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# DISCRETE SPECTRA FOR SOME COMPLEX INFINITE BAND MATRICES

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**Abstract.** Under suitable assumptions the eigenvalues for an unbounded discrete operator $A$ in $l_2$, given by an infinite complex band-type matrix, are approximated by the eigenvalues of its orthogonal truncations. Let

$$\Lambda(A) = \{\lambda \in \mathrm{Lim}_{n\to\infty}\lambda_n : \ \lambda_n \text{ is an eigenvalue of } A_n \text{ for } n \geq 1\},$$

where $\mathrm{Lim}_{n\to\infty}\lambda_n$ is the set of all limit points of the sequence $(\lambda_n)$ and $A_n$ is a finite dimensional orthogonal truncation of $A$. The aim of this article is to provide the conditions that are sufficient for the relations $\sigma(A) \subset \Lambda(A)$ or $\Lambda(A) \subset \sigma(A)$ to be satisfied for the band operator $A$.

**Keywords:** unbounded operator, band-type matrix, complex tridiagonal matrix, discrete spectrum, eigenvalue, limit points of eigenvalues.

**Mathematics Subject Classification:** 47B36, 47B37, 47A25, 47A75, 15A18.

## 1. INTRODUCTION AND NOTATIONS

Special classes of infinite band matrices such as Jacobi matrices or tridiagonal matrices have been systematically studied in the literature. Tridiagonal real or complex matrices are linked to the orthogonal polynomials or formal orthogonal complex polynomials, second order differential equations, Mathieu equation and functions, and Bessel and have many uses (see e.g., in [1, 3–6, 10, 12, 13, 17, 18, 29, 30]). The band matrices are also interesting and worthy of extensive investigation because of a wide range of applications ([20, 26]). Sometimes band matrices can be treated as tridiagonal block matrices ([8, 9, 11, 22]). Recently the problem of asymptotic behaviour of the discrete spectrum for operators defined by 5-diagonal complex matrices has been discussed in [2]. Moreover, spectral properties with asymptotics for eigenvalues of selfadjoint band operators with compact resolvent have been investigated in [7].

 

Let us fix the notation $\mathbb{V} = \mathbb{N}$ or $\mathbb{V} = \mathbb{Z}$ and consider the Hilbert space

$$l_2 = l_2(\mathbb{V}) = \operatorname{span}\{e_n : n \in \mathbb{V}\}, \tag{1.1}$$

where $\{e_n : n \in \mathbb{V}\}$ is a canonical basis in this space such that

$$e_n = (\delta_n(k))_{k\in\mathbb{V}}, \quad \text{and} \quad \delta_n(k) = \begin{cases} 1, & n = k, \\ 0, & n \neq k. \end{cases}$$

Let $A$ be a linear operator in the Hilbert space $l_2$, for which the matrix representation is given by

$$(A_{i,j})_{i,j\in\mathbb{V}},$$

where $A_{i,j} = (Ae_j, e_i)$ and $A_{i,j} = 0$ for $|i - j| > q$, where $q \geq 1$ is a fixed integer. Then $A$ is a band type operator and the band-width of $A$ equals $q$.

Denote the set of indexes

$$J_n = \begin{cases} \{1, \ldots, n\} & \text{for } \mathbb{V} = \mathbb{N}, \\ \{-n, \ldots, 0, 1, \ldots, n\} & \text{for } \mathbb{V} = \mathbb{Z} \end{cases} \tag{1.2}$$

and a finite dimensional subspace $E_n \subset l_2$, where

$$E_n = \operatorname{span}\{e_k : \ k \in J_n\}. \tag{1.3}$$

Let $P_n$ be an orthogonal projection in $l_2$ on the subspace $E_n$. Then define an orthogonal truncation of $A$ as an finite dimensional linear mapping

$$A_n = P_n A|_{E_n} : E_n \to E_n, \tag{1.4}$$

where $n \geq 1$. Then $(A_{i,j})_{i,j\in J_n}$ is a matrix representation of $A_n$.

Define a set

$$\Lambda(A) = \{\lambda \in \operatorname{Lim}_{n\to\infty}\lambda_n : \ \lambda_n \text{ is an eigenvalue of } A_n \text{ for } n \geq 1\}, \tag{1.5}$$

where $\operatorname{Lim}_{n\to\infty}\lambda_n$ is the set of limit points of the sequence $(\lambda_n)_{n=1}^{\infty}$.

We focus on unbounded operators in $l_2$ with compact resolvent. The aim of this work is finding classes of band operators, for which the inclusions for the sets $\sigma(A)$ and $\Lambda(A)$ can be established. The inclusion $\sigma(A) \subset \Lambda(A)$ for a bounded self-adjoint operator $A$ on $l_2$ is a known result (see [1]), but this spectral property does not have to be true for non-selfadjoint or unbounded operators. This problem is directly related to the natural and useful problem of approximation of the eigenvalues for an operator by the eigenvalues of its orthogonal truncations (see e.g. [6, 10, 19, 21, 22, 24, 27–29]). Sufficient conditions for this inclusion for tridiagonal compact operators or tridiagonal operators with compact resolvent have been presented in the literature (see [15–17, 19, 21, 23, 25] and others). It has been shown that if the self-adjoint tridiagonal operator is compact or it is a compact perturbation of the diagonal operator then $\sigma(A) = \Lambda(A)$ ([14]).

In addition, the authors of [25] also found other conditions sufficient for the equality of the spectrum of $A$ and the set $\Lambda(A)$ to be true. Surprisingly, a very interesting and difficult problem is finding sufficient conditions on the operator $A$, for which the relation $\Lambda(A) \subset \sigma(A)$ occures. Solutions to this problem for Jacobi matrices and complex tridiagonal operators can be found, for example, in [1, 14, 16, 23] and [25].

Let us take the standard notations for band operators. Denote a diagonal operator in $l_2$

$$D = \operatorname{diag}(d(n))_{n\in\mathbb{V}}, \tag{1.6}$$

for which the subspace

$$Dom(D) = \{(f_n)_{n\in\mathbb{V}} \in l_2 : \ (d(n)f_n)_{n\in\mathbb{V}} \in l_2\}$$

is a domain, where $(d(n))_{n\in\mathbb{V}}$ is a complex sequence and

$$De_n = d(n)e_n \quad \text{for} \quad n \in \mathbb{V}. \tag{1.7}$$

Let $S$ be a shift operator in $l_2$ such that $Se_n = e_{n+1}$ for $n \in \mathbb{V}$. Then $S^*$ is the adjoint operator for $S$ and we observe that $S^k e_n = e_{n+k}$ and $(S^*)^k e_n = e_{n-k}$, where $e_{n-k} = (\delta_{n-k}(l))_{l\in\mathbb{V}}$, $n \in \mathbb{V}$ and $k \geq 1$.

Assume that $q \geq 1$ is an integer and $D_k = \operatorname{diag}(d_k(n))_{n\in\mathbb{V}}$ for $|k| \leq q$. Describe more precisely a band operator as a $(2q+1)$-diagonal operator given by the formula

$$A = D_0 + \sum_{k=1}^{q} (S^k D_k + D_{-k}(S^*)^k); \tag{1.8}$$

it means that if $f = \sum_{n\in\mathbb{V}} f_n e_n \in l_2$ then

$$Af = \sum_{n\in\mathbb{V}} \left( d_0(n)f_n + \sum_{k=1}^{q} (d_k(n-k)f_{n-k} + d_{-k}(n)f_{n+k}) \right) e_n, \tag{1.9}$$

where $d_k(j) = 0$ and $f_j = 0$ for $j \leq 0$ in the case $\mathbb{V} = \mathbb{N}$, and we assume that the operator $A$ acts on a maximal domain in $l_2$

$$Dom(A) = \left\{ f \in l_2 \colon \left( d_0(n)f_n + \sum_{k=1}^{q} (d_k(n-k)f_{n-k} + d_{-k}(n)f_{n+k}) \right)_{n\in\mathbb{V}} \in l_2 \right\}. \tag{1.10}$$

Denote

$$\rho(n) = \max\{|d_k(n+s)| : \ 0 < |k| \leq q, \ |s| \leq q, \ n+s \in \mathbb{V}\} \tag{1.11}$$

for $n \in \mathbb{V}$. We assume that $A$ has a strongly asymptotically dominated main diagonal,

$$\lim_{|n|\to\infty} |d_0(n)| = \infty \tag{1.12}$$

and

$$\lim_{|n|\to\infty} \frac{\rho(n)}{|d_0(n)|} = 0. \tag{1.13}$$

We are going to study this class of band-type operators in more detail.

## 2. SUFFICIENT CONDITIONS FOR DISCRETENESS OF $\sigma(A)$ AND THE INCLUSION $\sigma(A) \subset \Lambda(A)$

We consider a band operator A defined by (1.9) and (1.10), where the assumptions (1.12) and (1.13) are fulfilled. First, we will examine the problem of the existence of a purely discrete spectrum of the considered operator. Suppose that the sequence $(d_0(n))_{n\in\mathbb{V}}$ is a main diagonal of $A$. Let

$$B(n) = \{z \in \mathbb{C} : \ |d_0(n) - z| \leq 2(q+1)\rho(n)\} \tag{2.1}$$

denote the Gerschgorin-type disc in the complex plane, where $q$ equals to the band-width of $A$ and $\rho(n)$ is given by the formula (1.11) for $n \in \mathbb{V}$.

**Theorem 2.1.** *If $A$ is a band operator such that* (1.9)–(1.13) *are fulfilled and*

$$\bigcup_{n\in\mathbb{V}} B(n) \neq \mathbb{C},$$

*where $B(n)$ is given by* (2.1)*, then*

(1) $Dom(A) = Dom(D_0) = \{(f_n)_{n\in\mathbb{V}} \in l_2 : \ (d_0(n) f_n)_{n\in\mathbb{V}} \in l_2\}$,
(2) $(A-\lambda)^{-1}$ *is a compact operator on* $l_2$ *for* $\lambda \in \mathbb{C} \setminus \sigma(A)$,
(3) *the spectrum of $A$ is discrete and*

$$\sigma(A) \subset \bigcup_{n\in\mathbb{V}} B(n).$$

*Proof.* Let us take $\lambda \in \mathbb{C} \setminus \bigcup_{n\in\mathbb{V}} B(n)$. Then obviously

$$\frac{\rho(n)}{|d_0(n) - \lambda|} < \frac{1}{2(q+1)} \tag{2.2}$$

for all $n \in \mathbb{V}$ because of (2.1). Denote

$$L = \sum_{k=1}^{q} \left(S^k D_k + D_{-k}(S^*)^k\right) \tag{2.3}$$

and assume

$$Dom(L) = \left\{(f_n)_{n\in\mathbb{V}} \in l_2 : \left(\sum_{k=1}^{q} (d_k(n-k) f_{n-k} + d_{-k}(n) f_{n+k})\right)_{n\in\mathbb{V}} \in l_2\right\}. \tag{2.4}$$

It is clear that (1.11)–(1.13) imply that $Dom(D_0) \subset Dom(L)$.

Next we notice that

$$\begin{aligned} A - \lambda &\supseteq D_0 - \lambda + L \\ &= \left(I + L(D_0-\lambda)^{-1}\right)(D_0 - \lambda). \end{aligned} \tag{2.5}$$

Let us examine the operator norm of $L(D_0-\lambda)^{-1}$. If $f=\sum_{n\in\mathbb{V}} f_n e_n \in l_2$, then the following equalities are satisfied:

$$\begin{aligned} L(D_0-\lambda)^{-1}f &= \sum_{k=1}^{q}\sum_{n\in\mathbb{V}} \big(S^k D_k (D_0-\lambda)^{-1} f_n e_n + D_{-k}(S^*)^k (D_0-\lambda)^{-1} f_n e_n\big) \\ &= \sum_{k=1}^{q}\sum_{n\in\mathbb{V}} \left(\frac{d_k(n) f_n}{d_0(n)-\lambda} e_{n+k}\right) + \sum_{k=1}^{q}\sum_{n\in\mathbb{V}} \left(\frac{d_{-k}(n-k) f_n}{d_0(n)-\lambda} e_{n-k}\right) \\ &= \sum_{n\in\mathbb{V}} \left(\sum_{k=1}^{q} \frac{d_k(n-k) f_{n-k}}{d_0(n-k)-\lambda} + \sum_{k=1}^{q} \frac{d_{-k}(n) f_{n+k}}{d_0(n+k)-\lambda}\right) e_n. \end{aligned}$$

Therefore,

$$\begin{aligned} &\|L(D_0-\lambda)^{-1}f\| \\ &\le \left(\sum_{n\in\mathbb{V}} \left|\sum_{k=1}^{q} \frac{d_k(n-k) f_{n-k}}{d_0(n-k)-\lambda}\right|^2\right)^{\frac{1}{2}} + \left(\sum_{n\in\mathbb{V}} \left|\sum_{k=1}^{q} \frac{d_{-k}(n) f_{n+k}}{d_0(n+k)-\lambda}\right|^2\right)^{\frac{1}{2}} \\ &\le \left(\sum_{n\in\mathbb{V}} q \sum_{k=1}^{q} \frac{|d_k(n-k)|^2 |f_{n-k}|^2}{|d_0(n-k)-\lambda|^2}\right)^{\frac{1}{2}} + \left(\sum_{n\in\mathbb{V}} q \sum_{k=1}^{q} \frac{|d_{-k}(n)|^2 |f_{n+k}|^2}{|d_0(n+k)-\lambda|^2}\right)^{\frac{1}{2}} \\ &\le \left(\sum_{n\in\mathbb{V}} q \sum_{k=1}^{q} \frac{(\rho(n-k))^2 |f_{n-k}|^2}{|d_0(n-k)-\lambda|^2}\right)^{\frac{1}{2}} + \left(\sum_{n\in\mathbb{V}} q \sum_{k=1}^{q} \frac{(\rho(n+k))^2 |f_{n+k}|^2}{|d_0(n+k)-\lambda|^2}\right)^{\frac{1}{2}} \\ &\le \left(\sum_{n\in\mathbb{V}} q \sum_{k=1}^{q} \frac{1}{4(q+1)^2} |f_{n-k}|^2\right)^{\frac{1}{2}} + \left(\sum_{n\in\mathbb{V}} q \sum_{k=1}^{q} \frac{1}{4(q+1)^2} |f_{n+k}|^2\right)^{\frac{1}{2}} \\ &\le \frac{q\|f\|}{q+1}. \end{aligned}$$

Thus we observe

$$\|L(D_0-\lambda)^{-1}\| \le \frac{q}{q+1} < 1,$$

so the existence of the bounded operator $(I+L(D_0-\lambda)^{-1})^{-1}$ on $l_2$ is assured.

The assumption (1.12) implies that $\lim_{|n|\to\infty} \left|\frac{1}{d_0(n)-\lambda}\right| = 0$, so the diagonal operator $(D_0-\lambda)^{-1}$ is compact and hence

$$\big((I+L(D_0-\lambda)^{-1})(D_0-\lambda)\big)^{-1} = (D_0-\lambda)^{-1}\left(I+L(D_0-\lambda)^{-1}\right)^{-1}$$

is a compact operator on $l_2$.

Now we are going to prove that $A-\lambda$ is injective on $Dom(A)$. Suppose that there exists $f=(f_n)_{n\in\mathbb{V}} \in Dom(A)$, $\|f\|=1$ and $(A-\lambda)f=0$. Then, according to (1.9)

$$(d_0(n)-\lambda) f_n = -\sum_{k=1}^{q} \left(d_k(n-k) f_{n-k} + d_{-k}(n) f_{n+k}\right), \quad n\in\mathbb{V}. \tag{2.6}$$

From (2.6) we derive

$$\begin{aligned}|f_n| &\le \sum_{k=1}^{q}\left(\left|\frac{d_k(n-k)}{d_0(n)-\lambda}\right||f_{n-k}| + \left|\frac{d_{-k}(n)}{d_0(n)-\lambda}\right||f_{n+k}|\right)\\ &\le \frac{\rho(n)}{|d_0(n)-\lambda|}\sum_{k=1}^{q}(|f_{n-k}|+|f_{n+k}|)\end{aligned}$$

and using (2.2)

$$\begin{aligned}|f_n|^2 &\le \frac{1}{4(q+1)^2}\left(\sum_{k=1}^{q}(|f_{n-k}|+|f_{n+k}|)\right)^2\\ &\le \frac{q}{2(q+1)^2}\left(\sum_{k=1}^{q}|f_{n-k}|^2+\sum_{k=1}^{q}|f_{n+k}|^2\right)\end{aligned}$$

for $n \in \mathbb{V}$. Thus $\|f\|^2 \le \frac{q}{2(q+1)^2}\cdot 2q\|f\|^2$ and finally $\|f\| \le \frac{q}{q+1}\|f\| < \|f\|$, but this is impossible. So we conclude that if $(A-\lambda)f = 0$ then $f = 0$ and $A-\lambda$ is injective.

Therefore, from (2.5) we derive

$$(A-\lambda)^{-1} \supseteq \left((I+L(D_0-\lambda)^{-1})(D_0-\lambda)\right)^{-1}$$

but the operator on the right side of this relation is bounded on $l^2$ and

$$Dom\left(\left((I+L(D_0-\lambda)^{-1})(D_0-\lambda)\right)^{-1}\right) = l^2 = Dom\left((A-\lambda)^{-1}\right).$$

The equality of the domains of these operators entails the equality of the operators. Then we deduce that

$$A-\lambda = D_0-\lambda+L = \left(I+L(D_0-\lambda)^{-1}\right)(D_0-\lambda)$$

and

$$Dom(A) = Dom(A-\lambda) = Dom(D_0-\lambda) = Dom(D_0).$$

We also conclude that $(A-\lambda)^{-1}$ is a compact operator on $l_2$. Then $A$ is an operator with compact resolvent and it is clear that the spectrum of $A$ is discrete, because we observe that

$$\sigma(A) = \left\{\lambda + \frac{1}{\mu_n}:\ n \in \mathbb{N}\right\},$$

where $\mu_n \to 0$ as $n \to \infty$, and

$$\sigma\left((A-\lambda)^{-1}\right) = \{\mu_n \neq 0:\ n \in \mathbb{N}\} \cup \{0\},$$

and the multiplicity of the eigenvalue $\lambda + \frac{1}{\mu_n}$ of $A$ equals to the multiplicity of the eigenvalue $\mu_n$ for $n \ge 1$. Moreover, $\sigma(A) \subset \bigcup_{n\in\mathbb{V}} B(n)$ since $\lambda$ was chosen as any element from $\mathbb{C} \setminus \bigcup_{n\in\mathbb{V}} B(n)$. □

Let us take the notation for an angular set on the complex plane

$$\Delta(\alpha_1, \alpha_2) = \{z \in \mathbb{C} : \ \alpha_1 \le arg(z) \le \alpha_2\}, \tag{2.7}$$

where $0 < \alpha_2 - \alpha_1 < 2\pi$.

**Theorem 2.2.** *If $A$ is a band operator,* (1.9)*,* (1.10)*,* (1.12) *and* (1.13) *are satisfied and there exist $n_0 \in \mathbb{N}$ and $\alpha_1$, $\alpha_2$, where $0 < \alpha_2 - \alpha_1 < 2\pi$, such that*

$$d_0(n) \in \Delta(\alpha_1, \alpha_2)$$

*for $|n| \ge n_0$, then $\bigcup_{n\in\mathbb{V}} B(n) \ne \mathbb{C}$, $A$ has compact resolvent and the spectrum of $A$ is discrete.*

*Proof.* Without losing generality, multiplying operator $A$ by a suitable complex constant and rotating the set on the complex plane, we can assume $\alpha_2 = -\alpha_1 = \alpha \in (0, \pi)$. By extending the set $\Delta(-\alpha, \alpha)$ assume that $\alpha \in (\frac{3}{4}\pi, \pi)$.

Denote $R_n = 2(q + 1)\rho(n)$. According to the assumptions and (1.11) there exists $N_0 \ge n_0$ such that

$$\frac{R_n}{|d_0(n)|} < \frac{\sin\alpha}{2} \tag{2.8}$$

for $|n| \ge N_0$. Now put

$$M_0 = \max\{|d_0(n)| + R_n : \ |n| \le N_0\}.$$

If $\lambda < -M_0$ then it is easy to see that $\lambda \notin B(n)$, where $|n| \le N_0$ or $\Re e(d_0(n)) \ge 0$ and $B(n)$ is given by (2.1).

Let $\lambda < -M_0$ and $n \in \mathbb{V}$ satisfies $|n| > N_0$ and $\Re e(d_0(n)) < 0$. Therefore, $d_0(n) = |d_0(n)|e^{i\alpha_n}$, where $\alpha_n \in [-\alpha, -\frac{\pi}{2}) \cup (\frac{\pi}{2}, \alpha]$ and then $\cos\alpha \le \cos\alpha_n$. Next, let us assume the hypothesis $\lambda \in B(n)$. According to this hypothesis the inequality $|d_0(n) - \lambda| \le R_n$ is satisfied, then we get

$$\begin{aligned} -2\lambda|d_0(n)|\cos\alpha + \lambda^2 &\le -2\lambda|d_0(n)|\cos\alpha_n + \lambda^2 \\ &= -2\lambda\Re e(d_0(n)) + \lambda^2 \\ &\le R_n^2 - |d_0(n)|^2. \end{aligned} \tag{2.9}$$

From (2.8) and (2.9) we easy derive that

$$\begin{aligned} (|d_0(n)|\cos\alpha - \lambda)^2 &= |d_0(n)|^2 - 2\lambda|d_0(n)|\cos\alpha + \lambda^2 - |d_0(n)|^2\sin^2\alpha \\ &\le R_n^2 - |d_0(n)|^2\sin^2\alpha \\ &< -\frac{3}{4}|d_0(n)|^2\sin^2\alpha < 0. \end{aligned}$$

It is obvious that this cannot be true, so we conclude $\lambda \notin B(n)$, where $|n| > N_0$ and $\Re e(d_0(n)) < 0$. Finally we have proved

$$\{\lambda \in \mathbb{R} : \lambda < -M_0\} \subset \mathbb{C} \setminus \bigcup_{n\in\mathbb{V}} B(n).$$

Thus $\bigcup_{n\in\mathbb{V}} B(n) \ne \mathbb{C}$ and using Theorem 2.1 we conclude that the operator $A$ has compact resolvent and its the spectrum is discrete. □

It is very well known that if $A$ is a bounded and self-adjoint operator on $l_2$ then $\sigma(A) \subset \Lambda(A)$ (see [1]). Unfortunately, inclusion does not have to be true for non-selfadjoint or unbounded operators. In the case of tridiagonal operator some results are known. Classical result (see e.g. in [15] or [14]) says that if $J$ is a self-adjoint operator given by a real Jacobi matrix in $l_2(\mathbb{N})$, then $\sigma(J) \subset \Lambda(J)$. If $J$ is represented by a complex tridiagonal matrix and $J$ is a compact operator on $l_2$ then $\sigma(J) \subset \Lambda(J)$ ([19]), but if $J$ is additionally selfadjoint then $\sigma(J) = \Lambda(J)$ ([25]). Some positive results in the case unbounded tridiagonal operators, we can find, for example, in [14–17, 23] and [25].

The new result for unbounded band-type operators we present in the following theorem.

**Theorem 2.3.** *If $A$ is an operator defined by* (1.9) *and* (1.10)*, with compact resolvent, and* (1.12) *and* (1.13) *are fulfilled, then $\sigma(A)$ is discrete and $\sigma(A) \subset \Lambda(A)$.*

*Proof.* It is obvious that the spectrum $\sigma(A)$ is discrete since $A$ has compact resolvent. We generalize the proof from [17] or [23]. At first we assume without losing generality that there exists $A^{-1}$ as a compact operator on $l_2$ and $d_0(n) \neq 0$ for $n \in \mathbb{V}$. Therefore, we denote a diagonal operator

$$C = D_0^{-1/2}, \tag{2.10}$$

that means

$$Ce_n = \frac{1}{\sqrt{d_0(n)}}\, e_n, \quad n \in \mathbb{V},$$

where $\sqrt{d_0(n)}$ is a complex number such that $\left(\sqrt{d_0(n)}\right)^2 = d_0(n)$. Obviously, $C$ is compact on the Hilbert space $l_2$.

Suppose that the operator

$$L = \sum_{k=1}^{q} (S^k D_k + D_{-k}(S^*)^k)$$

acts on the domain $D(L)$ defined by (2.4). Observe that

$$A \supseteq C^{-2} + L = C^{-1}(I + CLC)C^{-1} \tag{2.11}$$

and

$$CLC = \sum_{k=1}^{q} (CS^k D_k C + CD_{-k}(S^*)^k C).$$

Let $n \in \mathbb{V}$, then

$$\begin{aligned} CS^k D_k Ce_n &= \frac{d_k(n)}{\sqrt{d_0(n)}} CS^k e_n = \frac{d_k(n)}{\sqrt{d_0(n)}\sqrt{d_0(n+k)}} e_{n+k} \\ &= \beta_k(n) e_{n+k} = \beta_k(n) S^k e_n = S^k B_k e_n, \end{aligned}$$

where

$$\beta_k(n) = \frac{d_k(n)}{\sqrt{d_0(n)}\sqrt{d_0(n+k)}}$$

and $B_k = \operatorname{diag}(\beta_k(l))_{l \in \mathbb{V}}$ is a diagonal operator in $l_2$.

Similarly

$$CD_{-k}(S^*)^k Ce_n = \frac{1}{\sqrt{d_0(n)}} CD_{-k}e_{n-k} = \frac{d_{-k}(n-k)}{\sqrt{d_0(n)}\sqrt{d_0(n-k)}} e_{n-k}$$
$$= \gamma_k(n-k)(S^*)^k e_n = G_k(S^*)^k e_n,$$

where

$$\gamma_k(n-k) = \frac{d_{-k}(n-k)}{\sqrt{d_0(n)}\sqrt{d_0(n-k)}}$$

and $G_k = \operatorname{diag}(\gamma_k(l))_{l\in\mathbb{V}}$.

Under the assumptions (1.12) and (1.13)

$$\lim_{|n|\to\infty} \beta_k(n) = \lim_{|n|\to\infty} \gamma_k(n) = 0.$$

This implies that the diagonal operators $B_k$ and $G_k$ for $k = 1, \dots, q$ are compact. Therefore,

$$CLC = \sum_{k=1}^{q} (S^k B_k + G_k(S^*)^k)$$

is also a compact operator on $l_2$.

We have assumed that $A$ is invertible on $l_2$, consequently $I + CLC$ is injective because of (2.11), then it is invertible as an operator on $l_2$ because of compactness of $CLC$. Therefore, (2.11) implies that

$$A^{-1} \supseteq C(I + CLC)^{-1}C, \tag{2.12}$$

but both of the operators are compact on $l_2$, so from the relation (2.12) we conclude that the operators $A^{-1}$ and $C(I + CLC)^{-1}C$ are equal. Then denote

$$T := A^{-1} = C(I + CLC)^{-1}C. \tag{2.13}$$

It is clear that $Ax = \lambda x$ if and only if $Tx = \frac{1}{\lambda}x$ for $\lambda \in \mathbb{C} \setminus \{0\}$ and $x \in l_2 \setminus \{0\}$.

Let $E_n$ be a canonical subspace for $l_2$ according to (1.3) and $P_n$ is the orthogonal projection on $E_n$. Then $P_n$ admits the block matrix representation

$$P_n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $I_n$ is the identity on $E_n$ and

$$l_2 = H_n^{'} \oplus E_n \oplus H_n^{''}, \tag{2.14}$$

where

$$H_n^{'} = \operatorname{span}\{e_k : \ k \in \mathbb{V} \text{ and } k < -n\}$$

and

$$H_n^{''} = \operatorname{span}\{e_k : \ k > n\}.$$

The operator $CLC$ is compact. Therefore,

$$\|P_nCLCP_n - CLC\| \to 0, \quad n \to \infty, \tag{2.15}$$

and also

$$\|(P_nCLCP_n + I) - (CLC + I)\| \to 0, \quad n \to \infty. \tag{2.16}$$

Notice that $I + CLC$ is invertible on $l_2$, so there exists $n_0$ such that the operator $P_nCLCP_n + I$ is also invertible for $n \geq n_0$ and

$$\|(P_nCLCP_n + I)^{-1} - (CLC + I)^{-1}\| \to 0, \ n \to \infty.$$

Denote

$$T_n = P_nC(P_nCLCP_n + I)^{-1}CP_n \tag{2.17}$$

and observe that

$$P_nCLCP_n + I = \begin{pmatrix} I_n^{'} & 0 & 0 \\ 0 & C_nA_nC_n & 0 \\ 0 & 0 & I_n^{''} \end{pmatrix},$$

where

$$C_n = P_nC|_{E_n} = \operatorname{diag}(1/\sqrt{d_0(k)}\,)_{k \in J_n}$$

is a finite dimensional diagonal matrix, $I_n^{'}$ means the identity operator on the subspace $H_n^{'}$ and $I_n^{''}$ is also the identity operator on $H_n^{''}$.

Next we observe that

$$\begin{aligned}\|T_n - T\| &= \|P_nTP_n - T + P_nC\left[(P_nCLCP_n + I)^{-1} - (ClC + I)^{-1}\right]CP_n\| \\ &\leq \|P_nTP_n - T\| + \|C\|^2\|(P_nCLCP_n + I)^{-1} - (CLC + I)^{-1}\|\end{aligned}$$

and from (2.15) and (2.16) we derive

$$\|T_n - T\| \to 0, \quad n \to \infty.$$

The equation (2.17) implies that

$$T_n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A_n^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in accordance with the decomposition (2.16) for $l_2$. So if $\lambda \neq 0$ is an eigenvalue of $A$, then $\mu = \frac{1}{\lambda}$ is an eigenvalue of $T$. From the projective method approach ([19, Thm. 18.1]) we derive that there exists a sequence $(\mu_n)_{n \geq n_0}$ such that $\mu_n$ is an eigenvalue of $T_n$ and $\mu = \lim_{n\to\infty} \mu_n$. Notice that if $\mu_n \neq 0$ for enough large $n$, then $\lambda_n = \frac{1}{\mu_n}$ is an eigenvalue of $A_n$. Moreover, $\lambda = \lim_{n\to\infty} \lambda_n$ and $\sigma(A) \subset \Lambda(A)$ finally. □

## 3. PARTIAL SOLUTIONS OF THE PROBLEM $\Lambda(A) \subset \sigma(A)$

It is difficult problem to find some sufficient conditions on the operator $A$ for which the inclusion $\Lambda(A) \subset \sigma(A)$ holds. If $A$ is a tridiagonal operator then the inclusion $\Lambda(A) \subset \sigma(A)$ holds under some conditions (see [1, 14, 16, 23, 25], and others).

Let us first formulate two technical lemmas, which generalize the methods used in [25] and [23]. We assume that the system $\{e_n : n \in \mathbb{V}\}$ is the canonical basis for $l_2 = l_2(\mathbb{V})$ and $P_n$ means the orthogonal projection on $E_n = \operatorname{span}\{e_k : k \in J_n\}$. The operator $A$ is defined by (1.9) and (1.10), and $A_n$ is given by (1.4).

**Lemma 3.1.** *Assume that for all bounded complex sequences of eigenvalues $(\lambda_n)_{n=1}^{\infty}$, where $\lambda_n \in \sigma(A_n)$, $n \geq 1$, and for all sequences of eigenvectors $(x_n)_{n=1}^{\infty}$ such that $x_n \in E_n$, $A_n x_n = \lambda_n x_n$ and $\|x_n\| = 1$ for $n \geq 1$,*

$$\lim_{n\to\infty} |d_k(n-s+1)(x_n, e_{n-s+1})| = 0 \tag{3.1}$$

*and*

$$\lim_{n\to\infty} |d_{-k}(-n-s)(x_n, e_{-n-s+k})| = 0, \quad \textit{in the case } \mathbb{V} = \mathbb{Z}, \tag{3.2}$$

*where $k = 1, \ldots, q$ and $s = 1, \ldots, k$. Then $\Lambda(A) \subset \sigma(A)$.*

*Proof.* Let $\lambda \in \Lambda(A)$. Without loss of generality we can assume

$$\lambda = \lim_{n\to\infty} \lambda_n, \tag{3.3}$$

where $\lambda_n$ is an eigenvalue of the orthogonal truncation $A_n$. Let $x_n \in E_n$ be an eigenvector of $A_n$ such that $A_n x_n = \lambda_n x_n$ and $\|x_n\| = 1$ for $n \geq 1$. Then

$$P_n A x_n = P_n A P_n x_n = A_n x_n = \lambda_n x_n.$$

Denote

$$x_n = \sum_{k\in J_n} f_k e_k,$$

where $f_k = (x_n, e_k)$ for $k \in J_n$. Then

$$A x_n = P_n A x_n + (I - P_n) A x_n = \lambda_n x_n + (I - P_n) A x_n. \tag{3.4}$$

Put also $f_s = 0$ for $s \in \mathbb{Z} \setminus J_n$. Then according to (1.9)

$$A x_n = \sum_{s\in\mathbb{V}} \left( \sum_{k=1}^{q} (d_k(s-k) f_{s-k} + d_{-k}(s) f_{s+k}) + d_0(s) f_s \right) e_s,$$

and

$$(I - P_n)Ax_n = \sum_{s\in\mathbb{V}\setminus J_n} \left( \sum_{k=1}^{q} (d_k(s-k)f_{s-k} + d_{-k}(s)f_{s+k}) + d_0(s)f_s \right) e_s$$
$$= \underbrace{\sum_{s\in\mathbb{V}\setminus J_n} \left( \sum_{k=1}^{q} d_k(s-k)f_{s-k} \right) e_s}_{I1}$$
$$+ \underbrace{\sum_{s\in\mathbb{V}\setminus J_n} \left( \sum_{k=1}^{q} d_{-k}(s)f_{s+k} \right) e_s}_{I2} + \underbrace{\sum_{s\in\mathbb{V}\setminus J_n} d_0(s)f_s e_s}_{I3} .$$

It is clear that $I3 = 0$ because $f_s = 0$ for $s \in \mathbb{V} \setminus J_n$. Moreover,

$$I1 = \sum_{s=n+1}^{n+q} \left( \sum_{k=s-n}^{q} d_k(s-k)f_{s-k} \right) e_s$$
$$= \sum_{j=1}^{q} \left( \sum_{k=j}^{q} d_k(n+j-k)f_{n+j-k} \right) e_{n+j}$$
$$= \sum_{j=1}^{q} \left( \sum_{k=j}^{q} d_k(n+j-k)(x_n, e_{n+j-k}) \right) e_{n+j}$$

and

$$I2 = \sum_{s=-n-q}^{-n-1} \left( \sum_{k=-n-s}^{q} d_{-k}(s)f_{s+k} \right) e_s$$
$$= \sum_{j=1}^{q} \left( \sum_{k=j}^{q} d_{-k}(-n-j)f_{-n-j+k} \right) e_{-n-j}$$
$$= \sum_{j=1}^{q} \left( \sum_{k=j}^{q} d_{-k}(-n-j)(x_n, e_{-n-j+k}) \right) e_{-n-j}.$$

Next notice that (3.4) implies that

$$Ax_n - \lambda_n x_n = I1 + I2, \tag{3.5}$$

and

$$\|Ax_n - \lambda_n x_n\|^2 = \|I1\|^2 + \|I2\|^2 \tag{3.6}$$

because $I1 \perp I2$, where the orthogonality holds in the Hilbert space $l_2$. Then

$$\|Ax_n - \lambda x_n\|^2 = \|Ax_n - \lambda_n x_n + (\lambda_n - \lambda)x_n\|^2 = \|I1 + I2 + (\lambda_n - \lambda)x_n\|^2$$
$$= \|I1\|^2 + \|I2\|^2 + |\lambda_n - \lambda|^2 \|x_n\|^2$$
$$= \|I1\|^2 + \|I2\|^2 + |\lambda_n - \lambda|^2$$

because $I1$, $I2 \perp x_n$ and $\|x_n\| = 1$.

Therefore,

$$\|Ax_n - \lambda x_n\|^2 = \sum_{j=1}^{q} |\sum_{k=j}^{q} d_k(n+j-k)(x_n, e_{n+j-k})|^2 + \sum_{j=1}^{q} |\sum_{k=j}^{q} d_{-k}(-n-j)(x_n, e_{-n-j+k})|^2 + |\lambda_n - \lambda|^2.$$

From (3.1), (3.2) and (3.3) we derive that the parts of the above sum tend to 0. So, we deduce that there exists $(x_n)_{n=1}^{\infty}$ such that $\|x_n\| = 1$ for $n \geq 1$ and, $\|(A-\lambda)x_n\| \to 0$, as $n \to \infty$. Finally we notice that $\lambda$ belongs to $\sigma(A)$. □

**Lemma 3.2.** *Let $A_n x_n = \lambda_n x_n$, where $x_n \in E_n$ and $\|x_n\| = 1$ for $n \geq 1$, and there exists a constant $M > 0$ such that $|\lambda_n| \leq M$ for $n \geq 1$. Assume $\lim_{|n|\to\infty} |d_0(n)| = +\infty$ and for a fixed integer $\gamma \geq 2$ define*

$$K_n = \max\{|d_k(n+j)| : \ k = \pm 1, \dots, \pm q, \ |j| \leq 2\gamma q \text{ and } n+j \in \mathbb{V}\}, \tag{3.7}$$

$$M_n = \begin{cases} \min\{|d_0(k)| : \ k \geq n - 2\gamma q\}, & n > 0, \\ \min\{|d_0(k)| : \ k \in \mathbb{V} \text{ and } k \leq n + 2\gamma q\}, & n \leq 0 \end{cases} \tag{3.8}$$

*for $n \in \mathbb{V}$. Then there exists $N_0$ such that*

$$|(x_n, e_{n+j})| \leq \left(\frac{4qK_n}{M_n}\right)^{\gamma}$$

*and*

$$|(x_n, e_{-n+j})| \leq \left(\frac{4qK_{-n}}{M_{-n}}\right)^{\gamma}, \quad \text{when} \quad \mathbb{V} = \mathbb{Z},$$

*for $n > N_0$ and $|j| \leq q$.*

*Proof.* If $|\lambda_n| \leq M$, then

$$|d_0(l) - \lambda_n| \geq |d_0(l)| - M \geq \frac{1}{2}|d_0(l)| \geq \frac{1}{2}M_l \tag{3.9}$$

for $l \in \mathbb{V}$ and $|l| \geq n_0$, where $n_0$ is large enough, because $|d_0(n)| \to \infty$ as $|n| \to \infty$ and (3.8).

If $x_n \in E_n$, then $x_n = \sum_{k\in J_n} f_k e_k$, where $f_k = (x_n, e_k)$ for $k \in \mathbb{V}$. The equation $A_n x_n = \lambda_n x_n$ implies

$$\begin{aligned}\sum_{k\in J_n} \lambda_n f_k e_k = P_n A\left(\sum_{k\in J_n} f_k e_k\right) &= \sum_{k\in J_n} f_k P_n A e_k\\ &= \sum_{k\in J_n} f_k P_n \left(D_0 + \sum_{j=1}^{q}(S^j D_j + D_{-j}(S^*)^j)\right) e_k\\ &= \sum_{k\in J_n} f_k d_0(k) e_k + \sum_{k\in J_n}\sum_{j=1}^{q} f_k d_j(k) P_n e_{k+j}\\ &\quad + \sum_{k\in J_n}\sum_{j=1}^{q} f_k d_{-j}(k-j) P_n e_{k-j}.\end{aligned}$$

So, if $l \in J_n$ then

$$\begin{aligned}\lambda_n f_l &= f_l d_0(l) + \sum_{k\in J_n}\sum_{j=1}^{q} f_k d_j(k)(P_n e_{k+j}, e_l)\\ &\quad + \sum_{k\in J_n}\sum_{j=1}^{q} f_k d_{-j}(k-j)(P_n e_{k-j}, e_l)\\ &= f_l d_0(l) + \sum_{j=1}^{q} f_{l-j} d_j(l-j)(P_n e_l, e_l) + \sum_{j=1}^{q} f_{l+j} d_{-j}(l)(P_n e_l, e_l)\end{aligned}$$

and

$$(d_0(l) - \lambda_n) f_l = -\sum_{j=1}^{q} \left(f_{l-j} d_j(l-j) + f_{l+j} d_{-j}(l)\right).$$

Therefore,

$$|d_0(l) - \lambda_n||f_l| \le \sum_{j=1}^{q} \left(|f_{l-j}||d_j(l-j)| + |f_{l+j}||d_{-j}(l)|\right)$$

and using (3.7) and (3.9) for $|l| \ge n_0 + q$ we obtain

$$\begin{aligned}|f_l| \le& \frac{1}{|d_0(l) - \lambda_n|}\sum_{j=1}^{q} K_l\left(|f_{l-j}| + |f_{l+j}|\right) \le \frac{2K_l}{M_l}\sum_{j=1}^{q}\left(|f_{l-j}| + |f_{l+j}|\right)\\ \le& \frac{4qK_l}{M_l}\max\{|f_{l+j}| : j = 0, \pm 1, \dots, \pm q\}.\end{aligned}$$

Assume $|l| \ge n_0 + 2q$, then $|l \pm j| \ge n_0 + q$ and

$$\begin{aligned}|f_{l\pm j}| &\le \frac{4qK_l}{M_l}\max\{|f_{l\pm j+s}| : s = 0, \pm 1, \dots, \pm q\}\\ &\le \frac{4qK_l}{M_l}\max\{|f_{l+s}| : s = 0, \pm 1, \dots, \pm 2q\},\end{aligned}\tag{3.10}$$

so we derive

$$|f_l| \leq \left(\frac{4qK_l}{M_l}\right)^2 \max\{|f_{l+s}| : s = 0, \pm 1, \dots, \pm 2q\}.$$

Finally we observe that

$$|f_l| \leq \left(\frac{4qK_l}{M_l}\right)^\gamma \max\{|f_{l+s}| : s = 0, \pm 1, \dots, \pm \gamma q\}$$

for $|l| \geq n_0 + \gamma q$. Moreover, we known that $|f_l| = |(x_n, e_l)| \leq 1$, so

$$|f_l| \leq \left(\frac{4qK_l}{M_l}\right)^\gamma \quad \text{for} \quad |l| \geq n_0 + \gamma q. \tag{3.11}$$

Given (3.7), (3.8), (3.10) and (3.11) we also observe that

$$|f_{l+j}| \leq \left(\frac{4qK_l}{M_l}\right)^\gamma, \quad \text{where} \quad |l| \geq n_0 + 2\gamma q \quad \text{and} \quad |j| \leq q. \tag{3.12}$$

Finally, from (3.12) we derive

$$|(x_n, e_{n-j})| = |f_{n-j}| \leq \left(\frac{4qK_n}{M_n}\right)^\gamma$$

and

$$|(x_n, e_{-n+j})| = |f_{-n+j}| \leq \left(\frac{4qK_{-n}}{M_{-n}}\right)^\gamma$$

for $n \geq N_0 = n_0 + 2\gamma q$ and $j = 0, \dots, q$. □

**Theorem 3.3.** *If $A$ is an operator given by* (1.9)*) and* (1.10)*, where* (1.12) *is satisfied, and there exists an integer $\gamma \geq 1$ such that*

$$\lim_{|n| \to \infty} \frac{K_n^{\gamma+1}}{M_n^\gamma} = 0,$$

*where $K_n$, $M_n$ are given by* (3.7) *and* (3.8)*, then $\Lambda(A) \subset \sigma(A)$.*

*Proof.* It is enough to use the estimates from the thesis of Lemma 3.2 and Lemma 3.1. □

**Corollary 3.4.** *Let $A$ be an operator given by* (1.9) *and* (1.10) *such that*

$$|d_k(n)| = O(|n|^{\beta_k}),\ |k| \leq q,\ \text{and}\ \frac{1}{|d_0(n)|} = O(|n|^{-\alpha})\ \text{as}\ |n| \to \infty.$$

*If $\alpha > \beta_k \geq 0$ for $k = \pm 1, \dots, \pm q$ then $\Lambda(A) \subset \sigma(A)$. Moreover, if there exist $n_0 \in \mathbb{N}$ and $\alpha_1$, $\alpha_2$, where $0 < \alpha_2 - \alpha_1 < 2\pi$, such that*

$$\alpha_1 \leq d_0(n) \leq \alpha_2 \ \ for\ \ |n| \geq n_0, \tag{3.13}$$

*then $A$ has compact resolvent and $\sigma(A) = \Lambda(A)$.*

*Proof.* If $\alpha > \beta_k$ then there exists an integer $\gamma \geq 1$ such that $\gamma\alpha > (\gamma+1)\beta$, where $\beta = \max_{|k|\leq q}\beta_k$. So

$$\frac{K_n^{\gamma+1}}{M_n^{\gamma}} = O\left(n^{(\gamma+1)\beta-\gamma\alpha}\right), \quad |n| \to \infty,$$

and we can apply Theorem 3.3 to obtain the inclusion $\Lambda(A) \subset \sigma(A)$. If (3.13) is satisfied additionally, then we apply Theorem 2.2 to conclude that $A$ has compact resolvent and $\sigma(A) = \Lambda(A)$. □

## 4. EXAMPLE

Denote by $W_2^2[0,1]$ a Sobolev space. We consider a differential operator in $L_2[0,1]$

$$T : D(T) \to L_2[0,1],$$

where

$$Ty(x) = -y''(x) - \psi(x)y(x)$$

for

$$y \in D(T) = \{\varphi \in L_2[0,1] :\ \varphi \in W_2^2[0,1],\ \varphi(0) = \varphi(1),\ \varphi'(0) = \varphi'(1)\}.$$

Assume that $\psi$ is a trigonometric polynomial given by the formula

$$\psi(x) = \sum_{k=-q}^{q} \hat{\psi}(k)e^{2\pi ikx},$$

for $x \in [0,1]$, where $\hat{\psi}(k) \in \mathbb{C}$, $k = -q, \ldots, 0, 1, \ldots, q$, are the Fourier coefficients for $\psi$.

The system $E = \{e_n :\ e_n(x) = e^{2\pi inx},\ n \in \mathbb{Z}\}$ is an orthonormal basis in the Hilbert space $L_2[0,1]$. Notice that $E \subset D(T)$ and for $n \in \mathbb{Z}$

$$Te_n(x) = \left((2n\pi)^2 - \hat{\psi}(0)\right) e_n(x) - \sum_{k=1}^{q} \left(\hat{\psi}(-k)e_{n-k}(x) + \hat{\psi}(k)e_{n+k}(x)\right).$$

Therefore, $T$ is unitary equivalent to the (2q+1)-diagonal operator

$$A = D_0 + \sum_{k=1}^{q}(S^k D_k + D_{-k}(S^*)^k), \tag{4.1}$$

where $D_0 = \mathrm{diag}((2n\pi)^2 - \hat{\psi}(0))_{n\in\mathbb{Z}}$, $D_k = -\hat{\psi}(-k)I$ and $D_{-k} = -\hat{\psi}(k)I$ for $k = 1, 2, \ldots$, are diagonal operators in $l_2(\mathbb{Z})$.

In [2], in the case of $q = 2$, authors proved that the spectrum of $T$ is discrete and there exist $K \in \mathbb{N}$ such that

$$\sigma(T) = \sigma_K \cup \{\mu_n :\ |n| > K\},$$

where $\sigma_K$ consists of no more then $2K+1$ eigenvalues and the asymptotic behaviour of the eigenvalues $\mu_n$, $|n|>K$, is given by the formula

$$\mu_n = (2\pi n)^2 - \hat{\psi}(0) + O\left(|n|^{-1}\right), \quad |n| \to \infty.$$

We observe that $\sigma(T)=\sigma(A)$ and $A$ defined here by (4.1) satisfies assumptions of Theorem 2.2, Theorem 2.3 and Theorem 3.3, so we conclude the spectrum of $T$ is discrete and $\sigma(T)=\Lambda(A)$.

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