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PROPERTIES OF EVEN ORDER LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS OF MIXED TYPE

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Abstract. This paper is concerned with oscillatory behavior of linear functional differential equations of the type

$$y^{(n)}(t) = p(t)y(\tau(t))$$

with mixed deviating arguments which means that its both delayed and advanced parts are unbounded subset of $(0, \infty)$. Our attention is oriented to the Euler type of equation, i.e. when $p(t) \sim a/t^n$.

Keywords: higher order differential equations, mixed argument, monotonic properties, oscillation.

Mathematics Subject Classification: 34K11, 34C10.

1. INTRODUCTION

We consider linear functional differential equations

$$y^{(n)}(t) = p(t)y(\tau(t)). \tag{E}$$

In this paper we assume that

(H_1) $p \in C([t_0, \infty))$, $p(t) > 0$, n is even,
(H_2) $\tau(t) \in C^1([t_0, \infty))$, $\tau'(t) > 0$, $\lim_{t\to\infty} \tau(t) = \infty$.

By a proper solution of Eq. (E) we mean a function $y : [T_y, \infty) \to \mathbb{R}$ which satisfies (E) for all sufficiently large t and $\sup\{|y(t)| : t \geq T\} > 0$ for all $T \geq T_y$. We make the standing hypothesis that (E) does possess proper solutions. A proper solution is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory. An equation itself is said to be oscillatory if all its proper solutions are oscillatory.

We recall some basic facts from the oscillation theory of (E). If $y(t)$ is a nonoscillatory solution of (E), then there exist an even integer $\ell \in \{0, 2, \dots, n\}$ and a $t_0 \ge T_y$ such that

$$\begin{aligned} y(t)y^{(i)}(t) > 0 \quad \text{on } [t_0, \infty) \text{ for } 0 \le i \le \ell, \\ (-1)^i y(t)y^{(i)}(t) > 0 \quad \text{on } [t_0, \infty) \text{ for } \ell \le i \le n. \end{aligned} \tag{1.1}$$

Such an $y(t)$ is said to be a solution of degree ℓ, and the set of all solutions of degree ℓ is denoted by $\mathcal{N}_\ell$. If we denote the set of all nonoscillatory solutions of (E) by $\mathcal{N}$, then we have

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2 \cup \ldots \cup \mathcal{N}_n.$$

During the past decades authors investigated the particular situation, namely when

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_n. \tag{1.2}$$

It is caused by the fact that if $\tau(t) \equiv t$, then (E) always has solutions of degree 0 and n, that is $\mathcal{N}_0 \ne \emptyset$ and $\mathcal{N}_n \ne \emptyset$ for ordinary differential equation

$$y^{(n)}(t) = p(t)y(t).$$

The situation when (1.2) holds is referred as Property B of (E).

The status for $\tau(t) \not\equiv t$ is different. In fact, it may happen that

$$\mathcal{N} = \mathcal{N}_0 \quad \text{or} \quad \mathcal{N} = \mathcal{N}_n$$

for (E) when the deviating argument is delayed $(\tau(t) < t)$ or advanced $(\tau(t) > t)$ and the deviation $|t - \tau(t)|$ is large enough, see e.g. [1–9].

The last step in the oscillation theory of (E) is the case when $\mathcal{N} = \emptyset$, that is, all proper solutions of (E) are oscillatory. Really, this may occur for functional differential equation involving both delay and advanced arguments of the form

$$y^{(n)}(t) = p_1(t)y(\tau_1(t)) + p_2(t)y(\tau_2(t))$$

or for equation (E) with $\tau(t)$ being of mixed type which means that its delayed part

$$\mathcal{D}_\tau = \{t \in (t_0, \infty) : \tau(t) < t\}$$

and advanced part

$$\mathcal{A}_\tau = \{t \in (t_0, \infty) : \tau(t) > t\}$$

are both unbounded subset of (t_0, ∞). Kusano [8] was the first who remarked it. To recall his result let us denote

$$\tau_*(t) = \min\{\tau(t), t\}.$$

We assume that there are two sequences $\{t_k\}$, $\{s_k\}$ such that

$$t_k \in \mathcal{D}_\tau, \quad t_k \to \infty \quad \text{as} \quad k \to \infty \tag{1.3}$$

and

$$s_k \in \mathcal{A}_\tau, \quad s_k \to \infty \quad \text{as} \quad k \to \infty. \tag{1.4}$$

Theorem 1.1. *Assume that there is a constant* $\varepsilon > 0$ *such that*

$$\int^{\infty} \big(\tau_*(t)\big)^{n-1-\varepsilon} p(t)\,\mathrm{d}t = \infty. \tag{1.5}$$

Suppose moreover that there exist two sequences $\{t_k\}$, $\{s_k\}$ *satisfying* (1.3) *and* (1.4). *If*

$$\limsup_{k\to\infty} \int_{\tau(t_k)}^{t_k} \big(\tau(t_k) - \tau(s)\big)^{n-1} p(s)\,\mathrm{d}s > (n-1)! \tag{1.6}$$

and

$$\limsup_{k\to\infty} \int_{s_k}^{\tau(s_k)} \big(\tau(t) - \tau(s_k)\big)^{n-1} p(t)\,\mathrm{d}t > (n-1)!, \tag{1.7}$$

then (E) *is oscillatory.*

The sense of conditions (1.5), (1.6) and (1.7) is to show that $\mathcal{N}_\ell = \emptyset$, $\ell = 2, 4, \dots, n-2$, $\mathcal{N}_0 = \emptyset$ and $\mathcal{N}_n = \emptyset$, respectively. It useful to notice that the condition (1.5) is such limiting that it cannot be applied to Euler type of equations when $p(t) \sim a/t^n$. In this paper, employing new nonotonicity of possible nonoscillatory solutions, we will be able to essentially improve all conditions (1.5)–(1.7).

Kusano's results have been improved by Baculikova [4], where introducing constants

$$\beta_0 = \min \frac{(t-\tau(t))^{n-1}}{(n-1)!} p(t), \quad t \in [\tau(\tau(t_k)), \tau(t_k)], \quad k = 1, 2, \dots,$$
$$\gamma_0 = \min \frac{(\tau(t)-t)^{n-1}}{(n-1)!} p(t), \quad t \in [\tau(s_k), \tau(\tau(s_k))] \quad k = 1, 2, \dots$$

and employing the new monotonicities

$$\mathrm{e}^{\beta_0 t} \Big(-y^{(n-1)}(t)\Big) \downarrow \quad \text{for } y \in \mathcal{N}_0,$$
$$\mathrm{e}^{-\gamma_0 t} \Big(y^{(n-1)}(t)\Big) \uparrow \quad \text{for } y \in \mathcal{N}_n.$$

earlier conditions (1.6) and (1.7) have been improved to

$$\limsup_{k\to\infty} \mathrm{e}^{\beta_0 \tau(t_k)} \int_{\tau(t_k)}^{t_k} p(s) \int_{\tau(s)}^{\tau(t_k)} \mathrm{e}^{-\beta_0 x} \big(x - \tau(s)\big)^{n-2}\,\mathrm{d}x\mathrm{d}s > (n-2)!$$

and

$$\limsup_{k\to\infty} \mathrm{e}^{-\gamma_0 \tau(s_k)} \int_{s_k}^{\tau(s_k)} p(t) \int_{\tau(s_k)}^{\tau(t)} \mathrm{e}^{\gamma_0 x} \big(\tau(t) - x\big)^{n-2}\,\mathrm{d}x\mathrm{d}t > (n-2)!,$$

respectively, and the progress has been demonstrated via

$$y^{(n)}(t) = p_0 y(t + \sin t). \tag{1.8}$$

Our criteria here use different monotonicities, namely

$$\begin{aligned} \tau^{\beta}(t) y(\tau(t)) \downarrow \quad &\text{for } \ y \in \mathcal{N}_0, \\ \tau^{-\gamma}(t) y(\tau(t)) \uparrow \quad &\text{for } \ y \in \mathcal{N}_n, \end{aligned}$$

and as an illustrative example the results will be applied to the Euler differential equation

$$y^{(n)}(t) = \frac{a}{t^n} y\big(t(1 + 0.5 \sin(\ln t))\big).$$

2. MAIN RESULTS

Our first considerations are intended to replace (1.5) with a couple of weaker conditions to conclude that $\mathcal{N}_\ell = \emptyset$, $\ell = 2, 4, \ldots, n-2$.

Theorem 2.1. *Assume that*

$$\int^{\infty} \big(\tau_*(t)\big)^{n-1} p(t) \, \mathrm{d}t = \infty \tag{2.1}$$

and

$$\begin{aligned} \limsup_{t \to \infty} \Bigg\{ \int\limits_{\tau_*(t)}^{t} \big(\tau_*(s)\big)^{n-1} p(s) \, \mathrm{d}s + \tau_*(t) \int\limits_{t}^{\infty} \big(\tau_*(s)\big)^{n-2} p(s) \, \mathrm{d}s \\ + \frac{1}{\tau_*(t)} \int\limits_{t_0}^{\tau_*(t)} \big(\tau_*(s)\big)^{n} p(s) \, \mathrm{d}s \Bigg\} > 2(n-2)!. \end{aligned} \tag{2.2}$$

Then for (E) *the classes* $\mathcal{N}_\ell = \emptyset$ *for all* $\ell = 2, 4, \ldots, n-2$.

Proof. Assume on the contrary that (E) possesses an eventually positive solution $y(t)$ of degree ℓ for some $\ell \in \{2, 4, \ldots, n-2\}$. Since $y^{(\ell-1)}(t)$ is positive and increasing, it easy to see that

$$y^{(\ell-2)}(t) \geq \int\limits_{t_1}^{t} y^{(\ell-1)}(s) \, \mathrm{d}s \geq y(t_1)(t - t_1)$$

Integrating $(\ell-2)$-times the last inequality, we are led to

$$y(t) \geq y(t_1) \frac{(t - t_1)^{\ell-1}}{(\ell-1)!} \geq c t^{\ell-1}, \quad c = \frac{y(t_1)}{2(\ell-1)!}.$$

Taking into account (2.1), one gets

$$\int^{\infty} t^{n-\ell} y^{(n)}(t)\,\mathrm{d}t = \int^{\infty} t^{n-\ell} p(t) y(\tau(t))\,\mathrm{d}t$$
$$\geq c \int^{\infty} t^{n-\ell} p(t) \tau^{\ell-1}(t)\,\mathrm{d}t$$
$$\geq \int^{\infty} \big(\tau_*(t)\big)^{n-1} p(t)\,\mathrm{d}t = \infty.$$

By Lemma 3.2 in [5], condition $\int^{\infty} t^{n-\ell} y^{(n)}(t)\,\mathrm{d}t = \infty$ guarantees that

$$\frac{y(t)}{t^{\ell}} \downarrow, \quad \frac{y(t)}{t^{\ell-1}} \uparrow \tag{2.3}$$

and

$$y(t) \geq \frac{t^{\ell}}{\ell!(n-\ell)!} \int_t^{\infty} s^{n-\ell-1} p(s) y(\tau(s))\,\mathrm{d}s + \frac{t^{\ell-1}}{\ell!(n-\ell)!} \int_{t_0}^{t} s^{n-\ell} p(s) y(\tau(s))\,\mathrm{d}s.$$

Since $y(t)$ is increasing, we see that $y(\tau(s)) \geq y(\tau_*(s))$ which in view of the above inequality yields

$$y(\tau_*(t)) \geq \frac{1}{2(n-2)!} \Bigg\{ \big(\tau_*(t)\big)^{\ell} \int_{\tau_*(t)}^{t} [\tau_*(s)]^{n-1} p(s) \frac{y(\tau_*(s))}{\big(\tau_*(s)\big)^{\ell}}\,\mathrm{d}s$$
$$+ \big(\tau_*(t)\big)^{\ell} \int_t^{\infty} \big(\tau_*(s)\big)^{n-2} p(s) \frac{y(\tau_*(s))}{\big(\tau_*(s)\big)^{\ell-1}}\,\mathrm{d}s$$
$$+ \big(\tau_*(t)\big)^{\ell-1} \int_{t_0}^{\tau_*(t)} \big(\tau_*(s)\big)^{n} p(s) \frac{y(\tau_*(s))}{\big(\tau_*(s)\big)^{\ell}}\,\mathrm{d}s \Bigg\}$$

Taking into account (2.3), we obtain

$$y(\tau_*(t)) \geq \frac{y(\tau_*(t))}{2(n-2)!} \Bigg\{ \int_{\tau_*(t)}^{t} \big(\tau_*(s)\big)^{n-1} p(s)\,\mathrm{d}s + \tau_*(t) \int_t^{\infty} \big(\tau_*(s)\big)^{n-2} p(s)\,\mathrm{d}s$$
$$+ \frac{1}{\tau_*(t)} \int_{t_0}^{\tau_*(t)} \big(\tau_*(s)\big)^{n} p(s)\,\mathrm{d}s \Bigg\}$$

which contradicts (2.2). □

Now we turn our attention for establishing conditions under which the class $\mathcal{N}_0 = \emptyset$.

Lemma 2.2. *Let there exist sequence* $\{t_k\}$ *satisfying* (1.3) *and* $\beta \geq 0$ *such that*

$$\tau(t) \int_{\tau(t)}^{t} p(s) \frac{(s-\tau(t))^{n-2}}{(n-2)!} \, \mathrm{d}s \geq \beta, \quad \textit{for} \quad t \in [\tau(t_k), t_k], \; k = 1, 2, \dots \tag{2.4}$$

Then every eventually positive solution $y(t) \in \mathcal{N}_0$ *satisfies*

$$\tau^{\beta}(t) y(\tau(t)) \downarrow \quad \textit{for} \quad t \in [\tau(t_k), t_k], \; k = 1, 2, \dots \tag{2.5}$$

Proof. Assume that $y(t) \in \mathcal{N}_0$ is an eventually positive solution of (E). Then (1.1) holds true with $\ell = 0$. We choose u, v such that $u < v \leq t_k$. It follows from (E) that

$$-y^{(n-1)}(u) \geq \int_u^v p(s) y(\tau(s)) \, \mathrm{d}s.$$

Integrating again in u over $[u, v]$ one gets

$$y^{(n-2)}(u) \geq \int_u^v \int_z^v p(s) y(\tau(s)) \, \mathrm{d}s \mathrm{d}z = \int_u^v p(s) y(\tau(s))(s-u) \, \mathrm{d}s.$$

Successive integration yields

$$-y'(u) \geq \int_u^v p(s) y(\tau(s)) \frac{(s-u)^{n-2}}{(n-2)!} \, \mathrm{d}s. \tag{2.6}$$

Setting $u = \tau(t)$ and $v = t$ we are led in view of (2.4) to

$$\begin{aligned} -\tau(t) y'(\tau(t)) &\geq \tau(t) \int_{\tau(t)}^{t} p(s) y(\tau(s)) \frac{(s-\tau(t))^{n-2}}{(n-2)!} \, \mathrm{d}s \\ &\geq y(\tau(t)) \tau(t) \int_{\tau(t)}^{t} p(s) \frac{(s-\tau(t))^{n-2}}{(n-2)!} \, \mathrm{d}s \geq \beta y(\tau(t)) \end{aligned}$$

for $t \in [\tau(t_k), t_k] \subset \mathcal{D}_\tau$, $k = 1, 2, \dots$ Consequently,

$$\left(\tau^{\beta}(t) y(\tau(t))\right)' \leq 0, \quad \text{for} \quad t \in [\tau(t_k), t_k], \; k = 1, 2, \dots,$$

and the proof is complete. □

Theorem 2.3. *Let there exist sequence* $\{t_k\}$ *satisfying* (1.3) *and* $\beta \ge 0$ *such that* (2.4) *holds. If*

$$\limsup_{k\to\infty} \tau^{\beta}(t_k) \int_{\tau(t_k)}^{t_k} p(s)\tau^{-\beta}(s)\big(s-\tau(t_k)\big)^{n-1}\,\mathrm{d}s > (n-1)!, \tag{2.7}$$

then $\mathcal{N}_0 = \emptyset$ *for* (E).

Proof. Assume on the contrary that (E) has an eventually positive solution $y(t)$ of degree 0. Then an integration of (2.6) yields to

$$y(u) \ge \int_u^v p(s)y(\tau(s))\frac{(s-u)^{n-1}}{(n-1)!}\,\mathrm{d}s.$$

Putting $u = \tau(t_k)$ and $v = t_k$ we obtain

$$\begin{aligned} y(\tau(t_k)) &\ge \int_{\tau(t_k)}^{t_k} p(s)\tau^{\beta}(s)y(\tau(s))\tau^{-\beta}(s)\frac{(s-\tau(t_k))^{n-1}}{(n-1)!}\,\mathrm{d}s \\ &\ge y(\tau(t_k))\tau^{\beta}(t_k)\int_{\tau(t_k)}^{t_k} p(s)\tau^{-\beta}(s)\frac{(s-\tau(t_k))^{n-1}}{(n-1)!}\,\mathrm{d}s \end{aligned}$$

This contradicts (2.7) and the proof is complete. □

For $\beta = 0$ the above mentioned theorem reduces to the following form:

Corollary 2.4. *Let there exist sequence* $\{t_k\}$ *satisfying* (1.3). *If*

$$\limsup_{k\to\infty} \int_{\tau(t_k)}^{t_k} p(s)\big(s-\tau(t_k)\big)^{n-1}\,\mathrm{d}s > (n-1)!, \tag{2.8}$$

then $\mathcal{N}_0 = \emptyset$ *for* (E).

Now we present criterion for elimination of the class $\mathcal{N}_n$.

Lemma 2.5. *Let there exist sequence* $\{s_k\}$ *satisfying* (1.4) *and* $\gamma \ge 0$ *such that*

$$\tau(t)\int_t^{\tau(t)} p(s)\frac{(\tau(t)-s)^{n-2}}{(n-2)!}\,\mathrm{d}s \ge \gamma, \quad \textit{for} \quad t \in [s_k, \tau(s_k)],\ k = 1, 2, \dots \tag{2.9}$$

Then every eventually positive solution $y(t) \in \mathcal{N}_n$ *satisfies*

$$\tau^{-\gamma}(t)y(\tau(t)) \uparrow \quad \textit{for} \quad t \in [s_k, \tau(s_k)],\ k = 1, 2, \dots \tag{2.10}$$

Proof. Suppose that $y(t) \in \mathcal{N}_n$ is an eventually positive solution of (E). Then (1.1) is satisfied with with $\ell = n$. We choose u, v such that $s_k \le u < v$. Then (E) implies that

$$y^{(n-1)}(v) \ge \int_u^v p(s)y(\tau(s))\,\mathrm{d}s.$$

Integrating again in v over $[u, v]$ we obtain

$$y^{(n-2)}(v) \ge \int_u^v \int_u^z p(s)y(\tau(s))\,\mathrm{d}s\mathrm{d}z = \int_u^v p(s)y(\tau(s))(v-s)\,\mathrm{d}s.$$

Repeating this procedure we are led to

$$y'(v) \ge \int_u^v p(s)y(\tau(s))\frac{(v-s)^{n-2}}{(n-2)!}\,\mathrm{d}s. \tag{2.11}$$

Employing (2.9) and letting $u = t$ and $v = \tau(t)$ we see that

$$\begin{aligned}\tau(t)y'(\tau(t)) &\ge \tau(t)\int_t^{\tau(t)} p(s)y(\tau(s))\frac{(\tau(t)-s)^{n-2}}{(n-2)!}\,\mathrm{d}s \\ &\ge y(\tau(t))\tau(t)\int_t^{\tau(t)} p(s)\frac{(\tau(t)-s)^{n-2}}{(n-2)!}\,\mathrm{d}s \ge \gamma y(\tau(t))\end{aligned}$$

for $t \in [s_k, \tau(s_k)] \subset \mathcal{A}_\tau$, $k = 1, 2, \ldots$ Therefore

$$\big(\tau^{-\gamma}(t)y(\tau(t))\big)' \ge 0, \quad \text{for} \quad t \in [s_k, \tau(s_k)],\ k = 1, 2, \ldots,$$

and the proof is complete. □

Theorem 2.6. *Let there exist sequence* $\{s_k\}$ *satisfying* (1.4) *and* $\gamma \ge 0$ *such that* (2.9) *holds. If*

$$\limsup_{k\to\infty} \tau^{-\gamma}(s_k)\int_{s_k}^{\tau(s_k)} p(s)\tau^{\gamma}(s)\big(\tau(s_k)-s\big)^{n-1}\,\mathrm{d}s > (n-1)!, \tag{2.12}$$

then $\mathcal{N}_n = \emptyset$ *for* (E).

Proof. Assume on the contrary that (E) possesses an eventually positive solution $y(t)$ of degree n. Then an integration of (2.11) yields to

$$y(v) \ge \int_u^v p(s)y(\tau(s))\frac{(v-s)^{n-1}}{(n-1)!}\,\mathrm{d}s.$$

If we set $u = s_k$ and $v = \tau(s_k)$, then we get

$$y(\tau(s_k)) \geq \int\limits_{s_k}^{\tau(s_k)} p(s)\tau^{-\gamma}(s)y(\tau(s))\tau^{\gamma}(s)\frac{(\tau(s_k)-s)^{n-1}}{(n-1)!}\,\mathrm{d}s$$
$$\geq y(\tau(s_k))\tau^{-\gamma}(s_k)\int\limits_{s_k}^{\tau(s_k)} p(s)\tau^{\gamma}(s)\frac{(\tau(s_k)-s)^{n-1}}{(n-1)!}\,\mathrm{d}s.$$

This contradicts (2.12) and the proof is complete. □

For $\gamma = 0$ the above mentioned result simplifies to the following corollary.

Corollary 2.7. *Let there exist sequence* $\{s_k\}$ *satisfying* (1.4). *If*

$$\limsup_{k\to\infty}\int\limits_{s_k}^{\tau(s_k)} p(s)\big(\tau(s_k)-s\big)^{n-1}\,\mathrm{d}s > (n-1)!, \tag{2.13}$$

then $\mathcal{N}_n = \emptyset$ *for* (E).

If we pick up the previous results, we immediately obtain criterion for oscillation of (E).

Theorem 2.8. *Assume that all conditions of Theorems 2.1, 2.3 and 2.6 hold true. Then* (E) *is oscillatory.*

We support novelty of the paper with the following illustrative example.

Example 2.9. Consider the equation

$$y^{(n)}(t) = \frac{a}{t^n}y\big(t(1+0.5\sin(\ln t))\big), \tag{E_x}$$

where $a > 0$ is a constant. Clearly, the deviating argument $\tau(t)$ is of mixed type, $0.5t \leq \tau(t) \leq 1.5t$ and $\tau_*(t) \geq 0.5t$.

It is easy to see that condition (2.1) holds true and (2.2) reduces to

$$\lim_{t\to\infty}\left\{\int\limits_{t/2}^{t}\left(\frac{s}{2}\right)^{n-1}\frac{a}{s^n}\,\mathrm{d}s + \frac{t}{2}\int\limits_{t}^{\infty}\left(\frac{s}{2}\right)^{n-2}\frac{a}{s^n}\,\mathrm{d}s + \frac{2}{t}\int\limits_{t_1}^{t/2}\left(\frac{s}{2}\right)^{n}\frac{a}{s^n}\,\mathrm{d}s\right\}$$
$$> 2(n-2)!,$$

that is,

$$a > \frac{2^n(n-2)!}{\ln 2 + 1.5} \tag{2.14}$$

and this condition guarantees that $\mathcal{N}_\ell = \emptyset$ for (E_x) for all $\ell = 2, 4, \ldots, n-2$.

On the other hand, it is easy to see that if $t_k = \mathrm{e}^{-\pi/2+2k\pi}$, $k = 1, 2, \ldots$, then $t_k \in \mathcal{D}_\tau$ and $\tau(t_k) = t_k/2$. A simple computation shows that for $t \in [\tau(t_k), t_k]$

$$\begin{aligned}
&\tau(t) \int_{\tau(t)}^{t} \frac{a}{s^n} \frac{(s-\tau(t))^{n-2}}{(n-2)!}\,\mathrm{d}s = \frac{a\tau(t)}{(n-2)!} \int_{\tau(t)}^{t} \frac{1}{s^2} \left(\frac{s-\tau(t)}{s}\right)^{n-2} \mathrm{d}s \\
&= \frac{a}{(n-1)!} \left(1 - \frac{\tau(t)}{t}\right)^{n-1} = \frac{a}{(n-1)!} \left(-0.5\sin(\ln t)\right)^{n-1} \\
&\geq \frac{a}{(n-1)!} \left(-0.5\sin(\ln \tau(t_k))\right)^{n-1} = \frac{a}{(n-1)!} \left(0.5\cos(\ln 2)\right)^{n-1} = \beta.
\end{aligned}$$

Establishing constant β we can recall criterion (2.7):

$$\lim_{k\to\infty} (t_k/2)^\beta \int_{0.5t_k}^{t_k} \frac{a}{s^n} \big(s(1+0.5\sin(\ln s))\big)^{-\beta} \big(s-0.5t_k\big)^{n-1}\,\mathrm{d}s > (n-1)!$$

Substitution $s = t_k\, t$, transforms it to

$$a\, 0.5^\beta \int_{0.5}^{1} t^{-n-\beta} \big(1-0.5\cos(\ln t)\big)^{-\beta} (t-0.5)^{n-1}\,\mathrm{d}t > (n-1)!, \tag{2.15}$$

which guarantees that $\mathcal{N}_0 = \emptyset$.

Now, if we set $s_k = \mathrm{e}^{\pi/2+2k\pi}$, $k = 1, 2, \ldots$, then $s_k \in \mathcal{A}_\tau$ and $\tau(s_k) = 3s_k/2$. It is easy to verify that for $t \in [s_k, \tau(s_k)]$

$$\begin{aligned}
&\tau(t) \int_{t}^{\tau(t)} \frac{a}{s^n} \frac{(\tau(t)-s)^{n-2}}{(n-2)!}\,\mathrm{d}s = \frac{a\tau(t)}{(n-2)!} \int_{t}^{\tau(t)} \frac{1}{s^2} \left(\frac{\tau(t)-s}{s}\right)^{n-2} \mathrm{d}s \\
&= \frac{a}{(n-1)!} \left(\frac{\tau(t)}{t} - 1\right)^{n-1} = \frac{a}{(n-1)!} \left(0.5\sin(\ln t)\right)^{n-1} \\
&\geq \frac{a}{(n-1)!} \left(0.5\sin(\ln \tau(s_k))\right)^{n-1} = \frac{a}{(n-1)!} \left(\frac{\cos(\ln 1.5)}{2}\right)^{n-1} = \gamma
\end{aligned}$$

and so criterion (2.12) takes the form

$$\lim_{k\to\infty} (1.5s_k)^{-\gamma} \int_{s_k}^{3s_k/2} \frac{a}{s^n} \big(s(1+0.5\sin(\ln s))\big)^{\gamma} \big(1.5s_k - s\big)^{n-1}\,\mathrm{d}s > (n-1)!.$$

Applying the substitution $s = s_k\, t$, this transforms to

$$a\, 1.5^{-\gamma} \int_{1}^{1.5} t^{-n+\gamma} \big(1+0.5\cos(\ln t)\big)^{\gamma} (1.5-t)^{n-1}\,\mathrm{d}t > (n-1)!, \tag{2.16}$$

which guarantees that $\mathcal{N}_n = \emptyset$. For particular value of n we are able to evaluate (Matlab) integrals in (2.15) and (2.16). For example, choosing $n = 4$ conditions (2.14)–(2.16) reduces to

$$a > 14.6, \quad a > 177.1, \quad a > 339,$$

respectively. Therefore $a > 339$ guarantees oscillation of (E_x).

If it is desirable to obtain closed formula for oscillation of (E_x), then we apply criteria (2.8) and (2.13) instead of (2.7) and (2.12). Accordingly condition (2.8) reduces to

$$\frac{a}{(n-1)!}\int_{0.5}^{1} t^{-n}(t-0.5)^{n-1}\,\mathrm{d}t > 1.$$

If we denote the integral in the last inequality by $I(n)$, then integration by parts yields the following recursion formula

$$I(n) = I(n-1) - \frac{1}{n-1}\left(\frac{1}{2}\right)^{n-1}, \quad I(1) = \ln 2.$$

Hence we conclude that the condition

$$\frac{a}{(n-1)!}I(n) > 1, \tag{2.17}$$

with

$$I(n) = \ln 2 - \left(\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}\right)^2 + \ldots + \frac{1}{n-1}\left(\frac{1}{2}\right)^{n-1}\right)$$

ensures that $\mathcal{N}_0 = \emptyset$.

Similarly, condition (2.13) takes the form

$$\frac{a}{(n-1)!}\int_{1}^{1.5} t^{-n}(1.5-t)^{n-1}\,\mathrm{d}t > 1.$$

Now we denote the integral involved above by $J(n)$ and we see that

$$J(n) = \frac{1}{n-1}\left(\frac{1}{2}\right)^{n-1} - J(n-1), \quad J(1) = \ln 1.5$$

and so condition

$$\frac{a}{(n-1)!}J(n) > 1, \tag{2.18}$$

with

$$J(n) = \left(\left(\frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{2}\right)^2 + \ldots + \frac{1}{n-1}\left(\frac{1}{2}\right)^{n-1}\right) - \ln\frac{3}{2}$$

ensures that $\mathcal{N}_n = \emptyset$.

Finally it can be verified that condition (2.18) implies both conditions (2.17) and (2.14) so that it guarantees oscillation of (E_x).

As we mentioned in the motivation part, Kusano's Theorem 1 cannot be applied since condition (1.5) fails for (E_x). Nevertheless it is interesting to compare Kusano's criterion (1.7) for $\mathcal{N}_n = \emptyset$ with that of ours. Setting the corresponding values into (1.7) we obtain

$$\lim_{k\to\infty} \frac{a}{(n-1)!} \int_{s_k}^{1.5s_k} \frac{\big(t(1+0.5\sin(\ln t)) - 1.5s_k\big)^{n-1}}{t^n}\,\mathrm{d}t > 1. \tag{2.19}$$

Substitution $t = s_k \mathrm{e}^z$ simplifies (2.19) into

$$\frac{a}{(n-1)!} \int_0^{\ln 1.5} \big(1+0.5\cos z - 1.5\mathrm{e}^{-z}\big)^{n-1}\,\mathrm{d}z > 1. \tag{2.20}$$

The Jensen inequality implies that

$$\begin{aligned}
&\int_0^{\ln 1.5} \big(1+0.5\cos z - 1.5\mathrm{e}^{-z}\big)^{n-1}\,\mathrm{d}z \\
&\geq \frac{1}{(\ln 1.5)^{n-2}} \left(\int_0^{\ln 1.5} \big(1+0.5\cos z - 1.5\mathrm{e}^{-z}\big)\,\mathrm{d}z\right)^{n-1} \\
&= \frac{1}{(\ln 1.5)^{n-2}} \big(\ln 1.5 + 0.5\sin\ln 1.5 - 0.5\big)^{n-1}.
\end{aligned}$$

In this regard (2.20) is satisfied provided that

$$\frac{a}{(n-1)!} \frac{\big(\ln 1.5 + 0.5\sin\ln 1.5 - 0.5\big)^{n-1}}{(\ln 1.5)^{n-2}} > 1 \tag{2.21}$$

and for $n = 4$ this yields $a > 911$. So our progress is respectable.

Remark 2.10. It is easy to see that the above mentioned Baculikova's criterion cannot be applied to (E_x) and on the other hand our criterion fails for (1.8), so that both criteria are independent.

3. SUMMARY

In this paper we tried to extend oscillation theory of differential equations with mixed arguments. Our criteria are of high generality, easily verifiable and improve the known ones. Moreover our results are applicable also for n-odd when

$$\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_3 \cup \ldots \cup \mathcal{N}_n.$$

In this case the situation is simpler due to absence of class $\mathcal{N}_0$.

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REFERENCES

[1] B. Baculikova, *Oscillation of second-order nonlinear noncanonical differential equations with deviating argument*, Appl. Math. Lett. **91** (2019), 68–75.

[2] B. Baculikova, *Oscillatory behavior of the second order noncanonical differential equation*, Electron. J. Qual. Theory Differ. Equ. **89** (2019), 1–17.

[3] B. Baculikova, *Oscillation and asymptotic properties of second order half-linear differential equations with mixed deviating arguments*, Mathematics **2021**, 9(20), 2552.

[4] B. Baculikova, *Oscillation of even order linear functional differential equations with mixed deviating arguments*, Opuscula Math. **42** (2022), no. 4, 549–560.

[5] R. Koplatadze, G. Kvinkadze, I.P. Stavroulakis, *Properties A and B of n-th order linear differential equations with deviating argument*, Georgian Math. J. **6** (1999), no. 6, 553–566.

[6] R. Koplatadze, T.A. Chanturia, *On Oscillatory Properties of Differential Equations with Deviating Arguments*, Tbilisi Univ. Press, Tbilisi, 1977.

[7] T. Kusano, *On even order functional differential equations with advanced and retarded arguments*, J. Differential Equations **45** (1982), no. 1, 75–84.

[8] T. Kusano, *Oscillation of even order linear functional differential equations with deviating arguments of mixed type*, J. Math. Anal. Appl. **98** (1984), 341–347.

[9] G. Laddas, V. Lakshmikantham, J.S. Papadakis, *Oscillation of higher-order retarded differential equations generated by retarded argument*, Delay and functional differential equations and their applications (Proc. Conf., Park City, Utah, 1972), Academic Press, New York, 1972, 219–231.

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