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Matrices and operations on them - a short course

Abstract. This article provides basic information about matrices and operations on them. It allows students to get acquainted with the material required for the basic linear algebra course in a comprehensible way. Rather than focusing on providing formal definitions and theorems the emphasis is on understanding the subject matter and the ability to perform key matrix operations. Hence the theory presented is illustrated with many examples.
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1. Basic definitions

1.1. Matrices

Although the definition of a matrix is formally given in a more complicated way, (most often as a function of two variables), for common use it is enough to think of a matrix as a rectangular table (a table with no grid) composed of numbers or characters. In fact, this is a very common way of specifying matrices in elementary linear algebra textbooks such as in [3, 4].

For example:

$$\mathbf{A} = \begin{bmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2 & 4 & 6 & 8 \\ 9 & 7 & 5 & 3 \\ 11 & 21 & 31 & 41 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \alpha + 3x & 2.2 & \frac{1}{3} \\ \alpha & 1 & 0 \\ 0 & -3 & |3x| \end{bmatrix}. \tag{1}$$

Please note that some textbooks use round brackets to mark matrices:

$$\mathbf{A} = \begin{pmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 & 4 & 6 & 8 \\ 9 & 7 & 5 & 3 \\ 11 & 21 & 31 & 41 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} \alpha + 3x & 2.2 & \frac{1}{3} \\ \alpha & 1 & 0 \\ 0 & -3 & |3x| \end{pmatrix}.$$

We will stay with square brackets.

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1.2. Rows and columns

The elements of each matrix can be grouped into the **rows** – horizontally:

$$\left[\begin{array}{|cccc|} \hline 2 & 4 & 6 & 8 \\ \hline 9 & 7 & 5 & 3 \\ \hline 11 & 21 & 31 & 41 \\ \hline \end{array}\right]$$

or in **columns** – vertically:

$$\left[\begin{array}{|c|c|c|c|} \hline 2 & 4 & 6 & 8 \\ 9 & 7 & 5 & 3 \\ 11 & 21 & 31 & 41 \\ \hline \end{array}\right].$$

Due to this, each element of the matrix can be assigned a row number and a column number, which indicate where the element is placed in the matrix.
If

$$\mathbf{A} = \begin{bmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix},$$

then a_{21} element standing in the second row and the first column of the matrix **A** is 3.
For

$$\mathbf{B} = \begin{bmatrix} 2 & 4 & 6 & 8 \\ 9 & 7 & 5 & 3 \\ 11 & 21 & 31 & 41 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{bmatrix}$$

the element b_{34} is 41, because it is the element that stands in the 3rd row and 4th column of **B**.

1.3. Order (dimension)

The number of rows and the number of columns define the **order** of matrix (term dimension is also used).
The matrix **A** has 2 rows and 3 columns, so it is of order 2×3, which we write down $\mathbf{A}_{2\times 3}$.
The matrix **B** has 3 rows and 4 columns – what we write $\mathbf{B}_{3\times 4}$, and for the matrix **C** we have $\mathbf{C}_{3\times 3}$.

We say that **matrices are equal** if their dimensions are the same and the elements standing in the same positions are equal.

1.4. Zero matrix

The notation $[a_{ij}]_{n\times m}$ stands for a matrix of n-rows, m-columns and a_{ij} elements for $i = 1, 2, ...n$ and $j = 1, 2, ...m$:

$$[a_{ij}]_{n\times m} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix}.$$

The matrix, which is made up of zeros, regardless of the dimension, is called the **zero matrix** – often simply marked as **0**, for example:

$$\mathbf{0}=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix},\qquad \mathbf{0}=\begin{bmatrix}0&0\\0&0\end{bmatrix}.$$

Remark 1. Zero matrices of different order are not equal.

1.5. Square matrix

Matrices that have the same number of rows as columns are called **square matrices**, and the number of their rows (columns) is called **order** of the square matrix, e.g.:

$$\begin{bmatrix}x&y&z\\2x&3y&4z\\3&2&1\end{bmatrix}=\mathbf{D}_3 \qquad \text{– square matrix of order 3,}$$

$$\begin{bmatrix}1&0\\-2&5\end{bmatrix}=\mathbf{F}_2 \qquad \text{– square matrix of order 2.}$$

1.6. Diagonal matrix

Diagonal matrix is any square matrix in which different than zero elements only appear on the **main diagonal** (i.e. at the intersection of rows and columns with the same numbers). All other elements of the matrix are zeros, e.g.:

$$\begin{bmatrix}7&0&0&0&0\\0&0&0&0&0\\0&0&0.2&0&0\\0&0&0&-2&0\\0&0&0&0&11\end{bmatrix},\qquad \begin{bmatrix}a_{11}&0&0&0\\0&a_{22}&0&0\\0&0&a_{33}&0\\0&0&0&a_{44}\end{bmatrix},\qquad \begin{bmatrix}1&0&0\\0&12&0\\0&0&123\end{bmatrix}.$$

Remark 2. There can be an element equal to zero on the diagonal of the diagonal matrix.

1.7. Identity matrix

Diagonal matrices of the n degree, where all elements on the main diagonal are 1, are called **identity matrices** (sometimes – unit matrices) and are denoted as $\mathbf{I}_n$ (the designation $\mathbf{E}_n$ is also used) or simply $\mathbf{I}$:

$$\mathbf{I}_1=\begin{bmatrix}1\end{bmatrix},\qquad \mathbf{I}_2=\begin{bmatrix}1&0\\0&1\end{bmatrix},\qquad \mathbf{I}_3=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\qquad \mathbf{I}_4=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}.$$

1.8. Triangular matrix

Triangular matrix is a square matrix that has only zeros above or below the main diagonal. We then speak about the lower and upper-triangle matrices, respectively, e.g.,

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 3 & 2 & -5 \end{bmatrix}, \qquad \begin{bmatrix} \sin\alpha & 0 \\ \cos\alpha & 1 \end{bmatrix} \qquad \text{– lower triangular matrices,}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \qquad \begin{bmatrix} \pi & \sqrt{3} \\ 0 & 0 \end{bmatrix} \qquad \text{– upper triangular matrices.}$$

2. Transposition of the matrix

$$\mathbf{A}^T = ?$$

Transposed matrix is created from a given matrix by saving its rows as columns (and reverse) in order of elements. A matrix transposed to $\mathbf{A}$ is marked with $\mathbf{A}^T$, for example:

$$\text{If} \quad \mathbf{A} = \begin{bmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} \quad \text{then} \quad \mathbf{A}^T = \begin{bmatrix} 6 & 3 \\ 5 & 2 \\ 4 & 1 \end{bmatrix}.$$

The transposition of the matrix changes its order

$$\mathbf{A}_{2\times 3} \quad \to \quad \mathbf{A}^T_{3\times 2}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}.$$

3. Addition and subtraction of matrices

$$\mathbf{A} + \mathbf{B} = ? \qquad \mathbf{A} - \mathbf{B} = ?$$

Remark 3. Addition and subtraction matrices are not always possible.

For addition or subtraction to be possible, the matrices must have the same dimensions.

The result of adding (subtracting) the $\mathbf{A}$ and $\mathbf{B}$ matrices is a new matrix, with the same dimension as $\mathbf{A}$ and $\mathbf{B}$, and its elements are sums (differences) of the elements located in the appropriate positions of matrices **A** and **B**. For example, for

$$\mathbf{A} = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 1 & -2 \end{bmatrix}_{2\times 3} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & -4 & 3 \\ 2 & -1 & 0 \end{bmatrix}_{2\times 3}$$

we have

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \begin{bmatrix} -1+1 & 2+(-4) & 0+3 \\ 3+2 & 1+(-1) & -2+0 \end{bmatrix}_{2\times 3} = \begin{bmatrix} 0 & -2 & 3 \\ 5 & 0 & -2 \end{bmatrix}_{2\times 3},$$

$$\mathbf{D} = \mathbf{A} - \mathbf{B} = \begin{bmatrix} -1+1 & 2+(-4) & 0+3 \\ 3+2 & 1+(-1) & -2+0 \end{bmatrix}_{2\times 3} = \begin{bmatrix} -2 & 6 & -3 \\ 1 & 2 & -2 \end{bmatrix}_{2\times 3}.$$

Below are examples of operations that cannot be performed:

$$\mathbf{A} + \mathbf{B}^T = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 1 & -2 \end{bmatrix}_{2\times 3} + \begin{bmatrix} 1 & 2 \\ -4 & -1 \\ 3 & 0 \end{bmatrix}_{3\times 2} = ?,$$

$$\mathbf{A}^T - \mathbf{B} = \begin{bmatrix} -1 & 3 \\ 2 & 1 \\ 0 & -2 \end{bmatrix}_{3\times 2} - \begin{bmatrix} 1 & -4 & 3 \\ 2 & -1 & 0 \end{bmatrix}_{2\times 3} = ?.$$

If $\mathbf{A} = [a_{ij}]_{n\times m}$, $\mathbf{B} = [b_{ij}]_{n\times m}$ these addition and subtraction operations can be written down in a general way:

$$\mathbf{A} + \mathbf{B} = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}] = \mathbf{C} = [c_{ij}]_{n\times m},$$
$$\mathbf{A} - \mathbf{B} = [a_{ij}] - [b_{ij}] = [a_{ij} - b_{ij}] = \mathbf{D} = [d_{ij}]_{n\times m}.$$

The **neutral element of addition** and subtraction of the matrix is the zero matrix (of appropriate order), e.g.:

$$\mathbf{A} + \mathbf{0} = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 1 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 1 & -2 \end{bmatrix} = \mathbf{A},$$

$$\mathbf{A}^T - \mathbf{0} = \begin{bmatrix} -1 & 3 \\ 2 & 1 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 2 & 1 \\ 0 & -2 \end{bmatrix} = \mathbf{A}^T.$$

4. Multiplying the matrix by number

$$5 \cdot \mathbf{A} = ?$$

A matrix of any order can be multiplied by any real number. This operation consists of multiplying each element of a given matrix by a specified number, and as a result getting a new matrix of the same order, for example

$$-3\cdot\begin{bmatrix}\alpha+3x & 2.2 & \frac{1}{3}\\ \alpha & 1 & 0\\ 0 & -3 & |3x|\end{bmatrix}=\begin{bmatrix}-3\alpha-9x & -6.6 & -1\\ -3\alpha & -3 & 0\\ 0 & 9 & -9|x|\end{bmatrix}.$$

Generally for $\mathbf{A}=[a_{ij}]_{n\times m}$ we have:

$$\lambda\mathbf{A}=[\lambda a_{ij}]_{n\times m}=\begin{bmatrix}\lambda a_{11} & \lambda a_{12} & \dots & \lambda a_{1m}\\ \vdots & \vdots & \ddots & \vdots\\ \lambda a_{n1} & \lambda a_{n2} & \dots & \lambda a_{nm}\end{bmatrix}.$$

In multiplying the matrix by a number it is customary to write the number on the left- and the matrix on the right-hand side.

5. Multiplication of two matrices

$$\mathbf{A}\cdot\mathbf{B}=?$$

Remark 4. The multiplication of two matrices is not always possible.

It depends on their orders whether two matrices can be multiplied. For such operation to be possible, **the number of columns of the first** matrix must be the same as **the number of rows of the second** one.

The following example shows two matrices that can be multiplied if the correct order of matrices is maintained

$$\mathbf{A}=\begin{bmatrix}5 & 4\\ 3 & 2\\ 1 & 0\end{bmatrix}_{3\times 2},\qquad \mathbf{B}=\begin{bmatrix}0 & 6 & 8 & -4\\ -2 & 7 & 9 & 10\end{bmatrix}_{2\times 4}.$$

The result of the multiplication $\mathbf{A}\cdot\mathbf{B}$ is a new matrix $\mathbf{C}$ with as many rows as matrix $\mathbf{A}$ had and as many columns as matrix $\mathbf{B}$ had.

$$\mathbf{A}_{3\times 2}\cdot\mathbf{B}_{2\times 4}=\mathbf{C}_{3\times 4},$$

$$\begin{bmatrix}5 & 4\\ 3 & 2\\ 1 & 0\end{bmatrix}_{3\times 2}\cdot\begin{bmatrix}0 & 6 & 8 & -4\\ -2 & 7 & 9 & 10\end{bmatrix}_{2\times 4}=\begin{bmatrix}-8 & 58 & 76 & 20\\ -4 & 32 & 42 & 8\\ 0 & 6 & 8 & -4\end{bmatrix}_{3\times 4}.$$

How to calculate the matrix $\mathbf{C}$?

Element c_{11} (standing in the first row and first column) is calculated by multiplying the elements of the first row of the $\mathbf{A}$ matrix by the corresponding elements of the first column of the $\mathbf{B}$ matrix and adding the results obtained

$$\begin{bmatrix} \boxed{5} & \boxed{4} \\ 3 & 2 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \boxed{0} & 6 & 8 & -4 \\ \boxed{-2} & 7 & 9 & 10 \end{bmatrix} = \begin{bmatrix} \boxed{5 \cdot 0 + 4 \cdot (-2)} & . & . & . \\ . & . & . & . \\ . & . & . & . \end{bmatrix}.$$

Element c_{12} (first line, second column) is calculated by multiplying the first line of **A** by the second column of **B** and adding the results

$$\begin{bmatrix} \boxed{5} & \boxed{4} \\ 3 & 2 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & \boxed{6} & 8 & -4 \\ -2 & \boxed{7} & 9 & 10 \end{bmatrix} = \begin{bmatrix} -8 & \boxed{5 \cdot 6 + 4 \cdot 7} & . & . \\ . & . & . & . \\ . & . & . & . \end{bmatrix} \text{ and so on.}$$

For example, the c_{33} element is calculated by multiplying the third line of **A** by the third column of **B**

$$\begin{bmatrix} 5 & 4 \\ 3 & 2 \\ \boxed{1} & \boxed{0} \end{bmatrix} \cdot \begin{bmatrix} 0 & 6 & \boxed{8} & -4 \\ -2 & 7 & \boxed{9} & 10 \end{bmatrix} = \begin{bmatrix} -8 & 58 & . & . \\ . & . & . & . \\ . & . & \boxed{1 \cdot 8 + 0 \cdot 9} & . \end{bmatrix}.$$

In general terms, the c_{ij} element is calculated by multiplying the elements of the ith row of **A** by the corresponding elements of the jth column of **B**.

Finally, the multiplication $\mathbf{A} \cdot \mathbf{B}$ gives us:

$$\begin{bmatrix} 5 & 4 \\ 3 & 2 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 6 & 8 & -4 \\ -2 & 7 & 9 & 10 \end{bmatrix} =$$

$$= \begin{bmatrix} 5 \cdot 0 + 4 \cdot (-2) & 5 \cdot 6 + 4 \cdot 7 & 5 \cdot 8 + 4 \cdot 9 & 5 \cdot (-4) + 4 \cdot 10 \\ 3 \cdot 0 + 2 \cdot (-2) & 3 \cdot 6 + 2 \cdot 7 & 3 \cdot 8 + 2 \cdot 9 & 3 \cdot (-4) + 2 \cdot 10 \\ 1 \cdot 0 + 0 \cdot (-2) & 1 \cdot 6 + 0 \cdot 7 & 1 \cdot 8 + 0 \cdot 9 & 1 \cdot (-4) + 0 \cdot 10 \end{bmatrix}.$$

Let's note that multiplying the **A** and **B** in reverse order is not possible!
Multiplication $\mathbf{B}_{2\times\mathbf{4}} \cdot \mathbf{A}_{\mathbf{3}\times 2}$ is not possible because the number of columns in the first matrix and the number of rows in the second matrix are different!

This illustrates essential fact:

> The multiplication of the matrix is not commutative.

Therefore, it is very important to keep an eye on the order of multiplication of the matrix (both in the notation and in the calculations).

Even for square matrices, changing the order of multiplication usually leads to different results, e.g.:

$$\begin{bmatrix} 5 & 3 \\ 5 & 3 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix},$$

but

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 5 & 3 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The last example illustrates another "strange" property of the multiplication matrix, although the result of the multiplication is a zero matrix, none of the factors is a zero matrix.

It is also worth noting that you cannot reduce the equation $\mathbf{A} \cdot \mathbf{C} = \mathbf{B} \cdot \mathbf{C}$.
From the fact that $\mathbf{A} \cdot \mathbf{C} = \mathbf{B} \cdot \mathbf{C}$ it doesn't have to be that $\mathbf{A} = \mathbf{B}$, e.g.:

$$\mathbf{A} \cdot \mathbf{C} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix},$$

$$\mathbf{B} \cdot \mathbf{C} = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix},$$

that is $\mathbf{A} \cdot \mathbf{C} = \mathbf{B} \cdot \mathbf{C}$ although

$$\mathbf{A} \neq \mathbf{B} \quad !!$$

Remark 5. In general, a matrix multiplication is not commutative, but it is possible to find pairs of matrices for which the multiplication is commutative:

$$\begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} -1 & 6 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 6 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}.$$

The neutral element of multiplication of matrices is the identity matrix – the result of multiplication of a given matrix and the identity matrix of an appropriate order (such that multiplication is feasible) is the input matrix, e.g.:

$$\mathbf{A} \cdot \mathbf{I_2} = \begin{bmatrix} 5 & 4 \\ 3 & 2 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 3 & 2 \\ 1 & 0 \end{bmatrix} = \mathbf{A},$$

$$\mathbf{I_3} \cdot \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 & 4 \\ 3 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 3 & 2 \\ 1 & 0 \end{bmatrix} = \mathbf{A}.$$

For $\mathbf{A} = [a_{ij}]_{n\times p}$ i $\mathbf{B} = [b_{ij}]_{p\times m}$ the multiplication of matrices can be described by patterns:

$$\mathbf{A}\cdot\mathbf{B} = [a_{ij}]_{n\times p}\cdot[b_{ij}]_{p\times m} = [c_{ij}]_{n\times m},$$

where

$$\text{for all } i \ \in \{1,2,\ldots,n\}, j \in \{1,2,\ldots,m\}$$
$$c_{ij} = a_{i1}b_{ij} + a_{i2}b_{2j} + \ldots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}.$$

6. Properties of the matrix addition and multiplication

The addition and multiplication operations of the matrixes have the following properties:

- $\mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A}$ – commutative property of addition,
- $(\mathbf{A}+\mathbf{B})+\mathbf{C} = \mathbf{A}+(\mathbf{B}+\mathbf{C})$ – associative property of addition,
- $\mathbf{A}+\mathbf{0} = \mathbf{0}+\mathbf{A} = \mathbf{A}$, where $\mathbf{0}$ is a zero matrix of the appropriate order,
- $(\mathbf{A}\cdot\mathbf{B})\cdot\mathbf{C} = \mathbf{A}\cdot(\mathbf{B}\cdot\mathbf{C})$ – associative property of multiplication,
- $\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}$ – left distributive law,
- $(\mathbf{A}+\mathbf{B})\cdot\mathbf{C} = \mathbf{A}\cdot\mathbf{C}+\mathbf{B}\cdot\mathbf{C}$ – right distributive law,
- $\mathbf{A}\cdot\mathbf{I} = \mathbf{I}\cdot\mathbf{A} = \mathbf{A}$, where $\mathbf{I}$ is the identity matrix of the appropriate order.

7. Power of a square matrix

$$\mathbf{A}^n = ?$$

Raising the matrix to power is nothing more than multiplication of the matrix by itself, e.g.:

$$\mathbf{A}^2 = \mathbf{A}\cdot\mathbf{A}, \qquad \mathbf{A}^3 = \mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A}, \ \text{ and so on.}$$

It is not difficult to notice that in order to be able to perform a multiplication of the matrix by itself it must be square, thus it must have the same number of rows as the columns:

$$\mathbf{A}_{n\times n}\cdot\mathbf{A}_{n\times n} = \mathbf{A}^2.$$

For example:

$$\begin{bmatrix} -2 & 0 & 1 \\ 3 & 2 & 1 \\ 0 & -1 & 4 \end{bmatrix}^2 = \begin{bmatrix} -2 & 0 & 1 \\ 3 & 2 & 1 \\ 0 & -1 & 4 \end{bmatrix}\cdot\begin{bmatrix} -2 & 0 & 1 \\ 3 & 2 & 1 \\ 0 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & -1 & 2 \\ 0 & 3 & 9 \\ -3 & -6 & 15 \end{bmatrix},$$

$$\begin{bmatrix}-2&0&1\\3&2&1\\0&-1&4\end{bmatrix}^3=\begin{bmatrix}-2&0&1\\3&2&1\\0&-1&4\end{bmatrix}^2\cdot\begin{bmatrix}-2&0&1\\3&2&1\\0&-1&4\end{bmatrix}=$$

$$=\begin{bmatrix}4&-1&2\\0&3&9\\-3&-6&15\end{bmatrix}\cdot\begin{bmatrix}-2&0&1\\3&2&1\\0&-1&4\end{bmatrix}=\begin{bmatrix}-11&-4&11\\9&-3&39\\-12&-27&51\end{bmatrix}.$$

It is worthwhile to observe how the diagonal matrix is raised to a power

$$\begin{bmatrix}-2&0&0\\0&4&0\\0&0&3\end{bmatrix}^2=\begin{bmatrix}-2&0&0\\0&4&0\\0&0&3\end{bmatrix}\cdot\begin{bmatrix}-2&0&0\\0&4&0\\0&0&3\end{bmatrix}=\begin{bmatrix}4&0&0\\0&16&0\\0&0&9\end{bmatrix}.$$

Generally:

$$\begin{bmatrix}a_{11}&0&0&\cdots&0\\0&a_{22}&0&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&a_{nn}\end{bmatrix}^k=\begin{bmatrix}a_{11}^k&0&0&\cdots&0\\0&a_{22}^k&0&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&a_{nn}^k\end{bmatrix}. \tag{2}$$

8. Elementary Row Operations - ERO

Elementary Row Operations (in the following we will denote them **ERO**) play a very important role in determining the inverse of matrix, and in solving systems of linear equations. They can also be used in other calculations on matrices.

There are three Elementary Row Operations:

ERO 1 – the exchange of two rows of the matrix in places, e.g.:

$$\mathbf{A}=\begin{bmatrix}3&-2&1\\-1&0&3\\5&3&4\\0&1&-1\end{bmatrix}\overset{r_1\leftrightarrow r_2}{\sim}\begin{bmatrix}-1&0&3\\3&-2&1\\5&3&4\\0&1&-1\end{bmatrix}, \tag{3}$$

ERO 2 – multiplying the line of the matrix by a number different than zero, e.g.:

$$\begin{bmatrix}-1&0&3\\3&-2&1\\5&3&4\\0&1&-1\end{bmatrix}\overset{r_2\cdot 3}{\sim}\begin{bmatrix}-1&0&3\\9&-6&3\\5&3&4\\0&1&-1\end{bmatrix}, \tag{4}$$

ERO 3 – multiply a row in the matrix by a number and add it to another row, e.g.:

$$\begin{bmatrix} -1 & 0 & 3 \\ 3 & -2 & 1 \\ 5 & 3 & 4 \\ 0 & 1 & -1 \end{bmatrix} \overset{r_1 \cdot 3 + r_2}{\sim} \begin{bmatrix} -1 & 0 & 3 \\ 0 & -2 & 10 \\ 5 & 3 & 4 \\ 0 & 1 & -1 \end{bmatrix}. \tag{5}$$

The matrix that can be obtained from a given **A** matrix by using a certain number of ERO is called **the row equivalent matrix** to **A**, e.g.:

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 3 \\ 2 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 \\ 0 & 9 & 18 \\ 2 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

Any matrix can be transformed to row echelon form.

The given elementary operations can also be performed on columns of the matrix (we have Elementary Column Operations). The resulting matrix will be column equivalent. Unfortunately, ECO has far fewer uses, so we will only use ERO in our course.

Remark 6. The matrix obtained from a given **A** by using a certain number of ERO is not usually equal **A**.

9. Row Echelon Matrix and rank of matrix

ERO can be performed many times until a given matrix is brought to a matrix in **row echelon form**, i.e. a matrix that has the following attributes:

- all rows consisting entirely of zeros occur at the bottom of the matrix
- for two successive (nonzero) rows, the nonzero element in the higher row is farther to the left than the nonzero element in the lower row.

Examples of matrices in row echelon form:

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 7 & 14 & 4 & 5 \\ 0 & 2 & -1 & 0 & 2 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 9 \end{bmatrix}, \qquad \mathbf{A}_2 = \begin{bmatrix} 3 & -2 & 2 & 0 \\ 0 & 4 & -2 & -1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}. \tag{6}$$

A particular example of a matrix in row echelon form is a matrix in **reduced row echelon form**, whereby in addition:

- the first non-zero element in the row is equal to 1 (called a leading 1),
- the other elements in column with leading 1 are equal to zero.

The reduced row echelon form of a matrix is unique, unlike the row echelon form.

Examples matrices in reduced row echelon form are:

$$\mathbf{B}_1 = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 1 & 0 & -2 & 1 & 3 & -1 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{7}$$

The number of non-zero rows (rows in which not all elements are equal to zero) in a row echelon matrix is called **rank of matrix**.

The rank of matrix is very often defined in a more complicated way (using the so-called minors), but bringing a matrix to a row echelon matrix and counting its "steps" is one of the simplest methods of determining the rank.

The ranks of matrices in (6) and (7) are respectively:

$$R(\mathbf{A}_1) = 4; \qquad R(\mathbf{A}_2) = 4; \qquad R(\mathbf{B}_1) = 3; \qquad R(\mathbf{B}_2) = 4; \qquad R(\mathbf{B}_3) = 3.$$

The short course proposed above is based on an excerpt from an academic textbook [1]. This book describes examples of using matrix algebra in a wide variety of applications. It demonstrates why we need to know how to operate with matrices. More extensive material can be found in textbooks [2–5].

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10. Solved examples

Exercise 1. Given the following matrices

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 0 & 3 \\ -1 & 0 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} -3 & 1 & 2 \\ 1 & 0 & -1 \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} 0 & 1 & 2 \\ -1 & -3 & 1 \\ 2 & 2 & 0 \end{bmatrix}, \ \mathbf{D} = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix}$$

perform the indicated operation, if possible:

a) $\mathbf{A} + \mathbf{B}$, **b)** $\mathbf{A} - \mathbf{B}^T$, **c)** $\mathbf{C} + 3\mathbf{I}$,

d) $\mathbf{AB}$, **e)** $\mathbf{BA}$, **f)** $\mathbf{AC}$,

g) $\mathbf{BC}$, **h)** $\mathbf{CD}$, **i)** $\mathbf{DC}$.

Solution 1:

a) $\mathbf{A} + \mathbf{B} = \begin{bmatrix} 2 & -1 \\ 0 & 3 \\ -1 & 0 \end{bmatrix}_{3\times 2} + \begin{bmatrix} -3 & 1 & 2 \\ 1 & 0 & -1 \end{bmatrix}_{2\times 3}$ cannot be perform due to a mismatch of dimensions.

b) $\mathbf{A} - \mathbf{B}^T = \begin{bmatrix} 2 & -1 \\ 0 & 3 \\ -1 & 0 \end{bmatrix}_{3\times2} - \begin{bmatrix} -3 & 1 \\ 1 & 0 \\ 2 & -1 \end{bmatrix}_{3\times2} = \begin{bmatrix} 5 & -2 \\ -1 & 3 \\ -3 & 1 \end{bmatrix}_{3\times2}.$

c) Taking the matrix I as I_3 will allow the matrices' addition to be done.

$$\mathbf{C} + 3\mathbf{I}_3 = \begin{bmatrix} 0 & 1 & 2 \\ -1 & -3 & 1 \\ 2 & 2 & 0 \end{bmatrix}_{3\times3} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}_{3\times3} = \begin{bmatrix} -3 & 1 & 2 \\ -1 & -6 & 1 \\ 2 & 2 & -3 \end{bmatrix}_{3\times3}.$$

d) $\mathbf{AB} = \begin{bmatrix} 2 & -1 \\ 0 & 3 \\ -1 & 0 \end{bmatrix}_{3\times2} \cdot \begin{bmatrix} -3 & 1 & 2 \\ 1 & 0 & -1 \end{bmatrix}_{2\times3} = \begin{bmatrix} -7 & 2 & 5 \\ 3 & 0 & -3 \\ 3 & -1 & -2 \end{bmatrix}_{3\times3}.$

e) $\mathbf{BA} = \begin{bmatrix} -3 & 1 & 2 \\ 1 & 0 & -1 \end{bmatrix}_{2\times3} \cdot \begin{bmatrix} 2 & -1 \\ 0 & 3 \\ -1 & 0 \end{bmatrix}_{3\times2} = \begin{bmatrix} -8 & 6 \\ 3 & -1 \end{bmatrix}_{2\times2}.$

f) $\mathbf{AC} = \begin{bmatrix} 2 & -1 \\ 0 & 3 \\ -1 & 0 \end{bmatrix}_{3\times2} \cdot \begin{bmatrix} 0 & 1 & 2 \\ -1 & -3 & 1 \\ 2 & 2 & 0 \end{bmatrix}_{3\times3}$. Since the inner dimensions don't match, we can't do the multiplication.

g) $\mathbf{BC} = \begin{bmatrix} -3 & 1 & 2 \\ 1 & 0 & -1 \end{bmatrix}_{2\times3} \cdot \begin{bmatrix} 0 & 1 & 2 \\ -1 & -3 & 1 \\ 2 & 2 & 0 \end{bmatrix}_{3\times3} = \begin{bmatrix} 3 & -2 & -5 \\ -2 & -1 & 2 \end{bmatrix}_{2\times3}.$

h) $\mathbf{CD} = \begin{bmatrix} 0 & 1 & 2 \\ -1 & -3 & 1 \\ 2 & 2 & 0 \end{bmatrix}_{3\times3} \cdot \begin{bmatrix} 2 & 1 & 3 \end{bmatrix}_{1\times3}$. Since the inner dimensions don't match, we can't do the multiplication.

i) $\mathbf{DC} = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix}_{1\times3} \cdot \begin{bmatrix} 0 & 1 & 2 \\ -1 & -3 & 1 \\ 2 & 2 & 0 \end{bmatrix}_{3\times3} = \begin{bmatrix} 5 & 5 & 5 \end{bmatrix}_{1\times3}.$

Exercise 2. Given the matrices $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -1 & 0 & -2 \\ 1 & 2 & 3 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 2 & -1 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} 1 & 5 \\ -5 & 0 \\ 2 & 1 \end{bmatrix}$, calculate if possible:

a) $\mathbf{AI} + \mathbf{D}^T$, **b)** $\mathbf{AC}^T$, **c)** $\mathbf{DA} + \mathbf{AD}$,

d) $\mathbf{DD}^T$, **e)** $\mathbf{CC}^T + \mathbf{C}^T\mathbf{C}$, **f)** $\mathbf{A}^2 - 3\mathbf{I}$.

Solution 2:

a) $\mathbf{AI} = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}_{2\times2} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2} = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}_{2\times2} = \mathbf{A}$, because $\mathbf{I_2}$ is the identity matrix.

$\mathbf{D}^T = \begin{bmatrix} 1 & -5 & 2 \\ 5 & 0 & 1 \end{bmatrix}_{2\times3}$. The addition $\mathbf{A}_{2\times2} + \mathbf{D}^T_{2\times3}$ cannot be done due to the different dimensions of the matrices.

b) $\mathbf{AC}^T = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}_{2\times 2} \begin{bmatrix} 2 \\ -1 \end{bmatrix}_{2\times 2} = \begin{bmatrix} 2\cdot 2 + 3\cdot(-1) \\ (-1)\cdot 2 + 0\cdot(-1) \end{bmatrix}_{2\times 1} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}_{2\times 1}.$

c) $\mathbf{DA} = \begin{bmatrix} 1 & 5 \\ -5 & 0 \\ 2 & 1 \end{bmatrix}_{3\times 2} \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}_{2\times 2} = \begin{bmatrix} 1\cdot 2 + 5\cdot(-1) & 1\cdot 3 + 5\cdot 0 \\ -5\cdot 2 + 0\cdot(-1) & -5\cdot 3 + 0\cdot 0 \\ 2\cdot 2 + 1\cdot(-1) & 2\cdot 3 + 1\cdot 0 \end{bmatrix}_{3\times 2} = \begin{bmatrix} -3 & 3 \\ -10 & -15 \\ 3 & 6 \end{bmatrix}_{3\times 2}.$

$\mathbf{AD} = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}_{2\times 2} \begin{bmatrix} 1 & 5 \\ -5 & 0 \\ 2 & 1 \end{bmatrix}_{3\times 2}$ – multiplication cannot be performed because the number of columns and rows do not match.

d) $\mathbf{DD}^T = \begin{bmatrix} 1 & 5 \\ -5 & 0 \\ 2 & 1 \end{bmatrix}_{3\times 2} \begin{bmatrix} 1 & -5 & 2 \\ 5 & 0 & 1 \end{bmatrix}_{2\times 3} = \begin{bmatrix} 1\cdot 1 + 5\cdot 5 & 1\cdot(-5) + 5\cdot 0 & 1\cdot 2 + 5\cdot 1 \\ -5\cdot 1 + 0\cdot 5 & -5\cdot(-5) + 0\cdot 0 & -5\cdot 2 + 0\cdot 1 \\ 2\cdot 1 + 1\cdot 5 & 2\cdot(-5) + 1\cdot 0 & 2\cdot 2 + 1\cdot 1 \end{bmatrix}_{3\times 3} =$

$= \begin{bmatrix} 26 & -5 & 7 \\ -5 & 25 & -10 \\ 7 & -10 & 5 \end{bmatrix}_{3\times 3}.$

e) $\mathbf{CC}^T = \begin{bmatrix} 2 & -1 \end{bmatrix}_{1\times 2} \begin{bmatrix} 2 \\ -1 \end{bmatrix}_{2\times 1} = \begin{bmatrix} 2\cdot 2 + (-1)\cdot(-1) \end{bmatrix}_{1\times 1} = \begin{bmatrix} 5 \end{bmatrix}_{1\times 1}.$

$\mathbf{C}^T\mathbf{C} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}_{2\times 1} \begin{bmatrix} 2 & -1 \end{bmatrix}_{1\times 2} = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}_{2\times 2}.$

The addition $\mathbf{CC}^T + \mathbf{C}^T\mathbf{C}$ cannot be done due to the different dimensions of the matrices.

f) $\mathbf{A}^2 - 3\mathbf{I} = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}_{2\times 2} \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}_{2\times 2} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2} = \begin{bmatrix} 1 & 6 \\ -2 & -3 \end{bmatrix}_{2\times 2} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}_{2\times 2} = \begin{bmatrix} -2 & 6 \\ -2 & -6 \end{bmatrix}_{2\times 2}.$

References

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