1}p_s^{i2}z^{\phi(g_r^{i1},g_s^{i2})} \quad (10)$$

where $\phi(g_r^{i1},g_s^{i2})$ is the equivalent productivity of two components; it is $\phi(g_r^{i1},g_s^{i2})=\min(g_r^{i1},g_s^{i2})$ for series components or $\phi(g_r^{i1},g_s^{i2})=g_r^{i1}+g_s^{i2}$ for parallel components. Thus, we can evaluate the performance distribution of the entire system by repeatedly merging UGFs of all components.

In this paper, a modified UGF is provided to facilitate the expression and implementation of the AFP-searching algorithm.

Definition 3.2.1: A modified UGF (mUGF) of the i_{th} phase instead of the i_{th} component is defined as:

$$U_i(z)=\sum_{j=0}^{k_{mi}-1} j\bullet z^{[j]}, i\in\{1,2,\ldots,M\} \quad (11)$$

where j is the j_{th} state of the i_{th} phase, which refers to the number of newly failed components. $[j]$ is a vector that records the number of newly failed components in each phase.

Definition 3.2.2: A composition operator for two different phases is defined as:

$$U(z)=\Omega_{\varphi,\phi}(U_{i1}(z),U_{i2}(z)) = \sum_{r_1=0}^{k_{mi1}-1}\sum_{r_2=0}^{k_{mi2}-1}\varphi(r_1,r_2)z^{\phi([r_1],[r_2])}=\sum_{r_1}^{k_{mi1}-1}\sum_{r_2}^{k_{mi2}-1}(r_1+r_2)z^{[r_1,r_2]} \quad (12)$$

where $\varphi(r_1,r_2)=r_1+r_2$, and $\phi([r_1],[r_2])=[r_1,r_2]$. There are some remarks for (12):

Remark 1: $\varphi(\cdot)$ adds two elements together. $\phi(\cdot)$ merges two vectors together, its parameters are not exchangeable, $\phi([r_1],[r_2])\neq\phi([r_2],[r_1])$, because the result of $\phi(\cdot)$ indicates the execution sequence of phases.

Remark 2: We can repeatedly apply (12) to a sequence of phases to obtain a composite mUGF. In such a case, the term $r_1 z^{r_1}$ in $U_{i1}(z)$ must be rewritten as $r_1 z^{s_1}$, where $\boldsymbol{s_1}$ is a combined vector. The length of $\boldsymbol{s_1}$ is the number of the processed phases; the sum of all elements in $\boldsymbol{s_1}$ is $\boldsymbol{r_1}$.

Remark 3: When (12) is repeatedly used for a sequence of phases, the vector s_1 in $r_l z^{s_l}$ represents a failure subpath. Therefore, for a sequence of phases, the final result from (12) contains all failure paths, including all AFPs and non-AFPs. If all failure subpaths related to the non-AFPs are removed after each calculation, then the AFP set V can be obtained.

Therefore, an AFP-searching algorithm is described as follows:

Algorithm 1: An AFP-Searching Algorithm:

1. $U(z)=U_1(z)$;
2. For i=2,3,…,M;
   - 2.1 $U(z)=\Omega_{\varphi,\phi}(U(z),U_i(z))$;
   - 2.2 For each resulting term $r_l z^{s_l}$ in $U(z)$ from step 2.1;
     - 2.2.1 If $r_l\geq k_{mi}$, then remove the term from $U(z)$.

Finally, the AFP set is:

$$V=\{\Delta K_l^a=s_l : r_l z^{s_l} \text{ is a term in } U(z)\} \quad (13)$$

3.3. The time and space complexity of the method

Theorem 3.3.1: The cardinal number of the AFP set V in (5) satidfies:

$$C\leq\prod_{i=1}^{M}k_{mi}\leq(k_{mavr})^M \quad (14)$$

where k_{mavr}=mean$\{k_{mi}\}$.

Proof. The theorem can be proven using the inequality of arithmetic and geometric means.

For the reliability $R_e(m_i,n_i,\tau_i,L_i)$ of the i_{th} phase, we have $m_i\in\{0,1,\ldots,k_{mi}-1\}$ and $n_i\in\{n-k_{m(i-1)}+1,n-k_{m(i-1)}+2,\ldots,n\}$. Calculating and storing the values of all $R_e(m_i,n_i,\tau_i,L_i)$'s in advance are recommended to improve the computational efficiency.

The required storage space is mainly for storing the AFP set V and the values of all $R_e(m_i,n_i,\tau_i,L_i)$'s, whose space complexities are $O(C)$ and $O(\sum_{i=1}^{M}k_{mi(i-1)}k_{mi})$, respectively. According to (14) and the inequality of arithmetic and geometric means, the space complexity of the method is:

$$SC=O(C+\sum_{i=1}^{M}k_{m(i-1)}k_{mi})=O((k_{mavr})^M+Mk_{mavr}^2) \quad (15)$$

The computational time cost is mainly from three parts: the AFP-searching procedure, the calculations of all $R_e(m_i,n_i,\tau_i,L_i)$'s and (8), whose time complexities are, $O(\prod_{i=1}^{M}k_{mi})$ and $O(\sum_{i=1}^{M}k_{mi(i-1)}k_{mi})$ and $O(MC)$, respectively. Thus, the computational time complexity is:

$$TC=O(\prod_{i=1}^{M}k_{mi}+\sum_{i=1}^{M}k_{m(i-1)}k_{mi}+MC)=O(M(k_{mavr})^M+Mk_{mavr}^2) \quad (16)$$

By (15) and (16), the time and space complexity depends on the number of phases and k-values (the maximum allowed number of failed components) for the phases. Therefore, small k-values will decrease the time and space complexity of the method.

3.4. TFR model for an exactly *k*-out-*n*: F system

According to the regenerative process,

$$R_e(k,n,t)=R_a(k+1,n,t)-R_a(k,n,t) \quad (17)$$

where $R_a(k,n,t)$ is the reliability of load-sharing k-out-of-n: F systems. Here, the tampered failure rate (TFR) model is used to obtain a closed-form analytical expression of $R_e(k,n,t)$.

The TFR model was introduced by Bhattacharyya and Soejoeti [7] for the step-stress accelerated life test (ALT). Amari *et al*. [3] provided a closed-form analytical solution for the reliability of tampered failure rate load-sharing k-out-of-n: G systems with identical components, where all surviving components equally share the total system load. Several other models can be found in [1, 2, 23, 31]. Wang and Fei [31] studied the conditions for the coincidence of TFR, tampered random variable (TRV) and cumulative effect (CE) models.

According to the TFR model, if load L is imposed on a component, the failure rate (or hazard rate) of the component is:

$$h(t)=\delta(L)h_0(t) \quad (18)$$

where $h_0(t)$ is the baseline hazard rate and $\delta(L)$ is the tampered factor at load L.

For an equal load-sharing k-out-of-n system with identical components, let λ_i denote the failure rate of each surviving component when $(i$-1) (i=1,2,…,k) components have failed. We have:

$$\lambda_i = \delta(\frac{L}{n-i+1})h_0 \tag{19}$$

where L is the total load on the system and h_0 is a constant.

In particular, if the failure rate of each surviving component is directly proportional to the load imposed on it, then

$$\delta(\frac{L}{n-i+1}) = c \bullet \frac{L}{n-i+1} \tag{20}$$

where c is a constant. Therefore, the i_{th} failure of the system occurs at rate:

$$\alpha_i = (n-i+1)\lambda_i = (n-i+1)\delta(\frac{L}{n-i+1})h_0 = (n-i+1)c\frac{L}{n-i+1}h_0 = cLh_0 \tag{21}$$

In this case, all α_i's are equal. However, in other cases, the α_i's may be different. Amari *et al*. [3] have offered closed-form analytical expressions of $R_a(k,n,t)$ in three different cases. With (17) and the expressions of Amari *et al*. [3], the closed-form analytical expressions of the system reliability **of exactly tampered failure rate load-sharing k-out-of-n: F systems with i.i.d. components and the exponential failure time distributions** can be easily derived as follows.

Case (a): All α_i's are equal (say α).

$$R_e(k,n,t) = \frac{(\alpha t)^k \exp(-\alpha t)}{k!} \tag{22}$$

Case (b): All α_i's are distinct.

$$R_e(k,n,t) = \sum_{i=1}^{k}(A_i^{'} - A_i)\exp(-\alpha_i t) + A_{k+1}^{'}\exp(-\alpha_{k+1} t) \tag{23}$$

where:

$$A_i = \prod_{\substack{j=1 \\ j\neq i}}^{k} \frac{\alpha_j}{\alpha_j - \alpha_i}, A_i^{'} = \begin{cases} A_i \dfrac{\alpha_{k+1}}{\alpha_{k+1}-\alpha_i}, i < k+1 \\ \displaystyle\prod_{j=1}^{k} \frac{\alpha_j}{\alpha_j - \alpha_{k+1}}, i = k+1 \end{cases}$$

Case(c): α_i's are neither equal nor distinct. Specifically, assume that these α_i's take a (1<a<k) distinct values, $\beta_1,\beta_2,\ldots,\beta_a$. With possibly some renumbering of these α_i values, assume:

$$\begin{aligned} &\alpha_1 = \cdots = \alpha_{r_1} = \beta_1 \\ &\alpha_{r_1+1} = \cdots = \alpha_{r_1+r_2} = \beta_2 \\ &\quad\vdots \\ &\alpha_{r_1+r_2+\cdots+r_{a-1}+1} = \cdots = \alpha_{r_1+r_2+\cdots+r_a} = \beta_a \end{aligned} \tag{24}$$

where the assumptions are:

1) $a \geq 1$ and an integer
2) All β_i's are distinct
3) Sum of r_i's is equal to k
4) Each $r_i \geq 1$ and an integer.

There are two different cases under **case (c)** regarding α_{k+1}:

Case (c-1): $\forall i \in \{1,2,..,k\}$, $\alpha_{k+1} \neq \alpha_i$.

$$R_e(k,m,t) = B\sum_{j=1}^{a}\sum_{l=1}^{r_j}\frac{\alpha_{k+1}\Phi_{jl}^{1}(-\beta_j) - \Phi_{jl}(-\beta_j)}{(l-1)!(\beta_j)^{r_j-l+1}} \bullet \text{poif}(r_j - l;\beta_j t) + \alpha_k B\frac{\Phi_{(a+1)l}^{1}(-\alpha_{k+1})}{\alpha_{k+1}} \bullet \text{poif}(1;\alpha_{k+1}t) \tag{25}$$

Case (c-2): $\exists i \in \{1,2,...,k\}$, $\alpha_{k+1}=\alpha_i$, assume $\alpha_{k+1}=\alpha_s$.

$$\begin{aligned} R_e(k,m,t) = {} & B\sum_{\substack{j=1 \\ j\neq s}}^{a}\sum_{l=1}^{r_j}\frac{\alpha_{k+1}\Phi_{jl}^{2}(-\beta_j) - \Phi_{jl}(-\beta_j)}{(l-1)!(\beta_j)^{r_j-l+1}} \bullet \text{poif}(r_j - l;\beta_j t) \\ & + \sum_{l=1}^{r_s}\left(\frac{\dfrac{\Phi_{sl}^{2}(-\alpha_{k+1})\text{poif}(r_s+1-l;\alpha_{k+1}t)}{\alpha_{k+1}} - \Phi_{sl}(-\alpha_{k+1})\text{poif}(r_s - l;\alpha_{k+1}t)}{(l-1)!(\alpha_{k+1})^{r_s-l+1}}\right) \\ & + \frac{\Phi_{s(r_s+1)}^{2}(-\alpha_{k+1})}{r_s!\alpha_{k+1}} \bullet \text{poif}(1;\alpha_{k+1}t) \end{aligned} \tag{26}$$

where:

$$\begin{aligned} & B = \prod_{j=1}^{a}(\beta_j)^{r_j};\ \Phi_{jl}(t) \equiv D^{l-1}(\prod_{\substack{i=1 \\ i\neq j}}^{a}(\beta_i + t)^{-r_i}) \\ & \Phi_{jl}^{1}(t) = \begin{cases} \Phi_{jl}^{2}(t), & j \leq a \\ D^{l-1}(\prod_{i=1}^{a}(\beta_i + t)^{-r_i}), & j = a+1 \end{cases} \\ & \Phi_{jl}^{2}(t) = D^{l-1}(\frac{\prod_{\substack{i=1 \\ i\neq j}}^{a}(\beta_i + t)^{-r_i}}{\alpha_{k+1} + t}) \\ & \text{poif}(r_j - l;\beta_j t) = \sum_{i=0}^{r_j - l}\frac{\exp(-\beta_j t)(\beta_j t)^i}{i!} \end{aligned} \tag{27}$$

For the TFR models with an arbitrary baseline failure distribution $F(t)$, we can perform the time transformation suggested by Amari *et al*.[3]:

$$t' = \text{H}(t) = -\ln[1 - \text{F}(t)] \tag{28}$$

It can be proved that t'=H(t) follows an exponential distribution with a mean of 1 and a failure rate of 1. Thus, we can calculate $R_e(k,n,t)$ in the transformed scale by (25), (26) or (27), and the system failure rate α_i can be obtained by (18), (19) and (21) with $h_0(t')$=1. **Therefore, unless otherwise specified, only the TFR models with an exponential failure time baseline distribution are discussed in this paper.**

Moreover, the failure patterns of the same component may differ by phase, which means that the values of $R_e(k, n, t)$ may be different in two different phases for the same k, n and t. In such a case, (7) and (8) are still applicable, and we should calculate all possible $R_e(k, n, t)$'s for each phase.

If the functional components are never maintained at the end of each phase, meaning that the system failure rate depends on the previous phases, we can use the effective age of the systems [1] in the reliability evaluation. However, this evaluation is beyond the scope of this paper, and we will only provide a brief discussion.

For an AFP, the system reliability can be obtained through the effective age of the system at the end of the final phase; thus, the phased-mission reliability is evaluated by summing the reliabilities of all AFPs.

To obtain the effective age of the system at the beginning of the i_{th} phase, the following equation must be solved:

$$R_e(k_{m(i-1)}, n_{i-1}, \tau_{e(i-1)} + \tau_{i-1}, L_{i-1}) = R_e(k_{m(i-1)}, n_{i-1}, \tau_{ei}, L_i) \quad (29)$$

where $R_e(k_{m(i-1)}, n_{i-1}, \tau_{e(i-1)}+\tau_{i-1}, L_{i-1})$ ($i \in \{2,3,\ldots,M\}$) is the reliability at the end of the $(i\text{-}1)_{\text{th}}$ phase and τ_{ei} is the effective age of the system at the beginning of the i_{th} phases. Solving (29) may be complicated.

The reliability at the end of the i_{th} phase can be expressed as $R_e(k_{mi}, n_i, \tau_{ei}+\tau_i, L_i)$, where τ_i is the duration of the i_{th} phase. As a result, the phased-mission reliability is expressed as follows instead of (8):

$$R = \sum_{\Delta K_l^a \in V} R_e(\Delta k_{lM}, N - \sum_{i=0}^{M-1} \Delta k_{li}, \tau_{elM} + \tau_M, L_M) \quad (30)$$

where τ_{elM} is the effective age of the system at the beginning of the final phase for the l_{th} AFP.

4. Applications for operational scheduling optimization

Before further discussion, we apply a new assumption to the system:

Assumption A1. The submissions in different phases are independent, and the execution order of the submissions can change.

The system under assumption A1 can execute the submissions independently in random order. For example, a collection system collects different data at different places, and a computation/communication system selects tasks with different computation complexities and resource requirements from a waiting queue.

4.1. Two theorems to solve the problem under special conditions.

4.1.1. The durations of phases are constant.

Suppose $\bar{\boldsymbol{K}}$ is a set of k-values of M different phases for M different submissions. The vector is an execution sequence of phases, and k_{si} is the k-value of the i_{th} phase. $F(\bar{K})$ is defined as a set of execution sequences:

$$F(\bar{K}) = F(\{k_1, k_2, \ldots, k_M\}) = \{K_s = [k_{si}]_{1\times M} : \text{any sequence of } k_i\} \quad (31)$$

Lemma 4.1: Under assumption A1, for a specified $\bar{\boldsymbol{K}}$ and any, $K_s \in F(\bar{K})$ the AFP set on K_s is a subset of the AFP set on $K_* \in F(\bar{K})$, where $k_{*1} \le k_{*2} \le \ldots \le k_{*M}$. Therefore, the cardinal number of the AFP set V on K_s (Value of C) reaches the maximum value when K_s=K_*.

Proof. Consider an execution sequence $K_{sc}=[k_{sc1}, k_{sc2}, \ldots, k_{scM}]$ and $k_r=\min\{k_{sci}\}$, $r>1$. According to the discussion in section 3.1, the corresponding actual k-values K_m can be expressed as $K_m = [\underbrace{k_r, k_r, \ldots, k_r}_{r}, \underbrace{k_{m(r+1)}, \ldots, k_{mM}}_{M-r}]$. If the positions of $k_{(r\text{-}1)}$ and k_r (note that $k_{(r\text{-}1)} \ge k_r$) are swapped, there is a new execution sequence $K_{sc}'=[k_{sc1}, k_{sc2}, \ldots, k_r, k_{r\text{-}1}, \ldots, k_{scM}]$ and a new

$K'_m = [\underbrace{k_r, k_r, \ldots, k_r}_{r-1}, \underbrace{k_{m(r-1)}, k_{m(r+1)} \ldots, k_{mM}}_{M-r+1}]$,where $k_{m(r\text{-}1)} \ge k_r$. Comparing $\boldsymbol{K'_m}$ with $\boldsymbol{K_m}$, we note the only difference is that the value at the r_{th} position of $\boldsymbol{K'_m}$ is no smaller than that of $\boldsymbol{K_m}$. Therefore, it can be easily argued that for $\forall \boldsymbol{\Delta K_l^a} \in V(\boldsymbol{K_m})$, $\boldsymbol{\Delta K_l^a} \in V(\boldsymbol{K'_m})$. In other words, $V(\boldsymbol{K_m})$ is a subset of $V(\boldsymbol{K'_m})$, $V(\boldsymbol{K_m}) \subseteq V(\boldsymbol{K'_m})$.

Repeat the above procedure until k_r is moved to the first position, and perform the same procedure with the left k-values (without k_r). Finally, all k_{sci}'s are ranked in ascending order.

Hence, any $\boldsymbol{K_s}$ can turn to $\boldsymbol{K_*}$, where $k_{*1} \le k_{*2} \le \ldots \le k_{*M}$. With the course of the proof, it can be asserted that the AFP set on $\boldsymbol{K_s}$ is a subset of the AFP set on $\boldsymbol{K_*}$.

Theorem 4.1: If all phase durations τ_i and loads L_i are equal, $R(V(\boldsymbol{K_m}))$ in (8) reaches the maximum value when $V(\boldsymbol{K_m})$ takes the largest cardinal number.

Proof. Because all phase durations and loads are equal, for any two different phases, they have the same value of $R_e(k,n,\tau_i,L_i)$ if they have the same k and n. In other words, for all phases, $R_e(k,n,\tau_i,L_i)$ only depends on k and n. Therefore, for any identical AFP $\boldsymbol{\Delta K_l^a}$ in different AFP sets, they have the same value of the term $\prod_{i=1}^{M} R_e(\Delta k_{li}, N - \sum_{j=0}^{i-1} \Delta k_{lj}, \tau_i, L_i)$ in (8). Thus, from (8), larger AFP set indicates higher reliability. Moreover, according to **Lemma 4.1**, the AFP set on $\boldsymbol{K_s}$ is a subset of that on $\boldsymbol{K_*}$ with k-values in ascending order. Hence, the AFP set on contains all other AFP sets and takes the largest cardinal number. Therefore, **Theorem 4.1** is proven.

According to the above lemmas and theorem, if the failure probability of components varies little with time or all phase durations and loads are equal, the maximum phased-mission reliability can be acquired by sorting all k-values in ascending order.

4.1.2. The durations of phases are affected by the load and component capacity.

The duration of a mission in many practical systems, such as communication network systems [15], computer systems [10] and control systems [11], depends on the component capacity. Similar to **Theorem 4.1**, there is **Theorem 4.2** under the following **Assumption A2**:

Assumption A2: Suppose all components have the same capacity and the duration of a phase (submission) depends on the component capacity, which indicates that the duration of the i_{th} phase is $\tau_i = \frac{L_i}{n_i \mu}$ if no component fails in the phase and that the maximum duration is $\frac{L_i}{(n_i - \Delta k_i)\mu}$, if Δk_i components fail at the end of the phase. Therefore:

$$\frac{L_i}{n_i \mu} \le \tau_i \le \frac{L_i}{(n_i - \Delta k_i)\mu} \quad (32)$$

Theorem 4.2: Under **Assumption A2**, if all loads L_i are equal, $R(V(\boldsymbol{K_m}))$ in (8) reaches the maximum value when $V(\boldsymbol{K_m})$ takes the largest cardinal number.

Proof. Because all components have the same capacity and all phases have the same load, duration τ_i is determined if k and n are specified. Therefore, similar to the proof procedure of **Theorem 4.1**, the above theorem is proven.

4.2. General optimization problem

In the real world, the phase duration and load may be different in different phases. Therefore, there is an optimization problem as follows:

Max

$$R(V(K_m))$$

s.t.

$$\begin{aligned} &\text{A specified } \bar{K} = \{k_1, k_2, \ldots, k_M\}, \\ &\forall K_s = [k_{si}]_{1\times M} \in F(\bar{K}), \\ &K_m = [k_{mi} = \min(k_{si}, k_{s(i+1)}, \ldots, k_{sM})]_{1\times M} \end{aligned} \tag{33}$$

The decision variable in the above optimization problem is $\boldsymbol{K_m}$ generated by a specified execution sequence. $F(\bar{K})$ is the search space that represents all possible execution sequences. If the phase duration is affected by the load and component capacity, we set τ_i in (8) to the maximum duration of each phase:

$$\tau_i = \frac{L_i}{(n_i - \Delta k_i)\mu} \tag{34}$$

where L_i is the load in the i_{th} phase, μ is the component capacity, n_i is the number of surviving components at the beginning of the i_{th} phase, and Δk_i is the number of newly failed components in the i_{th} phase. As a result, the optimized reliability in (33) is a lower bound.

4.3. A genetic algorithm for the optimization problem

Many researchers have studied similar optimal problems, such as the standby element sequencing problem (SESP) [22], the redundancy allocation problem (RAP) [6] and the reliability-redundancy allocation problem (RRAP) [13]. Lad *et al.* [17] provided the latest review of optimal reliability design problems. The genetic algorithm (GA) is an effective optimization tool for these problems. The detailed information on development of GA theory and practice can be found in the works of Goldberg [14] and Back [5]. The GA used in this paper is named GENITOR, which was proposed by Whitley [32]. GENITOR outperforms the basic "generational" GA [28]. A successful application of GENITOR has been reported for optimal system operation scheduling problems by Levitin *et al.* [21].

The following algorithm is from the method of Levitin *et al.* [21]. Each solution is represented by a string of M integer numbers, each of which ranges from 1 to M. The crossover procedure is suggested in [27].

Algorithm 2 [21], **GA for the optimization problem**:

1. Generate an initial population that consists of N_s randomly constructed solutions and evaluate their fitness. We apply **Algorithm 1** to every solution. (33) is the objective function.
2. Randomly select two solutions and perform a crossover procedure to produce a new solution (offspring). A rank-based parent selection scheme is adopted in which all solutions in the population are ranked in increasing order of their fitness; the probability of selecting a solution as a parent is proportional to its rank.
3. Apply a mutation operation to the offspring. The mutation procedure exchanges the positions of two randomly chosen components of the string.
4. Decode the offspring to obtain the objective function value (fitness).
5. Apply a selection procedure that compares the offspring with the worst solution in the current population, and then selects the better solution and discards the worse one. Eliminate redundant equivalent solutions to decrease the population size.
6. Repeat step 2-5 N_{rep} times or until the population contains only a single solution or solutions with equal quality. Generate new random solutions to replenish the population and return to step 2 to run a new genetic cycle.
7. Terminate the GA after N_c genetic cycles.

5. Numerical examples

In this section, practical applications of the method are illustrated with four examples. The first example demonstrates the calculation of the phased-mission reliability, the second example studies the computation time of the method and the other two examples are related to the operation scheduling problem of phased-mission systems with independent submissions.

Example 1. The phased-mission reliability calculation

The following example is a k-out-of-n phased-mission system with 8 phases and 16 components. The failure rate of each surviving component is directly proportional to the load, which indicates $\delta(L) \propto L$; therefore, (22) is used. The parameters of the system are listed in Table 1.

Table 1. Parameters of Example 1

Phase	k	Duration	α
No.1	4	20	0.005
No.2	8	30	0.01
No.3	10	40	0.015
No.4	12	50	0.02
No.5	4	20	0.02
No.6	8	30	0.015
No.7	10	40	0.01
No.8	12	50	0.005

With (7) and (22), the phased-mission reliability is 0.7787.

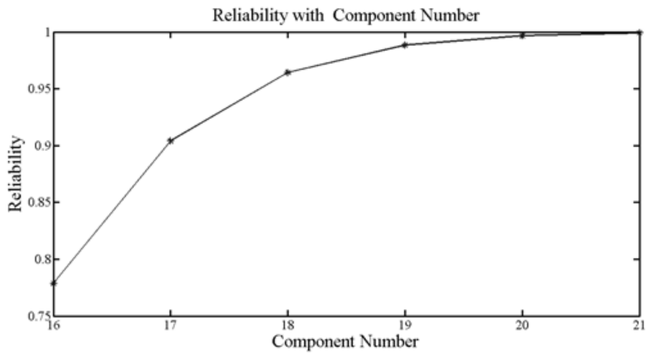

Fig. 1. Reliability with Component Number

The system is more reliable and robust if the number of components increases; however, the system cost also increases. Fig. 1 shows the reliability varies with the number of components. The reliability of a system with 16 or 17 components is more sensitive to the number of components. By adding only one component, the reliability increases from 0.7787 to 0.9041 for a system with 16 components and from 0.9041 to 0.9643 for a system with 17 components. In this example, the system with 20 or 21 components is a good choice because of the high reliability (0.9967 or 0.9991), good robustness and relatively low cost.

Example 2. Computation time vs. system size and the number of missions

The second example investigates the computation time of the reliability evaluation of the system as a function of the number of components and missions. We suppose that all missions have the same duration (= 40) and value of α (= 0.015). Table 2 shows four systems with different numbers of components and missions. For each system, we add one mission to the system each time and evaluate the reliability.

Fig. 2 shows that the computation time varies with the number of missions for each system. We find that the computation time is mainly affected by the number of missions, increasing exponentially as the number of missions increases. The same conclusion can be drawn from (16).

Table 2. Four Systems for Studying the Computation Time

	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8
12-component system	3	4	5	6	7	8	9	10
16-component system	3	4	5	6	7	8	10	12
20-component system	3	4	5	6	7	10	12	14
24-component system	3	4	5	6	10	12	14	16

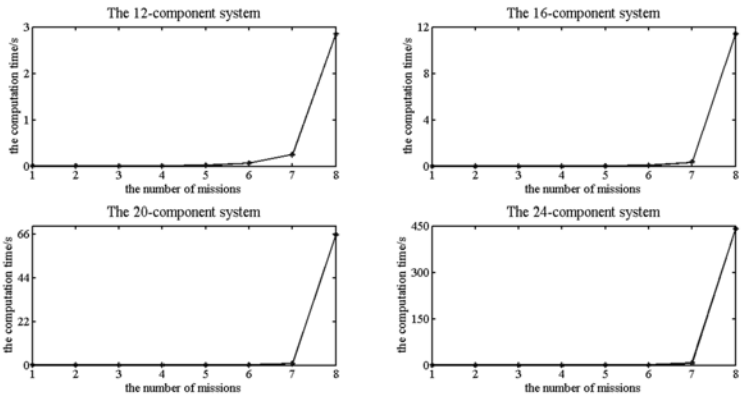

Fig. 2. Computation time for each system

Example 3. Reliability optimization

A. The durations of phases are constant.

We use the system in Example 1 but assume that the phase durations are constant. The parameters of GENITOR are listed in Table 3.

Table 3. Parameters of GENITOR for Example 3-A

N_s	10
N_{rep}	30
N_c	10
Mutation Probability	0.2

Fig. 3 shows the fast convergence of the algorithm, and the reliability increases from 0.8891 to 0.9732 after rescheduling. The best phase arrangement is [No. 1, No. 5, No. 6, No. 2, No. 3, No. 7, No. 4, No. 8]. Hence, a phase with higher required reliability is assigned an earlier execution time, which is consistent with the conclusion of the method (sorting all k-values in ascending order) of **Theorem 4.1**.

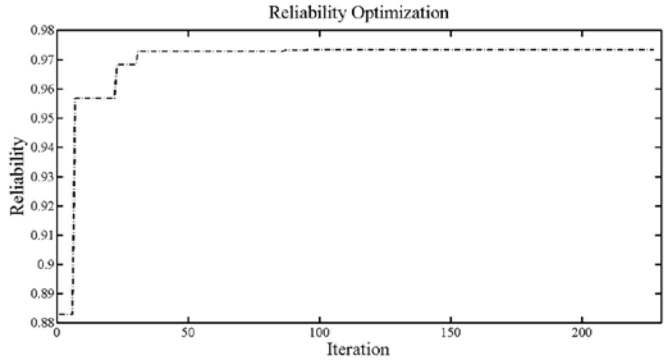

Fig. 3. Reliability Optimization (Constant Duration)

B. The durations of phases are affected by the load and component capacity.

Here, we present a system with 20 components in which the component capacity is 1 (μ=1). The system parameters are provided in Tables 4 and 5.

Table 4. Parameters of Example 3-B

Phase	K	α	Load
No. 1	4	0.01	40
No. 2	6	0.0225	90
No. 3	8	0.015	50
No. 4	10	0.0125	100
No. 5	11	0.0125	50
No. 6	13	0.025	100
No. 7	15	0.01	40
No. 8	17	0.0225	90

Table 5. Parameters of GENITOR for Example 3-B

N_s	10
N_{rep}	50
N_c	3
Mutation Probability	0.15

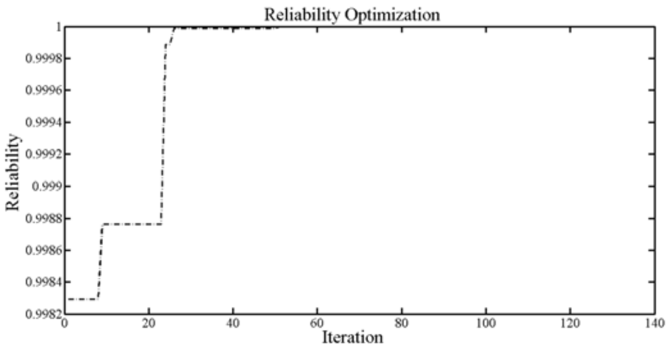

Fig. 4. Reliability Optimization (Non-Constant Duration)

Fig. 4 shows that the reliability increases from 0.998293718927419 to 0.999999568373216, and the best arrangement is [No. 1, No. 5, No. 3, No. 2, No. 7, No. 4, No. 6, No. 8]. Because the load varies in different phases, the result is different from the conclusion of Theorem 4.2, which is only suitable for all phases with similar loads. Fig. 5 shows the correctness of Theorem 4.2. The load is 100 in all phases.

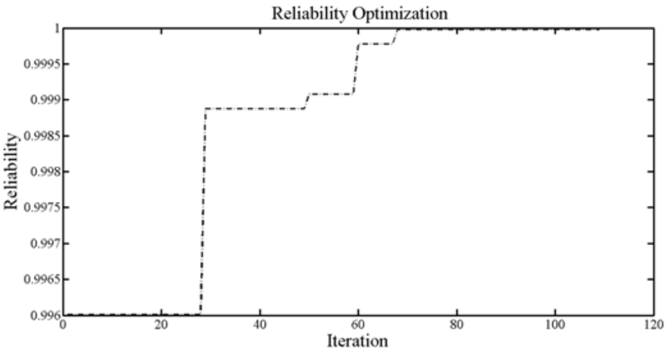

Fig. 5. Reliability Optimization (Load is 100 in All Phases)

The reliability increases from 0.996008009997850 to 0.999981789542223 in Fig. 5, and the best arrangement is [No. 1, No. 2, No. 3, No. 4, No. 6, No. 5, No. 7, No. 8], which is very close to the conclusion of **Theorem 4.2**, from which the best reliability is 0.999981905093990 by sorting all *k*-values in ascending order.

6. Conclusions

An AFP-based method is proposed for the phased-mission reliability analysis of equal load-sharing *k*-out-of-*n* phase-mission systems with identical components. Because the TRF load-sharing model for exactly *k*-out-*n* systems is introduced, the method can analyze systems having components with any type of baseline failure distributions. According to the time and space complexity analysis, this method is particularly suitable for systems with small *k*-values. Finally, the GENITOR algorithm, which integrates the method, is proposed to solve the operational scheduling problem of systems with independent submissions. If all phase durations and loads are equal, according to the theorems in section 4, the maximum reliability can be obtained by sorting all *k*-values in ascending order without using the GENTIOR algorithm. Moreover, an example in section 5 illustrates another application to select the optimal number of components to make the system reliable and robust. Our future work will consider the presented system with non-identical components and use the effective age to solve the system with dependence across phases.

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