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COMPUTING A MECHANISM FOR A BAYESIAN AND PARTIALLY OBSERVABLE MARKOV APPROACH

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The design of incentive-compatible mechanisms for a certain class of finite Bayesian partially observable Markov games is proposed using a dynamic framework. We set forth a formal method that maintains the incomplete knowledge of both the Bayesian model and the Markov system's states. We suggest a methodology that uses Tikhonov's regularization technique to compute a Bayesian Nash equilibrium and the accompanying game mechanism. Our framework centers on a penalty function approach, which guarantees strong convexity of the regularized reward function and the existence of a singular solution involving equality and inequality constraints in the game. We demonstrate that the approach leads to a resolution with the smallest weighted norm. The resulting individually rational and ex post periodic incentive compatible system satisfies this requirement. We arrive at the analytical equations needed to compute the game's mechanism and equilibrium. Finally, using a supply chain network for a profit maximization problem, we demonstrate the viability of the proposed mechanism design.

Keywords: dynamic mechanism design, partially observable Markov chains, games with private information, Bayesian equilibrium, regularization.

1. Introduction

1.1. Brief review. Mechanism design emerges as a framework for incentive compatible design of resource allocation processes in the engineering and economic theories. It has been already extended to other disciplines (Gallien, 2006; Kakade *et al.*, 2013; Nocedal and Wright, 2006). The main focus of the literature is on efficient or optimal mechanism design with selfish agents. A key difficulty in practical mechanism design is that they are developed by trial and-error techniques (Hartline and Lucier, 2015), which is not a method for finding the best solution. We develop a theory of designing mechanisms for frameworks with dynamic private information and propose a mechanism that maximizes profit for a particular class of finite Bayesian partially observable Markov games (BPOMGs).

The literature has usually reported the design of (economics) mechanisms without careful consideration of incentives and regardless of the differences with the real-world. As a consequence, the design of mechanisms needs to be related to an engineering approach.

There exists an extensive literature on mechanism design. We refer the reader to Bergemann and Said (2011) for a detailed survey on the mechanism design literature. The main focus is usually on one of the following approaches:

- models where mechanisms depend on informativeness of the agent with private information (Courty and Li, 2000; Battaglini, 2005);
- environments considering a continuum of agents with independent and private information (Atkeson

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and Lucas, 1992);

- games with repetitions in which players observe public outcome with imperfect signals and actions.

More precisely, Eső and Szentes (2007) extended the study of analyzing a monopolist selling an indivisible good to multiple players for optimal information revelation in auctions, employing a two-period state representation to orthogonalize an agent's future information. Board (2007) presented a multi-agent framework with an infinite-time horizon. Kakade *et al.* (2013) took into consideration a bandit auctions and proved that the optimal mechanism is a version of the dynamic pivot mechanism. Garg and Narahari (2008) suggested a mechanism design for single leader Bayesian Stackelberg problems. Gershkov and Moldovanu (2009) described a mechanism design approach for dynamic revenue maximization in which the agents arrive stochastically over time. Athey and Segal (2013) developed a Bayesian incentive-compatible mechanism for a dynamic environment with quasilinear payoffs in which agents observe private information and decisions are made over countably many periods. Pavan *et al.* (2014) considered a Myersonian mechanism design approach, which studies dynamic quasilinear environments where private information arrives over time and decisions are made over multiple periods. Board and Skrzypacz (2016,) suggested an approach for mechanism design where players take into account one piece of private information arriving stochastically over time. Hartline and Lucier (2015) presented a technique for transforming a potentially nonoptimal algorithm approach to optimization into a Bayesian incentive-compatible mechanism that improves the social welfare and revenue.

1.2. Main results. This paper presents a dynamic framework for incentive-compatible mechanism design for a particular class of finite BPOMGs. The main results are summarized as follows:

1. Provides a formal approach that preserve both, the incomplete information of the Bayesian model and the incomplete information over the states of the Markov system,
2. Proposes a method involving Tikhonov's regularization technique for converging to a Bayesian Nash equilibrium and the corresponding mechanism for the game.
3. Ensures the strong convexity of the reward function by a penalty approach and the existence of a solution involving equality and inequality constraints.
4. Derives analytical expressions for computing the mechanism and the equilibrium of the game.

1.3. Organization of the paper. The paper is structured paper as follows. In Section 2, we formalize the Bayesian Markov games defining incentive compatibility and mechanism design. In Section3, we discuss our approach for designing mechanisms in BPOMGs. The concepts of mechanism and equilibrium for this game are formalized in Section 4. In Section 5 the dynamic approach is presented and the convergence analysis is developed in Section 6. Section 7 provides an application example of the proposed method involving a supply chain network for profit maximization. Section 8 concludes with some remarks. Appendix contains the proofs of the lemmas and theorems.

2. Preliminaries: Bayesian Markov games

We consider a discrete-time repeated game played by a set $\mathcal{N} = \{1, \dots, n\}$ of players indexed by $l \in \mathcal{N}$. Following standard practice, we shall say that at each time $t \geq 1$, the player l is privately informed about her type $\theta_t^l \in \Theta_t^l$. Players simultaneously take an action (make decisions) $a_t^l \in A_t^l$. Let $A_t = \times_{l=1}^n A_t^l$. We write Θ_t for $\times_{l\in\mathcal{N}}\Theta_t^l$, Θ_t^{-l} for $\times_{h\in\mathcal{N},h\neq l}\Theta_t^h$ and $\Delta(A)$ for the set of all probability distributions over A. We assume that A_t^l and Θ_t^l are finite sets for all l.

A *social choice function* κ is a mapping from the profile of types to lotteries over alternatives if all players report their type $\theta_t^l \in \Theta_t$, i.e., $\kappa : \Theta_t \to \Delta(A_t)$. A social choice function represents the goals of the game, e.g., to maximize revenues, etc. Finally, each player has a known *valuation function* $u^l(a_t^l, \theta_t^l) \geq 0$, which determines the actual value (utility) for the player l. Each $u^l : A_t^l \times \Theta_t^l \to \mathbb{R}^+$ is a mapping from the alternatives and the set of type profiles to the set of non-negative real numbers.

We assume that the type θ_t^l of player l follows a controllable Markov process on the state space Θ_t^l. A *controllable homogeneous Markov chain* is a sequence of θ-valued random variables θ_t, $t \in T$, satisfying the *Markov condition*:

$$p^l(\theta_{t+1}^l|\theta_t^l, a_t^l, \theta_{t-1}^l, a_{t-1}^l \dots, \theta_0^l, a_0^l) = p^l\left(\theta_{t+1}^l|\theta_t^l a_t^l\right), \quad (1)$$

which represents the probability associated with the transition function (or stochastic kernel) from state θ_t^l to state θ_{t+1}^l, under an alternative a_t^l. The common prior (initial distribution) of the state-alternative process for each player l is denoted by $\{(\Theta_t, A_t)\,|t \in T\}$. In this case, the process described by a Markov chain is completely described by the transition function $p^l\left(\theta_{t+1}^l|\theta_t^l a_t^l\right)$ and the initial distribution vector $P^l\left(\theta_0^l\right) \in \Delta(\Theta_0^l)$ such that $P_t^l\left(\theta_t^l\right) \in \Delta(\Theta_t^l)$, where $\Delta(\Theta_t^l)$ denotes the set of all probability distributions over Θ_t^l. The Markov chains are mutually independent. We assume that each chain

$(P^l, p(\theta^l_{t+1}|\theta^l_t, a^l_t))$ is irreducible and aperiodic, and that P^l is its unique invariant distribution.

Definition 1. A *mechanism* μ in the above environment assigns a set of possible messages M^l_t to the player l.

At each time t, the player l sends a message m^l_t from this set and the mechanism μ responds with a (possibly randomized) decision that may depend on the entire history of messages sent up to time t, and on past decisions. Hence, the mechanism $\mu(a_t|m_t)$ has inputs a_t that is the current allocation for the players and $m_t = (m^1_t, \ldots, m^n_t)$, which is the joint set of messages made by the player l.

Let us consider an *allocation rule* g, which represents the probability measure associated with the occurrence of an alternative a_t from the profile of messages m_t at time $t \in T$. It is a mapping from the message m_t to lotteries over alternatives $\Delta(A_t)$, i.e., $g : M_t \to \Delta(A_t)$.

Formally, a mechanism μ is a pair (M_t, g) where M_t is the set of messages for player l and $g : M_t \to \Delta(A_t)$ is the allocation rule. The mechanism μ is interpreted as the probability that a_t will be the outcome if the profile of types θ_t and messages m_t are the players' types.

Let $\mathcal{U}$ be a set of available mechanisms. We have that

$$\mathcal{U}_{\text{adm}} = \Big\{\mu(a_t|m_t) \geq 0 \mid \sum_{a_t \in A_t} \mu(a_t|m_t) = 1,\ m_t \in \Theta_t\Big\}. \quad (2)$$

We write $\mu(g, M^1_t \times \cdots \times M^n_t) \in \mathcal{U}_{\text{adm}}$ and $g(m_t) = \Delta(A_t)$, where $m_t \in M_t$, $\mathcal{U}_{\text{adm}}$ is the set of feasible mechanisms.

Note that the *mechanism is common for all players*, i.e., it is independent of the individual index $l \in \mathcal{N}$.

A dynamic mechanism induces a dynamic game with incomplete information. The following sequence of events takes place in each period t. At the beginning of each period t, the designer announces publicly the mechanism to the players and each agent privately learns her current type $\theta^l_t \in \Theta^l_t$ drawn from $p^l\left(\theta^l_{t+1}|\theta^l_t, a^l_t\right)$. Next, players sent messages m^l_t simultaneously, and the profile of messages is publicly observed m_t. The mechanism selects a decision $a_t \in A_t$ according to $\mu(a_t|m_t)$ and the implemented alternative a_t is publicly announced.

With this assumption, the public history in period t is a sequence of messages m^l_t and alternatives $a^l_t(\theta_t)$ until period $t-1$. The state of an observer is the vector $h^l_t = (m^l_0, a^l_0, \ldots, m^l_{t-1}, a^l_{t-1}, m^l_t)$, which is a trajectory of length t called the *history* (public history) where:

(i) $\mathbb{H}_t$ is the set of possible states in period t, capturing all information relevant to the decision by the observer in that period, and

(ii) each $m_t = (m^l_0, \ldots, m^l_t)$ is a report profile of the player l.

The public history h_t stands for a generic element of $\mathbb{H}_t$, which is the set of possible public histories in period t. The sequence of reports by the players is part of the public history and we assume that the past reports of each player are observable to all the players. The *private history* h^l_t of player l in period t consists of the sequence of private observations and the public history until period t, that is,

$$h^l_t = (\theta^l_0, m^l_0, a^l_0, \ldots \theta^l_{t-1}, m^l_{t-1}, a^l_{t-1}, \theta^l_t).$$

The set of possible private histories for each player l in period t is denoted by $\mathcal{H}^l_t = \Theta^l_t \times \mathbb{H}^l_{t-1}$. A (behavioral) *strategy* $\sigma^l(m^l_t|\theta^l_t)$ for player l is a mapping $\sigma^l : \mathcal{H}^l \times \Theta^l \to \Delta(M^l)$. The set of all feasible policies is denoted by $\mathcal{S}^l_{\text{adm}}$. Therefore, we have

$$\mathcal{S}^l_{\text{adm}} = \Big\{\sigma^l(m^l_t|\theta^l_t) \geq 0 \mid \sum_{m^l_t \in \Theta^l_t} \sigma^l(m^l_t|\theta^l_t) = 1,\ \theta^l_t \in \Theta^l_t\Big\}. \quad (3)$$

The valuation functions $u^l(a^l_t, \theta^l_t)$ and the transition functions $p^l\left(\theta^l_{t+1}|\theta^l_t a^l_t\right)$ are all common knowledge at t. The common prior initial distribution vector $P^l(\Delta\left(\Theta^l_0\right))$ and the transition function $p^l\left(\theta^l_{t+1}|\theta^l_t a^l_t\right)$ are assumed to be independent across players. The interaction between players induces a Markov game given by the quintuple $\Gamma = (\mathcal{N}, \Theta^l, A^l, P^l, U^l)_{l\in\mathcal{N}}$, where the average payoff of player l is the expected value of the summed payoff obtained under the mechanism μ, the strategy σ, and is defined as

$$\begin{aligned} U^l_T(\mu, \sigma) &= \sum_{t\in T} \sum_{\theta^l_t \in \Theta^l_t} \sum_{m^l_t \in \Theta^l_t} \sum_{a^l_t \in A^l_t} u^l(a^l_t(\theta_t), \theta^l_t) \\ &\quad \times p^l(\theta^l_{t+1}|\theta^l_t a^l_t) \prod_{\iota\in\mathcal{N}} \mu(a_t|m_t)\sigma^\iota(m^\iota_t|\theta^\iota_t)P^\iota(\theta^\iota_t) \\ &= \sum_{t\in T} \sum_{\theta^l_t \in \Theta^l_t} \sum_{m^l_t \in \Theta^l_t} \sum_{a^l_t \in A^l_t} W^l(a^l_t, \theta^l_t) \\ &\quad \times \prod_{\iota\in\mathcal{N}} \mu(a_t|m_t)\sigma^\iota(m^\iota_t|\theta^\iota_t)P^\iota(\theta^\iota_t) \end{aligned}$$

where

$$W^l(a^l_t, \theta^l_t) = u^l(a^l_t(m^l_t), \theta^l_t)p^l(\theta^l_{t+1}|\theta^l_t a^l_t).$$

3. Bayesian partially observable Markov games

We consider the case where the process is not directly observable. Let us associate with Θ^l_t the observation set Ξ^l_t which takes values in a finite space. The stochastic process Ξ_t^l is called the *observation process*. By observing Ξ_t at time t information regarding the true value of Θ_t is obtained.

A *mechanism* μ in the above environment assigns a set of possible messages M_t^l to the player l. At each time t, the player l sends a message m_t^l from this set and the mechanism μ responds with a (possibly randomized) decision that may depend on the entire history of messages sent up to time t, and on past decisions. Hence, the mechanism $\mu(a_t|m_t)$ has inputs a_t that is the current allocation for the players and $m_t = (m_t^1, \dots, m_t^n)$, which is the joint set of messages made by the player l.

Consider an *allocation rule* g, which represents the probability measure associated with the occurrence of an alternative a_t from the profile of messages m_t at time $t \in T$. It is is a mapping from the message m_t to lotteries over alternatives $\Delta(A_t)$, i.e., $g : M_t \to \Delta(A_t)$.. Formally, a mechanism μ is a pair (M_t, g), where M_t is the set of messages for player l and $g : M_t \to \Delta(A_t)$ is the allocation rule. Let $\mathcal{U}$ be a set of available mechanisms. We have that

$$\mathcal{U}_{adm}=\Big\{\mu(a_t|m_t) \geq 0 \mid \sum_{a_t\in A_t} \mu(a_t|m_t)=1,\ m_t \in \Xi_t\Big\} \quad (4)$$

which differs from Eqn. (2).

A realization of the Bayesian partially observable system at time t is given by the infinite sequence $(\theta_0^l, \vartheta_0^l, m_0^l, a_0^l, \theta_1^l, \vartheta_1^l, m_1^l, a_1^l, \dots) \in \Omega^l := \left(\Theta^l, \Theta^l, \Xi^l, A^l\right)^\infty$, where θ_0^l has a given distribution $P^l\left(\theta_0^l\right) \in \Delta\Theta_0^l$ and A_t is a control sequence in A determined by a control policy. To define a (behavioral) strategy, we cannot use the states $\theta_0^l, \theta_1^l, \dots$. Then, we introduce the observable histories $h_0^l \in \mathbb{H}_0^l$ and $h_t := (\theta_0^l, \vartheta_0^l, m_0^l, a_0^l, \dots, \theta_{t-1}^l, \vartheta_{t-1}^l, m_{t-1}^l, a_{t-1}^l, m_t^l) \in H_t$ for all $t \geq 1$ and $\mathcal{H}_t^l = \Xi_t \times \mathbb{H}_{t-1}$, if $t \geq 1$. A (behavioral) *strategy* $\sigma^l(\vartheta_t^l|\theta_t^l)$ for player l is a mapping $\sigma^l : \Theta^l \to \Delta(\Theta^l)$. The set of all feasible policies is denoted by $\mathcal{S}_{\text{adm}}^l$. We have that

$$\mathcal{S}_{\text{adm}}^l=\Big\{\sigma^l(\vartheta_t^l|\theta_t^l) \geq 0 \mid \sum_{\vartheta_t^l\in\Theta_t^l} \sigma^l(\vartheta_t^l|\theta_t^l) = 1,\ \theta_t^l \in \Theta_t^l\Big\}, \quad (5)$$

which differs from Eqn. (3).

Now, the probability $p^l(m_t^l|\vartheta_t^l, a_t^l)$ is called the *observation kernel* and denotes the relationship between the type $\vartheta_t^l \in \Theta_t^l$ and the message $m_t^l \in M_t^l$ when an action $a_t^l \in A_t^l$ is chosen at time t. The observation kernel is a stochastic kernel on Ξ_t^l.

The dynamics of the game with a set $\mathcal{N} = \{1, \dots, n\}$ of players (indexed by $l = \overline{1,n}$) is described as follows. At time $t = 0$, the initial state θ_0^l has a given *a-priori* distribution $P^l\left(\theta_0^l\right) \in \Delta\Theta_0^l$, and the initial message m_0^l is generated according to the initial observation kernel $p^l(m_t^l|\vartheta_t^l a_t^l)$. If at time t the state of the system is Θ_t and the action $A_t^l \in A^l$ is applied, then each strategy is allowed to randomize, with distribution $\sigma^l(\vartheta_t^l|\theta_t^l)$ considering the mechanism $\mu(a_t|m_t)$ over the action choices A_t^l. These choices induce immediate utilities $U_t^l(\mu, \sigma)$ where the system tries to maximize the corresponding one-step valuation functions $u^l(a_t^l, \theta_t^l, m_t^l)$. Next, the system moves to a new state θ_{t+1}^l according to the transition probabilities $p^l\left(\theta_{t+1}^l|\theta_t^l a_t^l\right)$. Then, the observation Ξ_t is generated by the observation kernel $p^l(m_t^l|\vartheta_t^l a_t^l)$. Based on the obtained utility, the systems adapt its strategy computing $\sigma^\iota(\vartheta_t^\iota|\theta_t^\iota)$ and the mechanism $\mu(a_t|m_t)$ for the next selection of the actions. The valuation functions $u^l(a_t^l, \theta_t^l, m_t^l)$ and the transition functions $p^l\left(\theta_{t+1}^l|\theta_t^l a_t^l\right)$ are all common knowledge at $t = 0$. The common prior initial distribution vector $P^l\left(\theta_0^l\right) \in \Delta\Theta_0^l$ and the transition function $p^l\left(\theta_{t+1}^l|\theta_t^l a_t^l\right)$ are assumed to be independent across players. The interaction between players induces a Bayesian partially observable Markov game where the average payoff of player l is the expected value defined as

$$\begin{aligned}
&U_T^l(\mu, \sigma)\\
&= \sum_{t\in T} \sum_{\theta_t^l\in\Theta_t^l} \sum_{\vartheta_t^l\in\Theta_t^l} \sum_{m_t^l\in\Xi_t^l} \sum_{a_t^l\in A_t^l} u^l(a_t^l(\theta_t), \theta_t^l)\\
&\quad\times p^l(\theta_{t+1}^l|\theta_t^l a_t^l) \prod_{\iota\in\mathcal{N}} \mu(a_t|m_t)\\
&\quad\times \sigma^\iota(\vartheta_t^\iota|\theta_t^\iota) p^\iota(m_t^\iota|\vartheta_t^l a_t^l) P^\iota(\theta_t^\iota)\\
&= \sum_{t\in T} \sum_{\theta_t^l\in\Theta_t^l} \sum_{\vartheta_t^l\in\Theta_t^l} \sum_{m_t^l\in\Xi_t^l} \sum_{a_t^l\in A_t^l} W^l(a_t^l, \theta_t^l, m_t^l)\\
&\quad\times \prod_{\iota\in\mathcal{N}} \mu(a_t|m_t) \sigma^\iota(\vartheta_t^\iota|\theta_t^\iota) p^\iota(m_t^\iota|\vartheta_t^\iota a_t^\iota) P^\iota(\theta_t^\iota),
\end{aligned}$$

with

$$W^l(a_t^l, \theta_t^l, m_t^l) = u^l(a_t^l(m_t^l), \theta_t^l, m_t^l) p^l(\theta_{t+1}^l|\theta_t^l a_t^l).$$

Given a direct mechanism μ and a history $\mathcal{H}_t^i$, the following sequence of events takes place in period t and for any $i \in \mathcal{N}$:

1. Each player l privately observes her current type θ_t^l drawn from $P^l\left(\theta_t^l\right)$.
2. Each player, considering $p^l(m_t^l|\vartheta_t^l a_t^l)$, sends a message $m_t^l \in M_t^l$ using a mechanism $\mu(a_t|m_t) \in \mathcal{U}_{\text{adm}}$ with the strategy $\sigma^l(\vartheta_t^l|\theta_t^l)$.
3. The mechanism selects an alternative $a_t \in A_t$ according to $\mu(a_t|m_t) \in \mathcal{U}_{\text{adm}}$.
4. Then, the allocation is realized and they obtain a reward.

5. Finally, the state changes to θ^l_{t+1} by taking $p^l(\theta^l_{t+1}|\theta^l_t a^l_t)$.

The realization of this goal implies the fulfillment of the strong form of the strategy and mechanism such that the limiting average payoff in the ergodic case is

$$U^l(\mu,\sigma) = \lim_{T\to\infty} T^{-1}\, U^l_T(\mu,\sigma).$$

4. Mechanism and the equilibrium

In this section, we provide an analytical method for computing a mechanism. We restrict ourselves to the case where $p^l(m^l_t|\vartheta^l_t a^l_t) = p^l(m^l_t|\vartheta^l_t)$.

4.1. Problem formulation. We assume that players know their payoffs. Each player maximizes the individual payoff function $U^l(\mu,\sigma)$ realizing the rule given by

$$(\mu^*,\sigma^*_{\mu^*}) \in \operatorname{Arg}\max_{\mu\in\mathcal{U}_{\text{adm}}} \sum_{l\in\mathcal{N}} U^l(\mu,\sigma^{l*}_\mu) \tag{6}$$

for a given mechanism $\mu_{k|m}$ and the strategies σ^*_μ satisfy the *Bayesian Nash equilibrium* fulfilling for all admissible σ the condition

$$U^l(\mu,\sigma^*_\mu) \geq U^l(\mu,\sigma^l,\sigma^{-l*}). \tag{7}$$

Here

- the mechanism μ is unique for all the players,
- the strategy $\sigma^* = \left(\sigma^{1*},\dots,\sigma^{n*}\right)$ is referred to as a *Bayesian Nash equilibrium* with $\sigma^{-l*} = \left(\sigma^{1*},\dots,\sigma^{l-1*},\sigma^{l+1,*},\dots,\sigma^{n*}\right)$.

4.2. Auxiliary problem. Now, introduce the z-variable:

$$z^l(\theta^l_t\vartheta^l_t m^l_t a^l_t) := \mu(a_t|m_t)\sigma^l(\vartheta^l_t|\theta^l_t)p^l(m^l_t|\vartheta^l_t)P^l(\theta^l_t). \tag{8}$$

Let us define $Q^l = \left[p^l(m^l_t|\vartheta^l_t)\right]^{-1}$.

Problem 1. We will try to find a mechanism $\mu(a_t|m_t)$ and Bayesian strategies $\sigma(\vartheta^l_t|\theta^l_t)$ which solve the following individual nonlinear programming problem:

$$\tilde{U}^l(\mu,\sigma) = \sum_{l\in\mathcal{N}} \bar{U}^l(z) \to \max_{z\in\mathcal{Z}_{\text{adm}}} \tag{9}$$

$$\begin{aligned}\bar{U}^l(z) = &\sum_{\theta^l_t\in\Theta^l_t}\sum_{\vartheta^l_t\in\Theta^l_t}\sum_{m^l_t\in\Xi^l_t}\sum_{a^l_t\in A^l_t} W^l(a^l_t,\theta^l_t,m^l_t)\\ &\times\prod_{\iota\in\mathcal{N}} z^\iota(\theta^\iota_t\vartheta^\iota_t m^\iota_t a^\iota_t),\end{aligned} \tag{10}$$

where $z^\iota(\theta^\iota_t\vartheta^\iota_t m^\iota_t a^\iota_t)$ is given by (8) and $\mathcal{Z}_{\text{adm}} = \bigtimes_{l=1}^n Z^l_{\text{adm}}$ with

$$\begin{aligned}
&Z^l_{\text{adm}}\\
&:= \Big\{ z^l(\theta^l_t,\vartheta^l_t,m^l_t,a^l_t) \geq 0 \mid\\
&\sum_{\theta^l_t\in\Theta^l_t}\sum_{\vartheta^l_t\in\Theta^l_t}\sum_{m^l_t\in\Xi^l_t}\sum_{a^l_t\in A^l_t} z^l(\theta^l_t\vartheta^l_t m^l_t a^l_t) = 1,\\
&\sum_{\vartheta^l_t\in\Theta^l_t}\sum_{m^l_t\in\Xi^l_t}\sum_{a^l_t\in A^l_t} z^l(\theta^l_t\vartheta^l_t m^l_t a^l_t) = P^l(\theta^l_t) > 0,\\
&\sum_{\theta^l_t\in\Theta^l_t}\sum_{\vartheta^l_t\in\Theta^l_t}\sum_{m^l_t\in\Xi^l_t}\sum_{a^l_t\in A^l_t} [\delta_{\theta^l_t\theta^l_{t+1}} - p^l(\theta^l_{t+1}|\theta^l_t a^l_t)]\\
&\times z^l(\theta^l_t\vartheta^l_t m^l_t a^l_t) = 0,\ \theta^l_{t+1}\in\Theta^l_t,\\
&\sum_{\theta^l_t\in\Theta^l_t}\sum_{\vartheta^l_t\in\Theta^l_t}\sum_{\varrho^l_t\in\Xi^l_t}\sum_{a^l_t\in A^l_t} [\delta_{\varrho^l_t m^l_t} - p^l(m^l_t|\vartheta^l_t)]\\
&\times z^l(\theta^l_t\vartheta^l_t\varrho^l_t a^l_t) = 0,\ m^l_t\in\Xi^l_t\\
&\sum_{\theta^l_t\in\Theta^l_t}\sum_{\vartheta^l_t\in\Theta^l_t}\sum_{m^l_t\in\Xi^l_t}\sum_{a^l_t\in A^l_t} Q^l(\rho^l_t|\vartheta^l_t) z^l(\theta^l_t\vartheta^l_t m^l_t a^l_t) \geq 0,\\
&\rho^l_t\in\Xi^l_t\Big\}.
\end{aligned} \tag{11}$$

Notice that the following relations hold:

$$\sum_{a_t\in A_t}\mu(a_t|m_t) = 1,\qquad \sum_{\vartheta^l_t\in\Theta^l_t}\sigma^l(\vartheta^l_t|\theta^l_t) = 1,$$
$$\sum_{m^l_t\in\Theta^l_t} p^l(m^l_t|\vartheta^l_t) = 1,\qquad \sum_{\theta^l_t\in\Theta^l_t} P^l(\theta^l_t) = 1.$$

It is easy to check that Z^l_{adm} includes the simplex Δ^l, namely, $z^l\in\Delta^l\subset Z^l_{\text{adm}}$:

$$\begin{aligned}\Delta^l := &\Big\{z^l(\theta^l_t\vartheta^l_t m^l_t a^l_t)\geq 0\mid\\ &\sum_{\theta^l_t\in\Theta^l_t}\sum_{\vartheta^l_t\in\Theta^l_t}\sum_{m^l_t\in\Xi^l_t}\sum_{a^l_t\in A^l_t} z^l(\theta^l_t\vartheta^l_t m^l_t a^l_t) = 1,\\ &\sum_{\vartheta^l_t\in\Theta^l_t}\sum_{m^l_t\in\Xi^l_t}\sum_{a^l_t\in A^l_t} z^l(\theta^l_t\vartheta^l_t m^l_t a^l_t) = P^l(\theta^l_t) > 0\Big\}.\end{aligned} \tag{12}$$

Write the solution of the problem (9) as z^{l*}, $l\in\mathcal{N}$.

4.3. Recovering of optimal mechanism and strategies from the z^* solution. The result below clarifies how we may recover the mechanism $\mu^*(a_t|m_t)$.

Lemma 1. *Suppose that the problem (9) is solved. Then the mechanism $\mu^*(a_t|m_t)$ can be recovered from $z^{l*}(\theta^l_t,\vartheta^l_t,m^l_t,a^l_t)$ as follows:*

$$\mu^*(a_t|m_t) = \frac{\sum\limits_{l\in\mathcal{N}}\sum\limits_{\theta^l_t\in\Theta^l_t}\sum\limits_{\vartheta^l_t\in\Theta^l_t} z^{l*}(\theta^l_t\vartheta^l_t m^l_t a^l_t)}{\sum\limits_{l\in\mathcal{N}}\sum\limits_{\theta^l_t\in\Theta^l_t}\sum\limits_{\vartheta^l_t\in\Theta^l_t}\sum\limits_{\alpha^l_t\in A^l_t} z^{l*}(\theta^l_t\vartheta^l_t m^l_t \alpha^l_t)}. \tag{13}$$

Proof. See Appendix, Section A1. ■

Variables $\sigma^{l*}(\vartheta_t^l|\theta_t^l)$ and $\bar{P}^{l*}(m_t^l)$ can be recovered as it is presented below.

Corollary 1. *The equilibrium (behavior) strategies $\sigma^{l*}(\vartheta_t^l|\theta_t^l)$ are given by*

$$\sigma^{l*}(\vartheta_t^l|\theta_t^l)=\frac{\sum\limits_{m_t^l\in\Xi_t^l}\sum\limits_{a_t^l\in A_t^l} z^{l*}(\theta_t^l\vartheta_t^l m_t^l a_t^l)}{\sum\limits_{\varrho_t^l\in\Theta_t^l}\sum\limits_{m_t^l\in\Xi_t^l}\sum\limits_{a_t^l\in A_t^l} z^{l*}(\theta_t^l\varrho_t^l m_t^l a_t^l)}, \quad (14)$$

$l \in \mathcal{N}$ and the corresponding distributions $\bar{P}^{l*}(m_t^l)$ are as follows:

$$\bar{P}^{l*}(m_t^l) = \sum_{\theta_t^l\in\Theta_t^l}\sum_{\vartheta_t^l\in\Theta_t^l}\sum_{a_t^l\in A_t^l} z^l(\theta_t^l\vartheta_t^l m_t^l a_t^l), \quad l \in \mathcal{N}. \quad (15)$$

4.4. Necessary conditions for ergodicity. The next theorem presents the necessary egodicity conditions that the solutions of the problem (9) must satisfy.

Theorem 1. *If the strategy $\sigma^{l*}(\vartheta_t^l|\theta_t^l)$ and the mechanism $\mu^*(a_t|m_t)$ are solutions to the problem (9) and, hence, correspond to a Nash equilibrium (7), then variables $z^{l*}(\theta_t^l,\vartheta_t^l,m_t^l,a_t^l)$ for all $l\in\mathcal{N}$ satisfy the following ergodicity constraints:*

$$\sum_{\alpha_t^l\in\Theta_t^l}\sum_{\kappa_t^l\in\Theta_t^l}\sum_{\beta_t^l\in\Xi_t^l}\sum_{\gamma_t^l\in A_t^l}[\delta_{\alpha_t^l\theta_{t+1}^l} - p^l(\theta_{t+1}^l|\alpha_t^l\gamma_t^l)] \times z^{l*}(\alpha_t^l\kappa_t^l\beta_t^l\gamma_t^l) = 0, \quad \theta_{t+1}^l\in\Theta_t^l, \quad (16)$$

$$\sum_{\varrho_t^l\in\Xi_t^l}\sum_{\alpha_t^l\in\Theta_t^l}\sum_{\kappa_t^l\in\Theta_t^l}\sum_{\gamma_t^l\in A_t^l}[\delta_{\varrho_t^l\beta_t^l} - p^l(\beta_t^l|\kappa_t^l)] \times z^{l*}(\alpha_t^l\kappa_t^l\varrho_t^l\gamma_t^l) = 0, \quad \beta_t^l\in\Xi_t^l, \quad (17)$$

$$\sum_{\alpha_t^l\in\Theta_t^l}\sum_{\kappa_t^l\in\Theta_t^l}\sum_{\beta_t^l\in\Xi_t^l}\sum_{\gamma_t^l\in A_t^l} Q^l(\beta_t^l|\rho_t^l) z^{l*}(\alpha_t^l\kappa_t^l\beta_t^l\gamma_t^l) \geq 0, \quad \rho_t^l\in\Xi_t^l. \quad (18)$$

Proof. See Appendix, Section A2. ■

4.5. Bayesian Nash equilibrium.

Lemma 2. *The obtained mechanism $\mu^*(a_t|m_t)$ and the strategies $\sigma^{l*}(\vartheta_t^l|\theta_t^l)$ satisfy the Bayesian-Nash equilibrium given in Eqn. (7).*

Proof. We have

$$\begin{aligned}
&\max_{z\in\mathcal{Z}_{\text{adm}}}\tilde{U}(z)\\
&= \tilde{U}(z^*) = \sum_{l\in\mathcal{N}}\tilde{U}^l(z^*) = \sum_{l\in\mathcal{N}} U^l(\mu^*,\sigma^*_{\mu^*})\\
&= \sum_{l\in\mathcal{N}}\sum_{t\in T}\sum_{\theta_t^l\in\Theta_t^l}\sum_{\vartheta_t^l\in\Theta_t^l}\sum_{m_t^l\in\Xi_t^l}\sum_{a_t^l\in A_t^l} W^l(a_t^l,\theta_t^l,m_t^l)\\
&\quad\times(\mu^*(a_t|m_t))^n\sigma^{l*}(\vartheta_t^l|\theta_t^l)p^{l*}(m_t^l|\vartheta_t^l a_t^l)P^{l*}(\theta_t^l)\\
&\quad\times\prod_{\iota\neq l\in\mathcal{N}}\sigma^{\iota*}(\vartheta_t^\iota|\theta_t^\iota)p^{\iota*}(m_t^\iota|\vartheta_t^\iota a_t^\iota)P^{\iota*}(\theta_t^\iota),\\
&= \sum_{l\in\mathcal{N}}\sum_{t\in T}\max_{\sigma^l\in\mathcal{S}^l_{\text{adm}}}\sum_{m_t^l\in\Xi_t^l}\sum_{a_t^l\in A_t^l}(\mu^*(a_t|m_t))^n\\
&\quad\times\sum_{\theta_t^l\in\Theta_t^l}\sum_{\vartheta_t^l\in\Theta_t^l} W^l(a_t^l,\theta_t^l,m_t^l)\\
&\quad\times\sigma^l(\vartheta_t^l|\theta_t^l)p^l(m_t^l|\vartheta_t^l a_t^l)P^l(\theta_t^l)\\
&\quad\times\prod_{\iota\neq l\in\mathcal{N}}\sigma^{\iota*}(\vartheta_t^\iota|\theta_t^\iota)p^{\iota*}(m_t^\iota|\vartheta_t^\iota a_t^\iota)P^{\iota*}(\theta_t^\iota)\\
&= \sum_{l\in\mathcal{N}}\sum_{t\in T}\sum_{m_t^l\in\Xi_t^l}\sum_{a_t^l\in A_t^l}(\mu^*(a_t|m_t))^n\\
&\quad\times\sum_{\theta_t^l\in\Theta_t^l}\sum_{\vartheta_t^l\in\Theta_t^l}\Big(W^l(a_t^l,\theta_t^l,m_t^l)\\
&\quad\times\sigma^l(\vartheta_t^l|\theta_t^l)p^l(m_t^l|\vartheta_t^l a_t^l)P^l(\theta_t^l)\\
&\quad\times\prod_{\iota\neq l\in\mathcal{N}}\sigma^{\iota*}(\vartheta_t^\iota|\theta_t^\iota)p^{\iota*}(m_t^\iota|\vartheta_t^\iota a_t^\iota)P^{\iota*}(\theta_t^\iota)\Big)\\
&= \sum_{l\in\mathcal{N}} U^l(\mu,\sigma^l,\sigma^{-l*}).
\end{aligned} \quad (19)$$

From this inequality it follows that

$$\sum_{l\in\mathcal{N}}\left(U^l(\mu^*,\sigma^*_{\mu^*}) - U^l(\mu^*,\sigma^l,\sigma^{-l*})\right)\geq 0. \quad (20)$$

Since the above inequality is valid for all admissible strategies σ, it is valid when $\sigma^j=\sigma^{j*}$ for $j\neq l$, implying

$$U^l(\mu^*,\sigma^*_{\mu^*}) - U^l(\mu^*,\sigma^l,\sigma^{-l*})\geq 0, \quad (21)$$

which coincides with Eqn. (7) when $\mu=\mu^*$. The lemma is proven. ■

5. Convergence analysis

5.1. Nash equilibrium problem. In this section, we prove the convergence of the proposed method to a mechanism and unique Bayesian Nash equilibrium (Clempner and Poznyak, 2015; 2016). Each player selects the best reply to the other players' strategies to reach the equilibrium. To this end, we first recall the definition of the (standard) Nash equilibrium problem.

We consider a game played by a set $\mathcal{N} = \{1, \dots, n\}$ players indexed by $l \in \mathcal{N}$. Each of player $l \in \mathcal{N}$ controls the variable

$$z^l(\theta_t^l \vartheta_t^l m_t^l a_t^l) := \mu(a_t|m_t)\sigma^l(\vartheta_t^l|\theta_t^l)p^l(m_t^l|\vartheta_t^l)P^l(\theta_t^l).$$

Consider a game whose strategies are denoted by $x^l \in \mathcal{X}^l$ where $\mathcal{X}^l$ is a convex and compact set, and $x^l := \operatorname{col}\left(z^l(\theta_t^l \vartheta_t^l m_t^l a_t^l)\right)$, where col is the column operator. Let $x = (x^1, \dots, x^n)^\top \in X$, the vector formed by all these decision variables, be the joint strategy of the players and $x^{-l} := \left(x^1, \dots, x^{l-1}, x^{l+1}, \dots, x^n\right)^\top \in X^{-l}$ be a strategy of the rest of the players adjoint to $x^l \in X^l$. To emphasize the l-th player's variables within the vector x, we sometimes write $x = (x^l, x^{-l})^\top \in \mathbb{R}^n$ where x^{-l} subsumes all the other players' variables. We consider a Nash equilibrium problem with n players and denote by $x = (x^l, x^{-l}) \in \mathbb{R}^n$ the vector representing the l-th player's strategy where $x \in \mathcal{X}_{\text{adm}}$ and $\mathcal{X}_{\text{adm}} = \mathcal{X}_{\text{adm}}^l \times \mathcal{X}_{\text{adm}}^{-l}$, such that

$$\mathcal{X}_{\text{adm}} := \{x : x \geq 0,\ \Phi_0 x = b_0,\ \Phi_1 x \leq b_1\}.$$

Let $f^l : \mathbb{R}^n \to \mathbb{R}$ be the l-th player's reward function (cost function). We assume that these reward functions are at least continuous, and we further assume that the functions $f^l\left(x^l, x^{-l}\right)$ are concave in the variable x^l. The players try to reach one of the non-cooperative equilibrium, that is, they try to find a strategy $x^* = \left(x^{1*}, \dots, x^{n*}\right)$ satisfying for any x^l and any $l \in \mathcal{N}$ that

$$\mathcal{F}(x) := \sum_{l \in \mathcal{N}} \left[\left(\max_{x^l \in X^l} f^l\left(x^l, x^{-l}\right)\right) - f^l\left(x^l, x^{-l}\right)\right]. \tag{22}$$

Note that

$$\mathcal{F}(x) := \sum_{l \in \mathcal{N}} \left[f^l\left(\bar{x}^l, x^{-l}\right) - f^l\left(x^l, x^{-l}\right)\right], \tag{23}$$

where

$$\bar{x}^l \in \operatorname{Arg} \max_{x^l \in X^l} f^l\left(x^l, x^{-l}\right) \tag{24}$$

and the function f^l satisfies

$$f^l\left(\bar{x}^l, x^{-l}\right) - f^l\left(x^l, x^{-l}\right) \geq 0 \tag{25}$$

for any $x^l \in X^l$ and $l \in \mathcal{N}$. Then a vector $x^* \in X$ is a *non-cooperative equilibrium,* or a solution of the Nash equilibrium problem if

$$x^* \in \operatorname{Arg} \max_{x \in X} \{\mathcal{F}(x)\}. \tag{26}$$

In the case where $\mathcal{F}(x)$ is a strictly concave function, we have that

$$x^* = \arg \max_{x \in X} \{\mathcal{F}(x)\}.$$

5.2. Penalty function approach. Considering the "slack" vectors $\zeta \in \Lambda$ with nonnegative components, that is, $\zeta_j \geq 0$, the original problem (26) can be rewritten as

$$\left.\begin{array}{c} \mathcal{F}(x) \to \max\limits_{x \in X_{\text{adm}},\, \zeta \geq 0} \\ \mathcal{X}_{\text{adm}} := \{x : x \geq 0,\ \Phi_0 x = b_0,\ \Phi_1 x - b_1 + \zeta = 0\}. \end{array}\right\} \tag{27}$$

Remark 1. Note that the problem (27) may have a nonunique solution and $\det\left(V_0^\intercal V_0\right) = 0$.

We are dealing with a *penalty function* solution method which considers (Clempner and Poznyak, 2018a; 2018b)

$$\begin{aligned} \psi_{\zeta,\rho}(x, \zeta) := -\mathcal{F}(x) + \zeta \Big[&\frac{1}{2}\|\Phi_0 x - b_0\|^2 \\ &+ \frac{1}{2}\|\Phi_1 x - b_1 + \zeta\|^2 \\ &+ \frac{\rho}{2}\left(\|x\|^2 + \|\zeta\|^2\right)\Big] \end{aligned} \tag{28}$$

with $\zeta, \rho > 0$, such that the optimization problem

$$\psi_{\zeta,\rho}(x, \zeta) \to \min_{x \in X_{\text{adm}},\, \zeta \geq 0} \tag{29}$$

has a unique equilibrium point. The following lemma proves this statement.

Lemma 3. *The optimization function given in Eqn. (28) is strongly convex if the Hessian matrix $\mathbb{H}$ associated with the penalty function given in Eqn. (28) is strictly positive.*

Proof. See Appendix, Section A3. ■

It is important to note that

$$\arg \min_{x \in X_{\text{adm}},\, \zeta \geq 0} \psi_{\zeta,\rho}(x, \zeta) = \arg \min_{x \in X_{\text{adm}},\, \zeta \geq 0} \Psi_{\omega,\rho}(x, \zeta)$$

such that $\omega := \zeta^{-1} > 0$ and

$$\begin{aligned} \Psi_{\omega,\rho}(x, \zeta) := -\omega\mathcal{F}(x) &+ \frac{1}{2}\|\Phi_0 x - b_0\|^2 \\ &+ \frac{1}{2}\|\Phi_1 x - b_1 + \zeta\|^2 \\ &+ \frac{\rho}{2}\left(\|x\|^2 + \|\zeta\|^2\right) \end{aligned} \tag{30}$$

Let $\mathcal{X}^* \subseteq \mathcal{X}_{\text{adm}}$ be the set of all solutions to the problem (27). The following proposition holds for the penalty function method.

Proposition 1. *The parameters $x^*(\omega, \rho)$ and $\zeta^*(\omega, \rho)$, which are the solutions of the optimization problem*

$$\Psi_{\omega,\rho}(x, \zeta) \to \min_{x \in X_{adm},\, \zeta \geq 0},$$

tend to the set $\mathcal{X}^*$ of all solutions of the original optimization problem (27), when the penalty parameter ω tends to zero.

Remark 2. We have that Proposition 1 holds if

$$d\{x^*(\omega,\rho),\zeta^*(\omega,\rho);X^*\} \underset{\zeta,\rho\downarrow 0}{\longrightarrow} 0, \tag{31}$$

where $d\{a;X^*\}$ is the Hausdorff distance defined as

$$d\{y;X^*\} = \min_{x^*\in X^*}\|y-x^*\|^2$$

The following theorem determines how the parameters ω and ρ tend to zero. We provide the convergence analysis of the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{\zeta_n\}_{n\in\mathbb{N}}$ in the following theorem. Define $y=(x,\zeta)^\intercal$ such that $\mathcal{Y}_{\text{adm}}=\mathcal{X}_{\text{adm}}\times\Lambda_{\text{adm}}$.

Theorem 2. *Let the Markov game be Lipschitz continuous; in particular, for any $y:=(x,\zeta)^\intercal$, let $\Psi_{\omega,\rho}(x,\zeta)=\Psi_{\omega,\rho}(y)$ be convex and differentiable with the gradient satisfying the Lipschitz condition given by*

$$\|\nabla\Psi_{\omega,\rho}(y)-\nabla\Psi_{\omega,\rho}(k)\|\le c\,\|y-k\|$$

for $c>0$ and all $y,k\in\mathcal{Y}_{adm}$, where $\mathcal{Y}_{adm}$ a is convex and compact set. In addition, assume that ω and ρ are time-varying, i.e.,

$$\omega=\omega_n,\quad \rho=\rho_n\quad (n=0,1,2,\dots),$$

where the parameters satisfy

$$0<\omega_n\downarrow 0,\quad \frac{\omega_n}{\rho_n}\downarrow 0\quad \text{as } n\to\infty. \tag{32}$$

Then, for any $\rho\in(0,1)$ and any vector $y_n^*:=(x_n^*=x^*(\omega_n,\rho_n),\zeta_n^*=\zeta^*(\omega_n,\rho_n))^\intercal\in\mathcal{Y}_{adm}$ the sequences

$$\begin{aligned}x_n^* &:= x^*(\omega_n,\rho_n)\underset{n\to\infty}{\longrightarrow}x^{**},\\ \zeta_n^* &:= \zeta^*(\omega_n,\rho_n)\underset{n\to\infty}{\longrightarrow}\zeta^{**}\end{aligned} \tag{33}$$

converge to a Bayesian Nash equilibrium point and a incentive-compatible mechanism such that $x^{**}\in\mathcal{X}^*$ is the solution of the original problem (27).

Proof. See Appendix, Section A4. ∎

The following theorem determines how the parameters ω and ρ tend to zero employing the Hausdorff distance defined in Eqn. (31).

Theorem 3. *Assume that the bounded set $\mathcal{X}^*$ of all solutions of the original optimization problem (27) is not empty and Slater's condition holds, that is, there exists a point $\mathring{x}\in\mathcal{X}_{adm}$ such that*

$$\Phi_1\mathring{x}<b_1. \tag{34}$$

Then, by Theorem 2, for any $\rho\in(0,1)$ the sequences

$$x_n^*:=x^*(\omega_n,\rho_n)\underset{n\longrightarrow\infty}{\longrightarrow}x^{**},$$

$$\zeta_n^*:=\zeta^*(\omega_n,\rho_n)\underset{n\to\infty}{\longrightarrow}\zeta^{**}$$

converge to a unique Bayesian Nash equilibrium point and a unique incentive-compatible mechanism, such that $x^{**}\in\mathcal{X}^*$ is the solution of the original problem (27) with the minimal weighted norm given by

$$\|x^{**}\|\le\|x^*\|\quad \textit{for all } x^*\in\mathcal{X}^* \tag{35}$$

and

$$\zeta^{**}=b_1-\Phi_1x^{**}. \tag{36}$$

Proof. See Clempner and Poznyak (2018a; 2018b). ∎

Corollary 2. *The resulting mechanism is unique, and it is incentive compatible, i.e., it satisfies Eqn. (6).*

6. Numerical example: A supply chain network

6.1. Description of the supply chain problem. Consider a multinational firm with several subsidiaries (Maskin and Riley, 1984; Asian and Nie, 2014). We develop a supply chain network model where subsidiaries, denoted by $l\in\mathcal{N}$, are involved in the competitive production of a homogeneous product for multiple demand markets and compete in a noncooperative manner (Zeifman *et al.*, 2020; Khoury *et al.*, 2022). We consider the problem of selling prices and optimal production levels for a supply chain in the presence of asymmetric information related to local ownership requirements. The profit-maximizing divisions select independently both the capacities associated with producing as well as the product quantities.

The market requests a new product and the firm decided to invest in this new product. This new product involves several parts that can be produced by different subsidiaries, which have different capacities. In the supply chain, each subsidiary can take the actions a of either accept or reject to produce and deliver the intermediate product. In addition, the subsidiary decides about the produced and delivered quantity q.

The new product "coordinates" the independent divisions. Therefore, a price system combined with rules of the intra-firm information flow must be defined. The subsidiary l produces an intermediate product which is employed by the parent (l+1) to produce a finished product along a vertically integrated supply chain. The last division in the supply chain sells the final product. Let θ denote the subsidiaries' willingness to pay for an intermediate product.

A key feature of the supply chain subsidiaries model is that there is both asymmetric information and no intention to reveal the willingness to pay. In order to promote the persuasion process, we consider that the intermediate product is available on any outside market and can be sold there.

We define the cost function of the subsidiary as a combination of $c^l(a_t^l\theta_t^l m_t^l)$ and $\upsilon^l(m_t^l|a_t^l))$ that represents the quantity produced of the intermediate good.

The problem is presented in terms of the direct revelation mechanism (Myerson, 1983; Jackson, 2003), where the revelation game is played after a mechanism is defined by the mean of the combination of the functions $U^l(\mu(a_t|m_t),\ \sigma^l(\vartheta_t^l|\theta_t^l),\ p^l(m_t^l|\vartheta_t^l),\ P^l(\theta_t^l))$ and $\upsilon^l(m_t^l|a_t^l)$. Bear in mind that the former function, $U^l(\mu(a_t|m_t),\ \sigma^l(\vartheta_t^l|\theta_t^l),\ p^l(m_t^l|\vartheta_t^l),\ P^l(\theta_t^l))$ represents a monetary transfer from the parent to the subsidiary. The latter function represents the quantity level, $\upsilon^l(m_t^l|a_t^l)$, depending on the reported type m_t^l and the action a_t. The profit function of the subsidiary, denoted by π^l, is defined as

$$\begin{aligned}
&\pi^l(\mu(a_t|m_t)\sigma^l(\vartheta_t^l|\theta_t^l)p^l(m_t^l|\vartheta_t^l)P^l(\theta_t^l)\upsilon^l(m_t^l|a_t^l))\\
&= \Bigg\{ \sum_{\theta_t^l\in\Theta_t^l}\sum_{\vartheta_t^l\in\Theta_t^l}\sum_{m_t^l\in\Xi_t^l}\sum_{a_t^l\in A_t^l} W^l(a_t^l,\theta_t^l,m_t^l)\\
&\quad\times \prod_{\iota\in\mathcal{N}} \mu(a_t|m_t)\sigma^\iota(\vartheta_t^\iota|\theta_t^\iota)p^\iota(m_t^\iota|\vartheta_t^\iota)P^\iota(\theta_t^\iota)\Bigg\}\upsilon^l(m_t^l|a_t^l)\\
&\quad- \Bigg\{\sum_{\theta_t^l\in\Theta_t^l}\sum_{m_t^l\in\Xi_t^l}\sum_{a_t^l\in A_t^l} c^l(a_t^l\theta_t^l m_t^l)\Bigg\}\upsilon^l(m_t^l|a_t^l),
\end{aligned}$$

where

$$\Upsilon_{\text{adm}} := \Bigg\{ \upsilon^l(m_t^l|a_t^l) \,\Bigg|\, \sum_{m_t^l\in\Xi_t^l}\sum_{a_t^l\in A_t^l}\upsilon^l(m_t^l|a_t^l)\le \upsilon^+ \Bigg\}$$

such that $0 < \upsilon^+$ is the maximum quantity able to be produced. This represents the *gross profit* of the subsidiary who reports m_t^l for the observed type ϑ_t^l when its true type is θ_t^l.

Simplifying, we have

$$\begin{aligned}
&\pi^l(z^l(\theta_t^l\vartheta_t^l m_t^l a_t^l)q^l(m_t^l|a_t^l))\\
&= \Bigg\{ \sum_{\theta_t^l\in\Theta_t^l}\sum_{\vartheta_t^l\in\Theta_t^l}\sum_{m_t^l\in\Xi_t^l}\sum_{a_t^l\in A_t^l} W^l(a_t^l,\theta_t^l,m_t^l)\\
&\quad\times \prod_{\iota\in\mathcal{N}} z^\iota(\theta_t^\iota\vartheta_t^\iota m_t^\iota a_t^\iota)\Bigg\}\upsilon^l(m_t^l|a_t^l)\\
&\quad- \Bigg\{\sum_{\theta_t^l\in\Theta_t^l}\sum_{m_t^l\in\Xi_t^l}\sum_{a_t^l\in A_t^l} c^l(a_t^l\theta_t^l m_t^l)\Bigg\}\upsilon^l(m_t^l|a_t^l).
\end{aligned}$$

We will also define the profit function

$$\begin{aligned}
&\tilde{\pi}^l(z^l(\theta_t^l a_t^l)q^l(\theta_t^l|a_t^l))\\
&= \Bigg\{ \sum_{\theta_t^l\in\Theta_t^l}\sum_{a_t^l\in A_t^l} W^l(a_t^l,\theta_t^l)\prod_{\iota\in\mathcal{N}} z^\iota(\theta_t^\iota a_t^\iota)\Bigg\}\upsilon^l(\theta_t^l|a_t^l)\\
&\quad- \Bigg\{\sum_{\theta_t^l\in\Theta_t^l}\sum_{a_t^l\in A_t^l} c^l(a_t^l\theta_t^l)\Bigg\}\upsilon^l(\theta_t^l|a_t^l).
\end{aligned}$$

The dynamics of the game is as follows. As soon as the mechanism is suggested, the subsidiary makes a report of its type m_t^l for the observed type ϑ_t^l when its true type is θ_t^l; then the parent determines the quantity $\upsilon^l(m_t^l|a_t^l)$ to be produced. Next, production takes place and revenues and the intermediate products are transferred. The parent pay-off represents the reward obtained from the sale of $\upsilon^l(m_t^l|a_t^l)$ units of the intermediate product minus the costs (of the parent). The finished product is sold in a monopolistic manner by the final subsidiary in the supply chain on the open market, i.e., taking into account its effect on price. Without loss of generality, it can be supposed, that the quantity of the finished product equals the quantity of the intermediate good.

We will now develop the profit function of the parent shareholders. We suppose that they own a fraction a of the subsidiary. This means that they share, on a pro-rata basis, the revenues and bear the same share of costs. The profit of the parent is as follows

$$\begin{aligned}
&\pi^l(z^l(\theta_t^l\vartheta_t^l m_t^l a_t^l)\upsilon^l(m_t^l|a_t^l))\\
&= \Big\{ \sum_{\theta_t^l\in\Theta_t^l}\sum_{\vartheta_t^l\in\Theta_t^l}\sum_{m_t^l\in\Xi_t^l}\sum_{a_t^l\in A_t^l} W^l(a_t^l,\theta_t^l,m_t^l)\\
&\quad\times \prod_{\iota\in\mathcal{N}} z^\iota(\theta_t^\iota\vartheta_t^\iota m_t^\iota a_t^\iota)\\
&\quad\times W^{l-1}(a_t^{l-1},\theta_t^{l-1},m_t^{l-1})\\
&\quad\times \prod_{\iota\in\mathcal{N}} z^\iota(\theta_t^\iota\vartheta_t^\iota m_t^\iota a_t^\iota)\Big\}\upsilon^l(m_t^l|a_t^l)\\
&\quad- \Big\{\sum_{\theta_t^l\in\Theta_t^l}\sum_{m_t^l\in\Xi_t^l}\sum_{a_t^l\in A_t^l} c^l(a_t^l,\theta_t^l,m_t^l)\Big\}\upsilon^l(m_t^l|a_t^l),\\
&\quad l=2,\dots,n,
\end{aligned}$$

where $l = 1$ represents the first division and $l = 2,\dots,n$ the rest of the divisions on the vertically integrated supply chain such that

$$\begin{aligned}
&\pi^1(z^1(\theta_t^1\vartheta_t^1 m_t^1 a_t^1)\upsilon^1(m_t^1|a_t^1))\\
&= \Big\{ \sum_{\theta_t^1\in\Theta_t^1}\sum_{\vartheta_t^1\in\Theta_t^1}\sum_{m_t^1\in\Xi_t^1}\sum_{a_t^1\in A_t^1} W^1(a_t^1,\theta_t^l,m_t^1)
\end{aligned}$$

$$\times z^1(\theta_t^1\vartheta_t^1 m_t^1 a_t^1) - C^1\Big\}\upsilon^1(m_t^1|a_t^1)$$
$$-\Big\{\sum_{\theta_t^1\in\Theta_t^1}\sum_{m_t^1\in\Xi_t^1}\sum_{a_t^1\in A_t^1} c^1(a_t^1,\theta_t^1,m_t^1)\Big\}\upsilon^1(m_t^1|a_t^1),$$
$$l=1$$

given C^1 denotes acquisition and production costs (per unit/marginal costs).

6.2. Problem formulation. We now formulate the problem of designing a mechanism. We suppose that the players have all of the bargaining ability and can enforce the best ex-post outcome while respecting the players' opportunity of cost and private information. The mathematical statement of the problem is as follows:

$$\pi(\mu(a_t|m_t)\sigma(\vartheta_t|\theta_t)p(m_t|\vartheta_t)P(\theta_t)\upsilon(m_t|a_t)) \longrightarrow \max_{\mu\in\mathcal{U}_{\text{adm}},\sigma\in\mathcal{S}_{\text{adm}},\upsilon\in\Upsilon_{\text{adm}}} \tag{37}$$

subject to

$$\tilde{\pi}^l(z^l(\theta_t^l a_t^l)\upsilon^l(\theta_t^l|a_t^l)) \geq \pi^l(z^l(\theta_t^l\vartheta_t^l m_t^l a_t^l)\upsilon^l(m_t^l|a_t^l)), \tag{38}$$

$$\pi^l(z^l(\theta_t^l\vartheta_t^l m_t^l a_t^l)\upsilon^l(m_t^l|a_t^l)) \geq 0, \tag{39}$$

where

$$\pi(\mu(a_t|m_t)\sigma(\vartheta_t|\theta_t)p(m_t|\vartheta_t)P(\theta_t)\upsilon(m_t|a_t)) = \sum_{l\in\mathcal{N}}\pi^l(\mu(a_t|m_t)\sigma^l(\vartheta_t^l|\theta_t^l) \times p^l(m_t^l|\vartheta_t^l)P^l(\theta_t^l)\upsilon^l(m_t^l|a_t^l)).$$

The revelation principle states that the equilibrium of any Bayesian game of incomplete information can be implemented as an equilibrium of the direct revelation game in which players report the truth using an incentive-compatible mechanism (Myerson, 1981). The constraint of Eqn. (39) is the individual rationality or participation constraint, which determines that the parent should not lose from the arrangement. Here, the constraint on the problem reflects the fact the parent must prefer to buy the product with the subsidiary over an outside option. Notice that there is a point for an equal surplus for the subsidiary and the parent, where the constraint of individual rationality holds with equality,

$$\pi^l(z^l(\theta_t^l\vartheta_t^l m_t^l a_t^l)\upsilon^l(m_t^l|a_t^l)) = 0. \tag{40}$$

The economic interpretation of Eqn. (40) is that the subsidiary chooses a quantity where the marginal benefit to the parent equals the subsidiary's marginal cost of production. That is, the socially efficient quantity of the good is offered. This maximizes the size of the economic pie to be split between the parties. Having done this, the subsidiary then uses the transfer to capture all of this surplus.

In order to solve the problem given in Eqn. (37), we employ the proposed method for determining the incentive compatibility constraint. The following result of the mechanism design literature is given in the following lemma.

Lemma 4. *The mechanism $\mu(a_t|m_t)$ is incentive compatible, i.e., it satisfies Eqn. (37).*

Proof. It follows immediately from Theorem 3 that

$$\begin{aligned}\mathbb{E}\Big\{&\pi^{l*}(\mu^*(a_t|m_t)\sigma^{l*}(\vartheta_t^l|\theta_t^l) \\ &\times p^l(m_t^l|\vartheta_t^l)P^{l*}(\theta_t^l)q^{l*}(m_t^l|a_t^l))\Big\} \\ &\geq \mathbb{E}\Big\{\pi^l(\mu(a_t|m_t)\sigma^l(\vartheta_t^l|\theta_t^l) \\ &\times p^l(m_t^l|\vartheta_t^l)P^l(\theta_t^l)\upsilon^l(m_t^l|a_t^l))\Big\}.\end{aligned} \tag{41}$$

■

6.3. Numerical results. We consider a "downstream" production, so that intrafirm trade flows from the subsidiary to the parent. The constraint given in Eqn. (38) is the incentive compatibility or truth-telling constraint imposed on the mechanism designer (parent); in other words, Eqn. (38) says that a player must prefer to report her type truthfully than to reject the contract entirely. We suppose that the parent can enforce the best ex-ante outcome while respecting the subsidiary's opportunity cost and private information.

The convergence of the strategies $z^l(\theta_t^l\vartheta_t^l m_t^l a_t^l)$ are given in Figs. 1, 2 and 3. Applying Eqns. (13), (14) and (15), we have that the resulting Bayesian strategies $\sigma^l(\vartheta_t^l|\theta_t^l)$ are the following:

$$\sigma^{(1)*} = \begin{bmatrix} 0.5542 & 0.1537 & 0.2455 & 0.1631 \\ 0.1486 & 0.1537 & 0.2648 & 0.1631 \\ 0.1486 & 0.1537 & 0.2447 & 0.1631 \\ 0.1486 & 0.5388 & 0.2450 & 0.5108 \end{bmatrix},$$

$$\sigma^{(2)*} = \begin{bmatrix} 0.5295 & 0.1530 & 0.1914 & 0.1561 \\ 0.1568 & 0.1530 & 0.4299 & 0.1561 \\ 0.1568 & 0.1530 & 0.1894 & 0.1561 \\ 0.1568 & 0.5409 & 0.1894 & 0.5318 \end{bmatrix},$$

$$\sigma^{(3)*} = \begin{bmatrix} 0.2363 & 0.2531 & 0.2419 & 0.1591 \\ 0.2575 & 0.2532 & 0.2510 & 0.1561 \\ 0.2576 & 0.2533 & 0.2536 & 0.1599 \\ 0.2486 & 0.2405 & 0.2534 & 0.5249 \end{bmatrix},$$

and the distribution vectors $P^{l*}(m_t^l)$ for each player l are

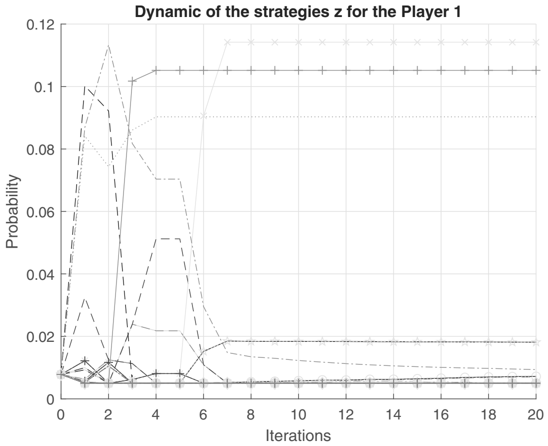

Fig. 1. Convergence of strategies $\sigma^{(1)*}(\vartheta|\vartheta)$ of Player 1.

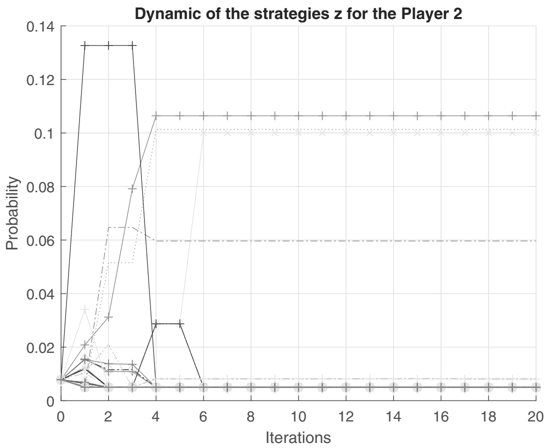

Fig. 2. Convergence of strategies $\sigma^{(2)*}(\vartheta|\vartheta)$ of Player 2.

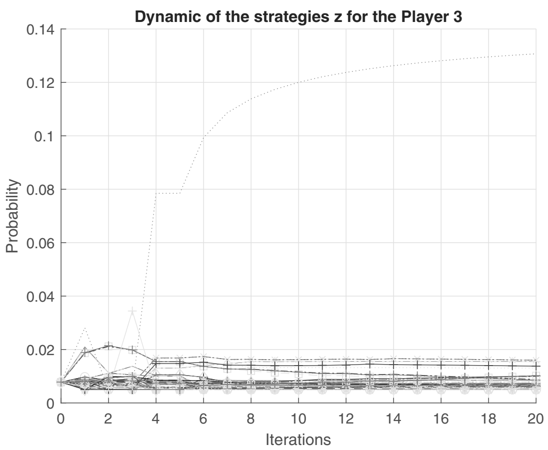

Fig. 3. Convergence of strategies $\sigma^{(3)*}(\vartheta|\vartheta)$ of Player 3.

given by

$$P^{(1)*} = \begin{bmatrix} 0.2126 \\ 0.2645 \\ 0.2537 \\ 0.2692 \end{bmatrix},$$

$$P^{(2)*} = \begin{bmatrix} 0.1600 \\ 0.3160 \\ 0.2563 \\ 0.2676 \end{bmatrix},$$

$$P^{(3)*} = \begin{bmatrix} 0.1816 \\ 0.2158 \\ 0.3005 \\ 0.3020 \end{bmatrix}.$$

The resulting mechanism is

$$\mu^*(a|m) = \begin{bmatrix} 0.4576 & 0.5424 \\ 0.3417 & 0.6583 \\ 0.6818 & 0.3182 \\ 0.6174 & 0.3826 \end{bmatrix}.$$

The convergence of the resulting quantities $\upsilon^l(m_t^l|a_t^l)$ is illustrated in Figs. 4, 5 and 6. The values of $\upsilon^l(m_t^l|a_t^l)$ are as follows:

$$\upsilon^{1*} = \begin{bmatrix} 10.9140 & 6.9828 \\ 8.2940 & 6.9235 \\ 5.3148 & 6.1103 \\ 6.5392 & 6.1096 \end{bmatrix},$$

$$\upsilon^{2*} = \begin{bmatrix} 6.1343 & 9.5842 \\ 6.1343 & 8.0331 \\ 7.1797 & 4.9834 \\ 6.8988 & 4.9828 \end{bmatrix},$$

$$\upsilon^{3*} = \begin{bmatrix} 6.1343 & 9.5842 \\ 6.1343 & 8.0331 \\ 7.1797 & 4.9834 \\ 6.8988 & 4.9828 \end{bmatrix}.$$

7. Conclusion

This paper presented an analytical method for computing incentive-compatible mechanisms for a class of controllable Markov games. For solving the problem, a new variable is considered that represents the product of the mechanism design, the strategy and the distribution vector. This variable makes the problem computationally tractable. We derive relations to analytically compute the variables of interest: the mechanism, the strategies and the distribution vector. We employed the notion of Bayesian Nash equilibrium as the equilibrium concept for our game. We also introduce a regularization parameter based on Tikhonov's approach for ensuring the convergence to a unique equilibrium point. We also show the convergence

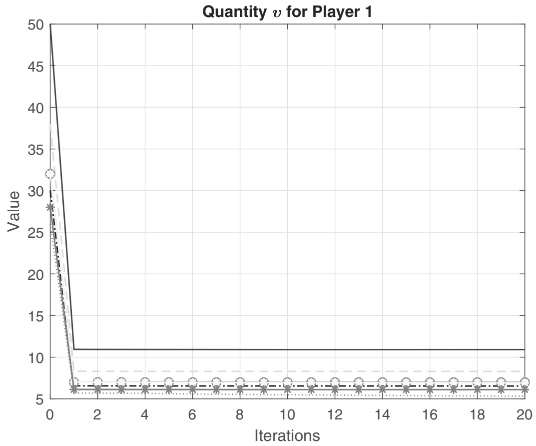

Fig. 4. Convergence of quantities $v^{1*}(m|a)$ of Player 1.

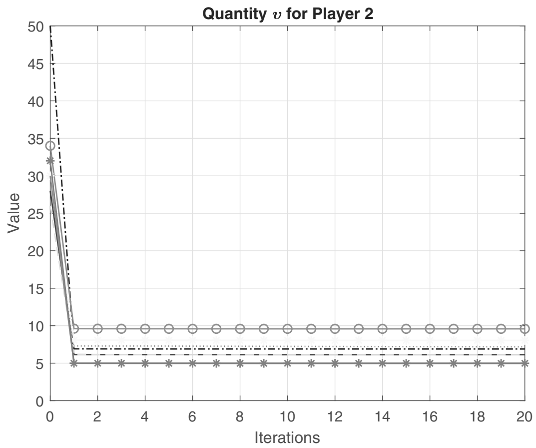

Fig. 5. Convergence of quantities $v^{2*}(m|a)$ of Player 2.

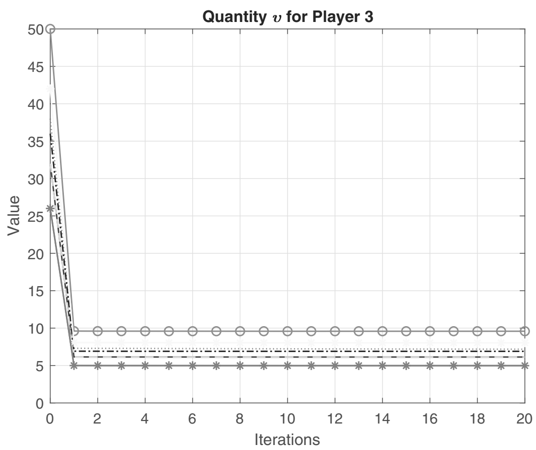

Fig. 6. Convergence of quantities $v^{3*}(m|a)$ of Player 3.

to a unique incentive-compatible mechanism and to a unique Bayesian Nash equilibrium of the game. The proposed approach is validated by a numerical example.

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Appendix

A1. Proof of Lemma 1

Observe that $\mu^*(a_t|m_t)$ can be obtained from Eqns. (11) and (12) as follows:

$$\begin{aligned}\sum_{l\in\mathcal{N}}\sum_{\theta_t^l\in\Theta_t^l}\sum_{\vartheta_t^l\in\Theta_t^l} & z^{l*}(\theta_t^l,\vartheta_t^l,m_t^l,a_t^l)\\ &:= \mu^*(a_t|m_t)\sum_{l\in\mathcal{N}}\sum_{\theta_t^l\in\Theta_t^l}\sum_{\vartheta_t^l\in\Theta_t^l}\sigma^{l*}(\vartheta_t^l|\theta_t^l)\\ &\qquad\times p^{l*}(m_t^l|\vartheta_t^l)P^{l*}(\theta_t^l).\end{aligned}$$

Hence

$$\mu^*(a_t|m_t) = \frac{\sum\limits_{l\in\mathcal{N}}\sum\limits_{\theta_t^l\in\Theta_t^l}\sum\limits_{\vartheta_t^l\in\Theta_t^l} z^{l*}(\theta_t^l,\vartheta_t^l,m_t^l,a_t^l)}{\sum\limits_{l\in\mathcal{N}}\sum\limits_{\theta_t^l\in\Theta_t^l}\sum\limits_{\vartheta_t^l\in\Theta_t^l}\sum\limits_{a_t^l\in A_t^l} z^{l*}(\theta_t^l,\vartheta_t^l,m_t^l,a_t^l)}. \tag{A1}$$

A2. Proof of Theorem 1

This means that the new variables $z^{l*}(\alpha_t^l\kappa_t^l\beta_t^l\gamma_t^l)$ should satisfy the following linear ergodicity constraints.

We prove the relation given in Eqn. (16) as follows:

$$\begin{aligned}P^{l*}(\theta_{t+1}^l) &= \sum_{\alpha_t^l\in\Theta_t^l}\Bigg\{\sum_{\kappa_t^l\in\Theta_t^l}\sum_{\beta_t^l\in\Xi_t^l}\sum_{\gamma_t^l\in A_t^l} p^l(\theta_{t+1}^l|\alpha_t^l\gamma_t^l)\\ &\qquad\times z^{l*}(\alpha_t^l\kappa_t^l\beta_t^l\gamma_t^l)\Bigg\}.\end{aligned}$$

Now,

$$\begin{aligned}\sum_{\theta_{t+1}^l\in\Theta_t^l}&\Big\{\sum_{\kappa_t^l\in\Theta_t^l}\sum_{\beta_t^l\in\Xi_t^l}\sum_{\gamma_t^l\in A_t^l} p^l(\theta_{t+1}^l|\alpha_t^l\gamma_t^l)\\ &\qquad\times z^{l*}(\theta_{t+1}\kappa_t^l\beta_t^l\gamma_t^l)\Big\}\\ &= \sum_{\alpha_t^l\in\Theta_t^l}\Big\{\sum_{\kappa_t^l\in\Theta_t^l}\sum_{\beta_t^l\in\Xi_t^l}\sum_{\gamma_t^l\in A_t^l} p^l(\theta_{t+1}^l|\alpha_t^l\gamma_t^l)\\ &\qquad\times z^{l*}(\alpha_t^l\kappa_t^l\beta_t^l\gamma_t^l)\Big\},\end{aligned}$$

which implies

$$\sum_{\alpha_t^l\in\Theta_t^l}\sum_{\kappa_t^l\in\Theta_t^l}\sum_{\beta_t^l\in\Xi_t^l}\sum_{\gamma_t^l\in A_t^l}[\delta_{\alpha_t^l\theta_{t+1}^l}-p^l(\theta_{t+1}^l|\alpha_t^l\gamma_t^l)]\times z^{l*}(\alpha_t^l\kappa_t^l\beta_t^l\gamma_t^l)=0,\quad \theta_{t+1}^l\in\Theta_t^l.$$

$$z^{l*}\in\Delta^l := \Big\{z^{l*}(\alpha_t^l\kappa_t^l\beta_t^l\gamma_t^l)\,\Big|\,\sum_{\alpha_t^l\in\Theta_t^l}\sum_{\kappa_t^l\in\Theta_t^l}\sum_{\beta_t^l\in\Xi_t^l}\sum_{\gamma_t^l\in A_t^l}[\delta_{\alpha_t^l\theta_{t+1}^l}-p^l(\theta_{t+1}^l|\alpha_t^l\gamma_t^l)]\times z^{l*}(\alpha_t^l\kappa_t^l\beta_t^l\gamma_t^l),\ \theta_{t+1}^l\in\Theta_t^l\Big\}.\quad (A2)$$

Equation (17) is fulfilled automatically since

$$\sum_{\varrho_t^l\in\Xi_t^l}\sum_{\alpha_t^l\in\Theta_t^l}\sum_{\kappa_t^l\in\Theta_t^l}\sum_{\gamma_t^l\in A_t^l}[\delta_{\varrho_t^l\beta_t^l}-p^l(\beta_t^l|\kappa_t^l)]\times z^{l*}(\alpha_t^l\kappa_t^l\varrho_t^l\gamma_t^l) = \sum_{\varrho_t^l\in\Xi_t^l}\sum_{\kappa_t^l\in\Theta_t^l}[\delta_{\varrho_t^l\beta_t^l}-p^l(\beta_t^l|\kappa_t^l)]\times\sum_{\alpha_t^l\in\Theta_t^l}\sum_{\gamma_t^l\in A_t^l}z^{l*}(\alpha_t^l\kappa_t^l\varrho_t^l\gamma_t^l).$$

We have that

$$\sum_{\alpha_t^l\in\Theta_t^l}\sum_{\gamma_t^l\in A_t^l}z^{l*}(\alpha_t^l\kappa_t^l\varrho_t^l\gamma_t^l) = \sum_{\alpha_t^l\in\Theta_t^l}\sum_{\gamma_t^l\in A_t^l}\mu(\gamma_t^l|\varrho_t^l)\sigma^l(\kappa_t^l|\alpha_t^l)p^l(\varrho_t^l|\kappa_t^l)P^l(\alpha_t^l) = p^l(\varrho_t^l|\kappa_t^l)\sum_{\alpha_t^l\in\Theta_t^l}\sum_{\gamma_t^l\in A_t^l}\mu(\gamma_t^l|\varrho_t^l)\sigma^l(\kappa_t^l|\alpha_t^l)P^l(\alpha_t^l)$$

and

$$\sum_{l\in\mathcal N}\sum_{\alpha_t^l\in\Theta_t^l}\sum_{\kappa_t^l\in\Theta_t^l}\sum_{\gamma_t^l\in A_t^l}z^{l*}(\alpha_t^l\kappa_t^l\varrho_t^l\gamma_t^l) = \sum_{l\in\mathcal N}\sum_{\alpha_t^l\in\Theta_t^l}\sum_{\gamma_t^l\in A_t^l}\mu(\gamma_t^l|\varrho_t^l)\times\left(\sum_{\kappa_t^l\in\Theta_t^l}\sigma^l(\kappa_t^l|\alpha_t^l)\right)p^l(\beta_t^l|\kappa_t^l)P^l(\alpha_t^l) = p^l(\varrho_t^l|\kappa_t^l)\sum_{l\in\mathcal N}\sum_{\alpha_t^l\in\Theta_t^l}\sum_{\gamma_t^l\in A_t^l}\mu(\gamma_t^l|\varrho_t^l)P^l(\alpha_t^l)$$

Then

$$\sum_{l\in\mathcal N}\sum_{\alpha_t^l\in\Theta_t^l}\sum_{\kappa_t^l\in\Theta_t^l}\sum_{\gamma_t^l\in A_t^l}z^{l*}(\alpha_t^l\kappa_t^l\varrho_t^l\gamma_t^l)\times\frac{\sum_{\alpha_t^l\in\Theta_t^l}\sum_{\gamma_t^l\in A_t^l}z^{l*}(\alpha_t^l\kappa_t^l\varrho_t^l\gamma_t^l)}{\sum_{l\in\mathcal N}\sum_{\alpha_t^l\in\Theta_t^l}\sum_{\kappa_t^l\in\Theta_t^l}\sum_{\gamma_t^l\in A_t^l}z^{l*}(\alpha_t^l\kappa_t^l\varrho_t^l\gamma_t^l)}$$

$$= \sum_{l\in\mathcal N}\sum_{\alpha_t^l\in\Theta_t^l}\sum_{\kappa_t^l\in\Theta_t^l}\sum_{\gamma_t^l\in A_t^l}z^{l*}(\alpha_t^l\kappa_t^l\varrho_t^l\gamma_t^l)\times\sum_{\alpha_t^l\in\Theta_t^l}\sum_{\gamma_t^l\in A_t^l}\mu(\gamma_t^l|\varrho_t^l)\sigma^l(\kappa_t^l|\alpha_t^l)p^l(\varrho_t^l|\kappa_t^l)P^l(\alpha_t^l)\times\Big\{\sum_{l\in\mathcal N}\sum_{\alpha_t^l\in\Theta_t^l}\sum_{\gamma_t^l\in A_t^l}\mu(\gamma_t^l|\varrho_t^l)\times\Big(\sum_{\kappa_t^l\in\Theta_t^l}\sigma^l(\kappa_t^l|\alpha_t^l)\Big)p^l(\varrho_t^l|\kappa_t^l)P^l(\alpha_t^l)\Big\}^{-1}$$

$$= \sum_{l\in\mathcal N}\sum_{\alpha_t^l\in\Theta_t^l}\sum_{\kappa_t^l\in\Theta_t^l}\sum_{\gamma_t^l\in A_t^l}z^{l*}(\alpha_t^l\kappa_t^l\varrho_t^l\gamma_t^l)\times\frac{p^l(\varrho_t^l|\kappa_t^l)\sum_{\alpha_t^l\in\Theta_t^l}\sum_{\gamma_t^l\in A_t^l}\mu(\gamma_t^l|\varrho_t^l)\sigma^l(\kappa_t^l|\alpha_t^l)P^l(\alpha_t^l)}{p^l(\varrho_t^l|\kappa_t^l)\sum_{l\in\mathcal N}\sum_{\alpha_t^l\in\Theta_t^l}\sum_{\gamma_t^l\in A_t^l}\mu(\gamma_t^l|\varrho_t^l)P^l(\alpha_t^l)}$$

$$= \sum_{l\in\mathcal N}\sum_{\alpha_t^l\in\Theta_t^l}\sum_{\kappa_t^l\in\Theta_t^l}\sum_{\gamma_t^l\in A_t^l}z^{l*}(\alpha_t^l\kappa_t^l\varrho_t^l\gamma_t^l)\times\frac{\sum_{\alpha_t^l\in\Theta_t^l}\sum_{\gamma_t^l\in A_t^l}\mu(\gamma_t^l|\varrho_t^l)\sigma^l(\kappa_t^l|\alpha_t^l)P^l(\alpha_t^l)}{\sum_{l\in\mathcal N}\sum_{\alpha_t^l\in\Theta_t^l}\sum_{\gamma_t^l\in A_t^l}\mu(\gamma_t^l|\varrho_t^l)P^l(\alpha_t^l)}$$

$$= \sum_{l\in\mathcal N}\sum_{\alpha_t^l\in\Theta_t^l}\sum_{\gamma_t^l\in A_t^l}\mu(\gamma_t^l|\varrho_t^l)\left(\sum_{\kappa_t^l\in\Theta_t^l}\sigma^l(\kappa_t^l|\alpha_t^l)\right)p^l(\varrho_t^l|\kappa_t^l)\times P^l(\alpha_t^l)\frac{\sum_{\alpha_t^l\in\Theta_t^l}\sum_{\gamma_t^l\in A_t^l}\mu(\gamma_t^l|\varrho_t^l)\sigma^l(\kappa_t^l|\alpha_t^l)P^l(\alpha_t^l)}{\sum_{l\in\mathcal N}\sum_{\alpha_t^l\in\Theta_t^l}\sum_{\gamma_t^l\in A_t^l}\mu(\gamma_t^l|\varrho_t^l)P^l(\alpha_t^l)}$$

$$= p^l(\varrho_t^l|\kappa_t^l)\sum_{l\in\mathcal N}\sum_{\alpha_t^l\in\Theta_t^l}\sum_{\gamma_t^l\in A_t^l}\mu(\gamma_t^l|\varrho_t^l)P^l(\alpha_t^l)\times\frac{\sum_{\alpha_t^l\in\Theta_t^l}\sum_{\gamma_t^l\in A_t^l}\mu(\gamma_t^l|\varrho_t^l)\sigma^l(\kappa_t^l|\alpha_t^l)P^l(\alpha_t^l)}{\sum_{l\in\mathcal N}\sum_{\alpha_t^l\in\Theta_t^l}\sum_{\gamma_t^l\in A_t^l}\mu(\gamma_t^l|\varrho_t^l)P^l(\alpha_t^l)}$$

$$= p^l(\varrho_t^l|\kappa_t^l)\sum_{\alpha_t^l\in\Theta_t^l}\sum_{\gamma_t^l\in A_t^l}\mu(\gamma_t^l|\varrho_t^l)\sigma^l(\kappa_t^l|\alpha_t^l)P^l(\alpha_t^l) = p^l(\varrho_t^l|\kappa_t^l)\sigma^l(\kappa_t^l|\alpha_t^l).$$

As a result,

$$\sum_{\varrho_t^l\in\Xi_t^l}\sum_{\kappa_t^l\in\Theta_t^l}[\delta_{\varrho_t^l\beta_t^l}-p^l(\beta_t^l|\kappa_t^l)]p^l(\varrho_t^l|\kappa_t^l)\sigma^l(\kappa_t^l|\alpha_t^l) = \sum_{\varrho_t^l\in\Xi_t^l}\sum_{\kappa_t^l\in\Theta_t^l}\delta_{\varrho_t^l\beta_t^l}p^l(\varrho_t^l|\kappa_t^l)\sigma^l(\kappa_t^l|\alpha_t^l) - \sum_{\varrho_t^l\in\Xi_t^l}\sum_{\kappa_t^l\in\Theta_t^l}p^l(\beta_t^l|\kappa_t^l)p^l(\varrho_t^l|\kappa_t^l)\sigma^l(\kappa_t^l|\alpha_t^l)$$

$$= \sum_{\kappa_t^l \in \Theta_t^l} p^l(\beta_t^l|\kappa_t^l)\sigma^l(\kappa_t^l|\alpha_t^l) - \sum_{\kappa_t^l \in \Theta_t^l} p^l(\beta_t^l|\kappa_t^l)\sigma^l(\kappa_t^l|\alpha_t^l) = 0.$$

Now, we prove the relation given in Eqn. (18). We have that $Q^l = \left[p^l(m_t^l|\vartheta_t^l)\right]^{-1}$. Then, we get that for any $\rho_t^l \in \Theta$ we have

$$\begin{aligned}
&\sum_{\alpha_t^l \in \Theta_t^l} \sum_{\kappa_t^l \in \Theta_t^l} \sum_{\beta_t^l \in \Xi_t^l} \sum_{\gamma_t^l \in A_t^l} Q^l(\beta_t^l|\rho_t^l) z^{l*}(\alpha_t^l \kappa_t^l \beta_t^l \gamma_t^l) \\
&= \sum_{\alpha_t^l \in \Theta_t^l} \sum_{\kappa_t^l \in \Theta_t^l} \sum_{\beta_t^l \in \Xi_t^l} Q^l(\beta_t^l|\rho_t^l)\sigma^{l*}(\kappa_t^l|\alpha_t^l) p^l(\beta_t^l|\kappa_t^l) \\
&\quad \times P^{l*}(\alpha_t^l) \sum_{\gamma_t^l \in A_t^l} \mu^*(\gamma_t|\beta_t) \\
&= \sum_{\alpha_t^l \in \Theta_t^l} \sum_{\kappa_t^l \in \Theta_t^l} \sigma^{l*}(\kappa_t^l|\alpha_t^l) P^{l*}(\alpha_t^l) \\
&\quad \times \sum_{\beta_t^l \in \Xi_t^l} Q^l(\beta_t^l|\rho_t^l) p^l(\beta_t^l|\kappa_t^l) \\
&= \sum_{\alpha_t^l \in \Theta_t^l} \sum_{\kappa_t^l \in \Theta_t^l} \sigma^{l*}(\kappa_t^l|\alpha_t^l) P^{l*}(\alpha_t^l) \delta_{\rho_t^l \kappa_t^l} \\
&= \sum_{\alpha_t^l \in \Theta_t^l} \sigma^{l*}(\rho_t^l|\alpha_t^l) P^{l*}(\alpha_t^l) \geq 0._t^l.
\end{aligned}$$

The theorem is proved.

A3. Proof of Lemma 3

Let us show that the Hessian matrix $\mathbb{H}$ associated with the penalty function given in Eqn. (28) is strictly positive definite for any positive ω and ρ. We have to prove that for any $x \in X_{\text{adm}}$ and $\zeta \geq 0$

$$\mathbb{H} = \begin{bmatrix} \frac{\partial^2}{\partial x^2}\Psi_{\omega,\rho}(x,\zeta) & \frac{\partial^2}{\partial \zeta \partial x}\Psi_{\omega,\rho}(x,\zeta) \\ \frac{\partial^2}{\partial x \partial \zeta}\Psi_{\omega,\rho}(x,\zeta) & \frac{\partial^2}{\partial \zeta^2}\Psi_{\omega,\rho}(x,\zeta) \end{bmatrix} > 0. \tag{A3}$$

For showing that Eqn. (A3) is true, in accordance with Schur's lemma, it is necessary and sufficient to prove only that

$$\begin{aligned}
&\frac{\partial^2}{\partial x^2}\Psi_{\omega,\rho}(x,\zeta) > 0, \\
&\frac{\partial^2}{\partial \zeta^2}\Psi_{\omega,\rho}(x,\zeta) > 0, \\
&\frac{\partial^2}{\partial x^2}\Psi_{\omega,\rho}(x,\zeta) > \frac{\partial^2}{\partial \zeta \partial x}\Psi_{\omega,\rho}(x,\zeta) \\
&\qquad \times \left[\frac{\partial^2}{\partial \zeta^2}\Psi_{\omega,\rho}(x,\zeta)\right]^{-1} \\
&\qquad \times \frac{\partial^2}{\partial x \partial \zeta}\Psi_{\omega,\rho}(x,\zeta).
\end{aligned} \tag{A4}$$

Hence, we have

$$\begin{aligned}
&\frac{\partial^2}{\partial x^2}\Psi_{\omega,\rho}(x,\zeta) \\
&= \omega \frac{\partial^2}{\partial x^2}\mathcal{F}(x) + \Phi_0^{\intercal}\Phi_0 + \Phi_1^{\intercal}\Phi_1 + \rho I_{N\times N} \\
&\geq \omega \frac{\partial^2}{\partial x^2}\mathcal{F}(x) + \rho I_{N\times N} \\
&\geq \rho \left(1 + \frac{\omega}{\rho}\lambda^-\right) I_{N\times N} > 0, \quad \forall\, \rho_n > 0,
\end{aligned}$$

$$\lambda^- := \min_{x \in X_{\text{adm}}} \lambda_{\min}\left(\frac{\partial^2}{\partial x^2}\mathcal{F}(x)\right),$$

$$\frac{\partial^2}{\partial \zeta^2}\Psi_{\omega,\rho}(x,\zeta) = (1+\rho)\, I_{M_1\times M_1} > 0.$$

Then, we need to satisfy

$$\begin{aligned}
&\frac{\partial^2}{\partial x^2}\Psi_{\omega,\rho}(x,\zeta) \\
&= \omega \frac{\partial^2}{\partial x^2}\mathcal{F}(x) + \Phi_0^{\intercal}\Phi_0 + \Phi_1^{\intercal}\Phi_1 + \rho I_{N\times N} \\
&> \frac{\partial^2}{\partial \zeta \partial x}\Psi_{\omega,\rho}(x,\zeta)\left[\frac{\partial^2}{\partial \zeta^2}\Psi_{\omega,\rho}(x,\zeta)\right]^{-1} \\
&\quad \times \frac{\partial^2}{\partial x \partial \zeta}\Psi_{\omega,\rho}(x,\zeta) \\
&= (1+\rho)^{-1}\,\Phi_1^{\intercal}\Phi_1,
\end{aligned}$$

or in an equivalent manner,

$$\omega \frac{\partial^2}{\partial x^2}\mathcal{F}(x) + \Phi_0^{\intercal}\Phi_0 + \frac{\rho}{1+\rho}\Phi_1^{\intercal}\Phi_1 + \rho I_{N\times N} > 0,$$

which is fulfilled for any $\rho > 0$; then

$$\begin{aligned}
&\left(\omega\lambda^- + \rho\right) I_{N\times N} + \Phi_0^{\intercal}\Phi_0 + \tfrac{\rho}{1+\rho}\Phi_1^{\intercal}\Phi_1 \\
&\geq \rho\left(1 + \frac{\omega}{\rho}\lambda^-\right) I_{N\times N} \\
&= \rho\,(1 + o(1))\, I_{N\times N} > 0
\end{aligned}$$

As a result, we have that $\mathbb{H} > 0$, which means that the penalty function in Eqn. (28) is strongly concave and it has a unique maximal point.

A4. Proof of Theorem 2

By the strictly convexity property showed in Lemma 3 for any $y := (x,\zeta)^{\intercal}$ and for any vector such that $y_n^* := (x_n^* = x^*(\omega_n,\rho_n), \zeta_n^* = \zeta^*(\omega_n,\rho_n))^{\intercal}$ for the function $\Psi_{\omega,\rho}(x,\zeta) = \Psi_{\omega,\rho}(y)$ we have

$$\begin{aligned}
&(y_n^* - y)^{\intercal}\frac{\partial}{\partial y}\Psi_{\omega_n,\rho_n}(y_n^*) \\
&= (x_n^* - x)^{\intercal}\frac{\partial}{\partial x}\Psi_{\omega_n,\rho_n}(x_n^*,\zeta_n^*) \\
&\quad + (\zeta_n^* - \zeta)^{\intercal}\frac{\partial}{\partial \zeta}\Psi_{\omega_n,\rho_n}(x_n^*,\zeta_n^*).
\end{aligned} \tag{A5}$$

Selecting in (A5) $x := x^* \in \mathcal{X}^*$ (x^* is one of admissible solutions such that $\Phi_0 x^* = b_0$) and $\zeta := (1+\rho)^{-1}(b_1 - \Phi_1 x_n^*)$, we obtain

$$\begin{aligned}0 \geq\ & \omega_n (x_n^* - x^*)^\intercal \frac{\partial}{\partial x}\mathcal{F}(x_n^*) + \|\Phi_0 (x_n^* - x^*)\|^2 \\ & + \|\Phi_1 (x_n^* - x^*)\|^2 + \rho_n (x_n^* - x^*)^\intercal x_n^* \\ & + (1+\rho)^{-1} \|\Phi_1 x_n^* - b_1 + (1+\rho)\zeta_n^*\|^2 \\ & + \rho_n (\zeta_n^* - b_1 - \Phi_1 x_n^*)^\intercal \zeta_n^*.\end{aligned}$$

Dividing both the sides of this inequality by ρ_n, we get

$$\begin{aligned}0 \geq\ & \frac{\omega_n}{\rho_n}(x_n^* - x^*)^\intercal \frac{\partial}{\partial x}\mathcal{F}(x_n^*) \\ & + \frac{1}{\rho_n}\Big(\|\Phi_0 x_n^* - b_0\|^2 + \|\Phi_1 (x_n^* - x^*)\|^2 \\ & + \|\Phi_1 x_n^* - b_1 + (1+\rho)\zeta_n^*\|^2\Big) \\ & + (x_n^* - x^*)^\intercal x_n^* + (\zeta_n^* - b_1 - \Phi_1 x_n^*)^\intercal \zeta_n^*.\end{aligned} \tag{A6}$$

Notice also that from (A5), taking $x = x_n^*$ and $\zeta = 0$, it follows that

$$\begin{aligned}0 &\geq (\zeta_n^*)^\intercal (\Phi_1 x_n^* - b_1 + (1+\rho)\zeta_n^*) \\ &= (\zeta_n^*)^\intercal (\Phi_1 x_n^* - b_1) + (1+\rho)\|\zeta_n^*\|^2 \\ &= \left[\left\|\sqrt{1+\rho}\,\zeta_n^* + \frac{(\Phi_1 x_n^* - b_1)}{2\sqrt{1+\rho}}\right\|^2 \right. \\ &\quad \left. - \left\|\frac{(\Phi_1 x_n^* - b_1)}{2\sqrt{1+\rho}}\right\|^2\right],\end{aligned}$$

implying

$$1 \geq \left\|e + 2(1+\rho)\zeta_n^* \|(\Phi_1 x_n^* - b_1)\|^{-1}\right\|^2,$$

$\|e\| = 1,$

which means that the sequence $\{\zeta_n^*\}$ is bounded. In view of this and taking into account that by (32)

$$\frac{\omega_n}{\rho_n} \underset{n\to\infty}{\longrightarrow} 0,$$

from (A6) it follows that

$$\begin{aligned}\text{Const} &= \limsup_{n\to\infty} \big(|(x_n^* - x^*)^\intercal x_n^*| \\ &\quad + |(\zeta_n^* - b_1 - \Phi_1 x_n^*)^\intercal \zeta_n^*|\big) \\ &\geq \limsup_{n\to\infty} \frac{1}{\rho_n}\Big(\|\Phi_0 x_n^* - b_0\|^2 + \|\Phi_1 (x_n^* - x^*)\|^2 \\ &\quad + (1+\rho_n)^{-1}\|\Phi_1 x_n^* - b_1 + (1+\rho_n)\zeta_n^*\|^2\Big).\end{aligned} \tag{A7}$$

From (A7) we can conclude that

$$\begin{aligned}&\|\Phi_0 x_n^* - b_0\|^2 + \|\Phi_1 (x_n^* - x^*)\|^2 \\ &\quad + (1+\rho_n)^{-1}\|\Phi_1 x_n^* - b_1 + (1+\rho_n)\zeta_n^*\|^2 \\ &\qquad = O(\rho_n)\end{aligned} \tag{A8}$$

and

$$\begin{aligned}\Phi_0 x_\infty^* - b_0 &= 0, \\ \Phi_1 x_\infty^* - \Phi_1 x^* &= \Phi_1 x_\infty^* - b_1 + \zeta_\infty^* = 0\end{aligned}$$

where $x_\infty^* \in \mathcal{X}^*$ is a partial limit of the sequence $\{x_n^*\}$, which may be not unique. The vector ζ_∞^* is also a partial limit of the sequence $\{\zeta_n^*\}$.

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