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A new strongly predefined time sliding mode controller for a class of cascade high-order nonlinear systems

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Many real-time systems can be described as cascade space-state models of different orders. In this paper, a new predefined controller is designed using a Strongly Predefined Time Sliding Mode Control (SPSMC) scheme for a cascade high-order nonlinear system. The proposed control scheme based-on SMC methodology is designed such that the system states reach zero within a determined time prior to performing numerical simulation. Moreover, Fixed Time Sliding Mode Control (FSMC) and Terminal Sliding Mode Control (TSMC) schemes are presented and simulated to provide a comparison with the proposed predefined time scheme. The numerical simulation is performed in Simulink/MATLAB for the proposed SPSMC and the other two existing methods on two examples: second and of third order to demonstrate the effectiveness of the proposed SPSMC method. The trajectory tracking of the ship course system is addressed as an example of a second-order system. Synchronization of two chaotic systems, Genesio Tesi and Coullet, is considered as an example of a third-order system. Also, by using two performance criteria, a thorough comparison is made between the proposed predefined time scheme, SPSMC, and the two no predefined time schemes, FSMC and TSMC.

Key words: predefined time, stability, sliding mode, trajectory, synchronization

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1. Introduction

Sliding Mode Control (SMC) method is a well-known nonlinear control method. Many control methods have been presented based-on this control scheme. This method has been used, combined with other control concepts such as fuzzy logic [1–3], backstepping [4], adaptive concept [5–7] in different studies. This control method has also been also used to address different real-time control problems. In [8], a combination of optimization control method, adaptive concept, and SMC method has been proposed to control the photovoltaic system in the smart grid system. In [9], the event-triggered control is addressed for fuzzy systems by utilizing observer-based SMC method. In [10], an adaptive robust finite-time controller has been proposed based-on SMC method to address chaos control. In [11], an adaptive backstepping SMC method has been proposed to develop an equilibrium position controller for an electrohydraulic elastic manipulator.

Promoting the concepts of stability has been the focus of much research for many years. Initially, the concepts of finite-time stability have been introduced by Bahat et al. [12] and used for different systems [13]. Subsequently, the fixed-time stability concept has been presented [14]. Recently, predefined time stability concepts have been introduced which has attracted the attention of researchers in different fields [15]. Different systems have been controlled by utilizing predefined time stability concepts [16–18].

By combining the predefined time stability concept and SMC method, a predefined SMC scheme has been introduced. This method is able to ensure system stability within a determined time prior to system control [19, 20]. Also, this method provides a robust control methodology to control those systems which are challenging to control using direct Lyapunov theorem. Indeed, SMC methodology provides a two-step control method such that it leads the system states to reach and remain zero; after ensuring the system convergence to a specific sliding surface. The combination of the SMC method and predefined time stability concepts are also used to control a variety of systems [21–24]. In [23], a novel controller based-on predefined time SMC method has been presented practically for an electro-pneumatic actuator.

Two concepts very closely related to each other have been presented as weakly predefined time and strongly predefined time stability, that researchers in the fields of control engineering use them according to the application. In the weakly predefined time method, the upper bound of the system stability time is presented. Also, in the strongly predefined time method, the stability time is clearly and precisely specified.

In this paper, a combination of the SMC method and predefined time stability concept is proposed based-on a strongly predefined time concept for a class of cascade high-order systems. the existing FSMC and TSMC methods are presented and simulated to provide no predefined time methods for comparing with the proposed predefined time method and to demonstrate the effectiveness of the predefined stability concept in the proposed SPSMC method. Furthermore, trajectory tracking and synchronization are addressed in the first and second examples, respectively, to reveal the proposed control method is proper and adjustable for different control problems. Our proposed controller can be used for a huge range of real-time applications, because the presented mathematical model of cascade high-order systems in this paper can describe many real-time systems. For example, it can describe the systems such as all double integrator systems such as ship course system [25], MEMS system [26], and Quadrotor [27], as well as other types of physical systems with cascade dynamics.

The remaining of this paper is organized as follows. In the second section, the mathematical preliminaries required for our proof are given. Then, the problem formulation is presented. In the third section, controller design by using SPSMC is given. Also, the other two no predefined time methods, FSMC and TSMC, are given. Section four provides two simulation examples with different control problems including trajectory tracking, and chaotic synchronization. The first numerical simulation example is considered for trajectory tracking of a cascade second-order (double integrator) of the ship course system. The second simulation example is considered for the synchronization of two chaotic Genesio Tesi and Coullet systems as cascade third-order systems. The last section is the conclusions of this paper.

2. Preliminaries

2.1. Fundamental definitions

Some standard definitions and lemmas are provided here that are used throughout the paper. Note that throughout the paper, the dot means differential with respect to time, and *sign* function signifies the signum function. Consider a system

$$\dot{x} = f(x; a), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state of the system and $f\colon \mathbb{R}^n \to \mathbb{R}^n$ and $a \in \mathbb{R}^b$ are the system parameters. Also, the system initial conditions are as $x(0) = x_0$.

Definition 1 *If the origin of Eq. (1) has global asymptotic stability as well as any solution $x(t, x_0)$ of (1) converges to the equilibrium point in a finite time, i.e. $\forall\, t \geqslant T(x_0)\colon x(t, x_0) = 0$, where $T\colon \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ is said settling time function, the origin of Eq. (1) has global finite-time stability [28].*

Definition 2 *If the origin of Eq. (1) has Finite-Time stability as well as the settling time function is bounded, i.e. $\exists\, T_{\max} \geqslant 0\colon \forall\, x_0\colon T(x_0) \leqslant T_{\max}$, then origin has fixed-time stability [29].*

Definition 3 *For the Fixed-Time stability when the parameters a of the system of Eq. (1) could be expressed in terms of bound of the settling time function $T_{\max}$, it is said the origin of Eq. (1) has predefined-time stability [30].*

Definition 4 *For the system parameters a and a non-empty set $M \subset \mathbb{R}^n$ is called globally strongly predefined-time attractive for Eq. (1), if any solution $x(t, x_0)$ converges to M at a finite time $t = t_0 + T(x_0)$, where the settling time function is $\sup T(x_0) = T_c\ \forall \in \mathbb{R}^n$ and T_c is said the strong predefined-time [17].*

Lemma 1 *Assume there exists a continuous radially unbounded Lyapunov function $V\colon \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ and real number $T_c > 0$ and $0 < q \leqslant 1$ such that $V(0) = 0$, $V(x) > 0$, $\forall\, x \neq 0$, if the derivative of V is as $\dot{V} = -\dfrac{1}{qT_c}\exp\left(V^q\right)V^{1-q}$, subsequently the origin of Eq. (1) is globally strongly Predefined-Time stable and the strong predefined time is $T(x_0) = T_c$ [17].*

Definition 5 *Consider $h \geqslant 0$ for $x \in \mathbb{R}^n$, define the function below:*

$$\|\lfloor x \rceil\|^h = \frac{x}{\|x\|^{1-h}}, \tag{2}$$

where $\|x\|$ the norm of x and this function for $h > 0$ is continuous and for $h = 0$ in $x = 0$ is discontinuous [21].

Definition 6 *For $x \in \mathbb{R}^n$, the predefined-time stabilizing function is defined as:*

$$\Phi_q\left(x; T_c\right) = \frac{1}{T_c q}\exp\left(\|x\|^q\right)\|\lfloor x \rceil\|^{1-q}, \tag{3}$$

where $T_c > 0$ and $0 < q \leqslant 1$ also $\dot{\Phi}_q\left(x; T_c\right) = \dfrac{\mathrm{d}\Phi_q\left(x; T_c\right)}{\mathrm{d}t}$ is available [21].

Lemma 2 *For every initial condition x_0, the system:*

$$\dot{x} = -\Phi_q(x; T_c), \tag{4}$$

where $T_c > 0$ and $0 < q \leqslant 1$ has global strong predefined-time stability with strong predefined-time T_c. That is, $x(t) = 0$ for all $t \geqslant t_0 + T_c$ in spite of the value of x_0 [21].

Definition 7 *The definition of $sgn(x)$ function is given as $sgn(x) = \begin{cases} 1 & x > 0, \\ 0 & x = 0, ; \\ -1 & x < 0 \end{cases}$ and $|x| = x sgn(x)$ is always true.*

Definition 8 *The $sig(x)$ function is mathematically related to the $sgn(x)$ function as $sig^a(x) = |x|^a sgn(x)$ [31].*

2.2. Problem formulation

Consider a class of cascade high-order nonlinear systems, as follows

$$\begin{aligned}\dot{x}_1 &= x_2\,,\\ \dot{x}_2 &= x_3\,,\\ \dot{x}_3 &= x_4\,,\\ &\vdots\\ \dot{x}_n &= f(t,x) + g(t,x)u,\end{aligned} \tag{5}$$

where $x = [x_1,\ x_2,\ \ldots,\ x_n]^T$ is the vector of the system states. Also, u is the only control input of the system. The following assumptions are considered for the system (5):

Assumption 1 *$f(t,x)$, $g(t,x)$ are nonlinear smooth functions (which means "sufficient differentiable function").*

Assumption 2 *$g(t,x)$ are bounded and invertible; i.e. the function $g(t,x)$ has a constant sign for all $t,\ x$.*

Assumption 3 *$x = [x_1,\ x_2,\ \ldots,\ x_n]^T$ all states of the system (5) are measurable.*

Note that the goal of this paper is to design the control input u using the SMC method such that the system is strongly predefined time stable.

3. Main results

Generally, two steps are required to achieve the predefined time stabilization of a nonlinear system by using the SMC technique. In the first step, the defined control input should ensure the existence of predefined time sliding motions, $s_{n-1} = 0$. Indeed, the system states reach sliding mode $s_{n-1} = 0$ in the predefined convergence time $T_1(x_0)$. Therefore, $s_{n-1} = 0$ is fulfilled for $t \geqslant T_1(x_0)$. In the second step, the strongly predefined time stability of sliding motion $s_{n-1} = 0$ should be proved. In other words, the system states (which have been proved to reach $s_{n-1} = 0$ for $t \geqslant T_1(x_0)$, in the first step) converge to zero in the predefined convergence time $T_2(x_0)$.

The sliding surfaces are given as follows

$$\begin{aligned} s_1 &= \dot{x}_1 + \Phi_{q_0}\left(x_1; T_{c_0}\right), \\ s_2 &= \dot{s}_1 + \Phi_{q_1}\left(s_1; T_{c_1}\right), \\ s_3 &= \dot{s}_2 + \Phi_{q_2}\left(s_2; T_{c_2}\right), \\ &\vdots \\ s_{n-1} &= \dot{s}_{n-2} + \Phi_{q_{n-2}}\left(s_{n-2}; T_{c_{n-2}}\right), \end{aligned} \tag{6}$$

where we have $0 < q_i \leqslant 1$, $T_{c_i} > 0$, $i = (0, 1, 2, \ldots, n-2)$. Also, the designed control input is as below

$$u = g^{-1}(t,x)\begin{pmatrix} -f(t,x) - \Phi_{q_0}^{(n-1)}\left(x_1; T_{c_0}\right) - \Phi_{q_1}^{(n-2)}\left(s_1; T_{c_1}\right) \\ -\ldots - \ddot{\Phi}_{q_{n-3}}\left(s_{n-3}; T_{c_{n-3}}\right) - \dot{\Phi}_{q_{n-2}}\left(s_{n-2}; T_{c_{n-2}}\right) \\ -\dfrac{1}{(2)^{(\gamma+1)/2} T_{c_f} q_f} e^{\left(V^{q_f}\right)} sig^{\gamma}\left(s_{n-1}\right) \end{pmatrix}, \tag{7}$$

where we have $0 < q_f \leqslant \frac{1}{2}$, $T_{c_f} > 0$, $0 \leqslant \gamma = 1 - 2q_f < 1$. Also, V is the defined candidate Lyapunov function as $V = \frac{1}{2} s_{n-1}^2$ (which satisfies the given conditions in Lemma 1).

Theorem 1 *Consider the system (5) with Assumptions 1 to 3, sliding surface (6). If designed control input (7) is applied to the system (5), then all system states reach zero as strongly predefined time. Also, strong predefined convergence time is as* $T(x_0) = \sum_{i=0}^{n-1} T_{c_i} + T_{c_f}$.

Proof. To prove the stability of the system as strongly predefined time by using SMC technique, two aforementioned steps are obtained as follows

Step 1. It is first necessary to prove that, by applying the control input of Eq. (7) to the system of Eq. (5), it reaches the sliding surface i.e. $s_{n-1} = 0$. For this purpose, the candidate Lyapunov function is defined as $V = \frac{1}{2} s_{n-1}^2$. By differentiating of this candidate function with respect to time we have

$$\dot{V} = s_{n-1}\dot{s}_{n-1}\,. \tag{8}$$

Prior to proceeding further, note that the time derivative of the sliding surface of Eq. (6) is as below

$$\dot{s}_{n-1} = f(t,x) + g(t,x)u + \Phi_{q_0}^{(n-1)}\left(x_1; T_{c_0}\right) + \Phi_{q_1}^{(n-2)}\left(s_1; T_{c_1}\right) + \ldots \\ + \ddot{\Phi}_{q_{n-3}}\left(s_{n-3}; T_{c_{n-3}}\right) + \dot{\Phi}_{q_{n-2}}\left(s_{n-2}; T_{c_{n-2}}\right). \tag{9}$$

Also, by substituting the designed control input of Eq. (7) into Eq. (9), we have

$$\dot{s}_{n-1} = -\frac{1}{(2)^{(\gamma+1)/2} T_{c_f} q_f} e^{\left(V^{q_f}\right)} sig^{\gamma}(s_{n-1}). \tag{10}$$

In consequence, by substituting Eq. (10) into Eq. (8), the Eq. (8) can be re-written as below

$$\dot{V} = s_{n-1}\left(-\frac{1}{(2)^{(\gamma+1)/2} T_{c_f} q_f} e^{\left(V^{q_f}\right)} sig^{\gamma}(s_{n-1})\right) \\ \rightarrow \quad \dot{V} = -\frac{1}{(2)^{(\gamma+1)/2} T_{c_f} q_f} e^{\left(V^{q_f}\right)} |s_{n-1}|^{\gamma+1}. \tag{11}$$

Since we have $|s_{n-1}| = \sqrt{2V}$, then, there comes

$$\dot{V} = -\frac{1}{(2)^{(\gamma+1)/2} T_{c_f} q_f} e^{\left(V^{q_f}\right)} (2V)^{(\gamma+1)/2} \\ \rightarrow \quad \dot{V} = -\frac{1}{T_{c_f} q_f} e^{\left(V^{q_f}\right)} (V)^{(\gamma+1)/2}. \tag{12}$$

Let $\gamma = 1 - 2q_f$ be, then there is

$$\dot{V} = -\frac{1}{T_{c_f} q_f} e^{\left(V^{q_f}\right)} (V)^{1-q_f}. \tag{13}$$

As a result, according to Lemma 1, by applying the control input (7) to the system (5), the system states reach the sliding surface; $s_{n-1} = 0$ as strongly predefined time. Also, according to Lemma 1, the strong predefined stabilization time is as $T_1(x_0) = T_{c_f}$. Hence, the first step of the proof is obtained.

Step 2. To obtain the second step of the proof, Lemma 2 is used. Indeed, according to Lemma 2 we have $s_{n-2} = 0$ as strongly predefined time. Similarly, we have $s_2 = 0$, and consequently we have $s_1 = 0$. Finally, x_1 reaches zero as strongly predefined time. In other words, it can be explained mathematically as follows

$$s_{n-1} \xrightarrow{T \leqslant T_{c_{n-1}}} 0 \rightarrow s_{n-2} \xrightarrow{T \leqslant T_{c_{n-2}}} 0 \rightarrow \ldots \rightarrow s_1 \xrightarrow{T \leqslant T_{c_1}} 0 \rightarrow x_1 \xrightarrow{T \leqslant T_{c_0}} 0. \tag{14}$$

Also, by converging x_1 to zero, it can be concluded that other states reach zero as strongly predefined time. In other words, we have

$$x_1 \xrightarrow{T \leqslant T_{c_0}} 0 \rightarrow \dot{x}_1 = x_2 \xrightarrow{T \leqslant T_{c_0}} 0 \rightarrow \ldots \rightarrow \dot{x}_{n-1} = x_n \xrightarrow{T \leqslant T_{c_0}} 0. \tag{15}$$

In consequence, all system states reach zero as strongly predefined time. Also, the strong predefined stabilization time is as $T_2(x_0) = \sum_{i=0}^{n-1} T_{c_i}$. This concludes the proof. □

Remark 1 *The strong predefined time stability is fulfilled for the system (5) at* $T(x_0) = T_1(x_0) + T_2(x_0) = \sum_{i=0}^{n-1} T_{c_i} + T_{c_f}$, *according to Steps 1 and 2 of the above Proof. Indeed, all system (5) states converge zero for* $t \geqslant T(x_0)$.

Remark 2 *The control parameters* $0 < q_i \leqslant 1$, $T_{c_i} > 0$, $0 < q_f \leqslant \frac{1}{2}$, $T_{c_f} > 0$, *and* $0 \leqslant \gamma = 1 - 2q_f < 1$ *in the sliding surface, the control input, and inequality related to* $T_1(x_0)$ *and* $T_2(x_0)$ *are arbitrary constants and chosen by the designer. Therefore, the control efforts and predefined stabilization time of the system is adjustable by selecting them properly.*

Proposition 1 *Consider the system (5) with Assumptions 1 to 3, sliding surface (16). If designed control input (17) is applied to the system (5), then all system states reach zero as fixed time (based-on previous study in [32]).*

$$\begin{aligned} s_1 &= \dot{x}_1 + A_1(x_1) + B_1(x_1), \\ s_2 &= \dot{s}_1 + A_2(s_1) + B_2(s_1), \\ s_3 &= \dot{s}_2 + A_3(s_2) + B_3(s_2), \\ &\vdots \\ s_{n-1} &= \dot{s}_{n-2} + A_{n-1}(s_{n-2}) + B_{n-1}(s_{n-2}), \end{aligned} \tag{16}$$

where we have $A_i(\xi) = a_{2i-1}\xi^{\frac{p_{2i-1}}{q_{2i-1}}}$, $B_i(\xi) = a_{2i}\xi^{\frac{p_{2i}}{q_{2i}}}$, $a_j > 0$, $0 < q_{2i-1} < p_{2i-1} < 2q_{2i-1}$ *and* $0 < p_{2i} < q_{2i}$. *Also,* p_j, q_j *are odd numbers.*

$$u = g^{-1}(t,x) \begin{pmatrix} -f(t,x) - A_1^{(n-1)}(x_1) - B_1^{(n-1)}(x_1) \\ -A_2^{(n-2)}(s_1) - B_2^{(n-2)}(s_1) \\ - \ldots - \ddot{A}_{n-2}(s_{n-3}) - \ddot{B}_{n-2}(s_{n-3}) \\ -\dot{A}_{n-1}(s_{n-2}) - \dot{B}_{n-1}(s_{n-2}) \\ -r_1 sig^{\gamma_1}(s_{n-1}) - r_2 sig^{\gamma_2}(s_{n-1}) \end{pmatrix}, \tag{17}$$

where $r_{1,2} > 0$ *and* $1 < \gamma_2$, $0 < \gamma_1 < 1$.

Its proof is similar to the proof of Theorem 1, and is omitted because of space limitations.

Proposition 2 *Consider the system (5) with Assumptions 1 to 3, sliding surface (16). If designed control input (17) is applied to the system (5), then all system states reach zero as fixed time (based-on previous study in [33]).*

$$
\begin{aligned}
s_1 &= \dot{x}_1 + A_1(x_1),\\
s_2 &= \dot{s}_1 + A_2(s_1),\\
s_3 &= \dot{s}_2 + A_3(s_2)\\
&\vdots\\
s_{n-1} &= \dot{s}_{n-2} + A_{n-1}(s_{n-2}),
\end{aligned}
\tag{18}
$$

where we have $A_i(\xi) = a_{2i-1}\xi^{\frac{p_{2i-1}}{q_{2i-1}}}$, $a_{2i-1} > 0$, *and* $0 < p_{2i-1} < q_{2i-1}$.

$$
u = g^{-1}(t,x)\Big(- f(t,x) - A_1^{(n-1)}(x_1) - A_2^{(n-2)}(s_1) - \ldots - \ddot{A}_{n-2}(s_{n-3}) - \dot{A}_{n-1}(s_{n-2}) - r_1 sig^{\gamma_1}(s_{n-1})\Big), \tag{19}
$$

where $r_1 > 0$, *and* $0 < \gamma_1 < 1$.

The proof is analogous to the proof of Theorem 1.

Remark 3 *Many practical systems in real-time are described by a cascade model as Eq. (5) such as ship course system [25, 34], Genesio Tesi and Coullet systems [35, 36], planetary gear-type inverted-pendulum (PIP) mechanism [37], van der Pol chaotic oscillator [38], etc. Indeed, the proposed method in this paper can be applied for a huge group of systems (described by cascade high-order model similar to Eq. (5)). Accordingly, the proposed method is simulated on two examples of cascade systems in the next section.*

4. Simulation and comparison

In this section, two simulation examples of second-order, and third-order (with different control goals) are simulated to verify the correctness of our proposed designs in the previous section. The simulation is done for the proposed SPSMC method and the FSMC and TSMC methods to demonstrate the effectiveness of predefined time stability concepts compared with fixed time and finite-time stability concepts. Additionally, trajectory tracking and synchronization are considered in the first and second examples, respectively, to demonstrate the proposed controller is proper and adjustable for different control problems. For the numerical simulation in this section, the Simulink environment of MATLAB software is utilized with numerical solver ode3 and step-size 0.01. Note that all simulations are performed in such a way that the settling time of all methods is almost identical. Also, all simulation conditions are considered identically. Hence, their control inputs and their errors are compared correctly.

In addition, two performance criteria are utilized to make a thorough comparison between the proposed SPSMC scheme and the other two schemes. The criteria Integral of the Square Value (ISV) and Integral of the Absolute value of the Error (IAE) have been defined in [6, 39, 40]. IAE is presented as below

$$IAE_{e_i} = \int_0^{t_f} |e_i| \mathrm{d}t, \tag{20}$$

whereas, ISV is presented as below

$$ISV_{u_i} = \int_0^{t_f} u_i^2 \mathrm{d}t, \tag{21}$$

where t_f denotes the total running time. The IAE is utilized to numerically measure the tracking performance for a total error curve. The ISV signifies energy consumption.

4.1. Trajectory tracking of ship course system

Consider the mathematical model of the second-order nonlinear ship course system given in [25, 34], as follows

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{\tau}\left(-H(x_2) + Ku\right), \end{aligned} \tag{22}$$

where $\tau = 0.21$ is the system time constant and $K = 78.41$ is the rudder gain and we have $H(x_2) = 2.2386x_2 + 1988.4x_2^3$. In accordance with Eqs. (6) and (7), the sliding surface and the control input of the system Eq. (22), will be as follows

$$\begin{aligned} s_1 &= \dot{e}_1 + \Phi_{q0}\left(e_1; T_{c0}\right), \\ u &= \frac{\tau}{K}\left(\frac{1}{\tau}H(e_2) - \dot{\Phi}_{q0}\left(e_1; T_{c0}\right) - \frac{1}{(2)^{(\gamma+1)/2} T_{c_f} q_f} e^{\left(V^{q_f}\right)} sig^{\gamma}(s_1) + \dot{x}_{2_d}\right), \end{aligned} \tag{23}$$

where the trajectory tracking errors are defined as $e_1 = x_1 - x_{1_d}$, $e_2 = x_2 - \dot{x}_{1_d}$. The control parameters are considered as $\gamma = 1 - 2q_f$, $q_0 = 0.09$, $T_{c_0} = 3.5$, $q_f = 0.09$, $T_{c_f} = 3.5$. By using the sliding surface in Eq. (23) and by applying the control input in Eq. (23) to the system in Eq. (22); the system (22) is controlled within a strong predefined convergence time with trajectory tracking goal. Also, both system states reach the desired tracking trajectories as $x_{1_d} = \sin\left(5t + \dfrac{\pi}{3}\right)$, $x_{2_d} = 5\cos\left(5t + \dfrac{\pi}{3}\right)$, within strongly predefined time. Also, the initial conditions

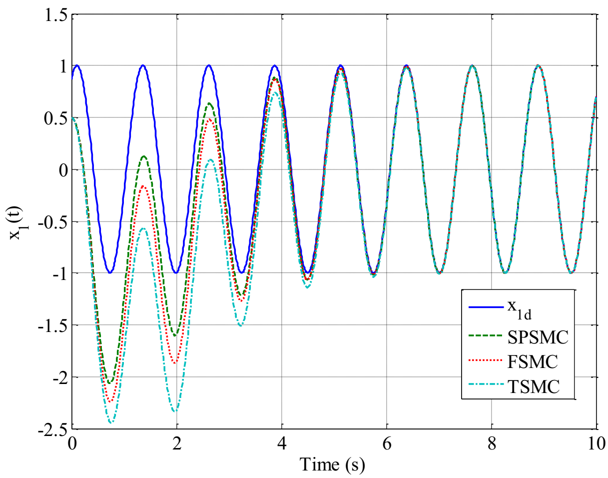

Figure 1: Time responses of $x_1(t)$ and $x_{1_d}(t)$

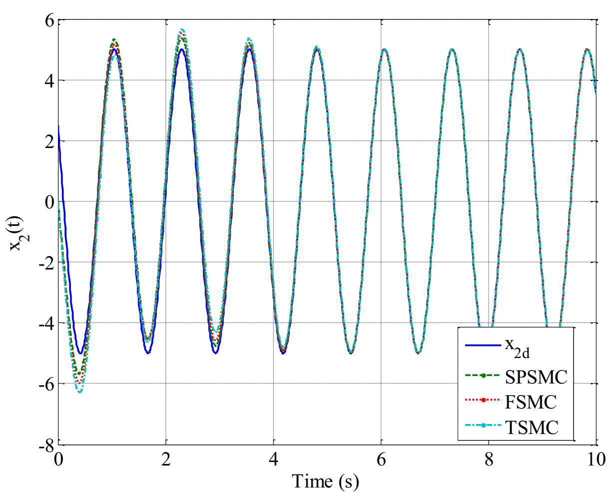

Figure 2: Time responses of $x_2(t)$ and $x_{2_d}(t)$

of the system states are chosen as $x_1(0) = 0.5$, $x_2(0) = 0$. Figs. 1 and 2 show time responses of the system states along with the desired tracking trajectory, after applying the proposed controller and the other two controllers. Fig. 3 represents time responses of the trajectory tracking errors, after applying the proposed controller and the other two controllers. Fig. 4 displays time responses of the designed control input effect, after applying the proposed controller and the other two controllers. It can be seen from Figs. 1 to 4 that the time responses of the proposed predefined time scheme, SPSMC, is more satisfactory with a better tracking performance than the no predefined time schemes, FSMC and TSMC.

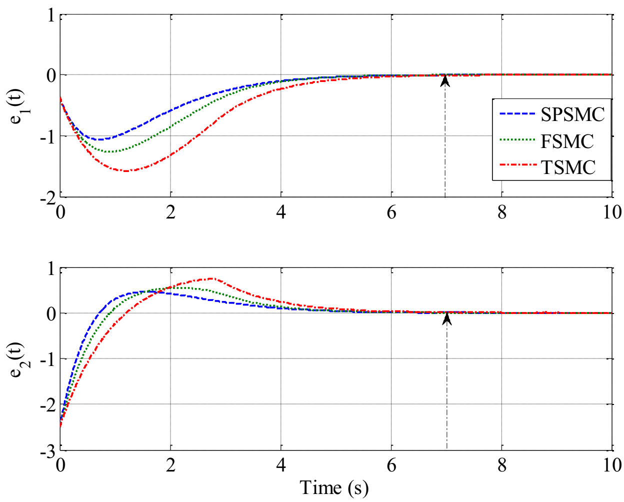

Figure 3: Time responses of $e_1(t)$ and $e_2(t)$

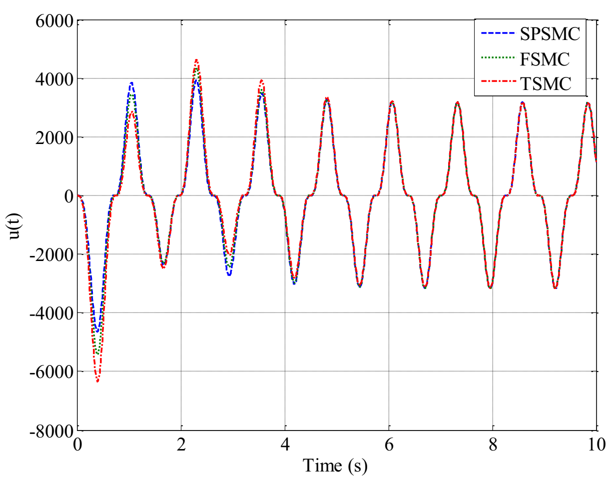

Figure 4: Time responses of $u(t)$

Table 1 and Fig. 5 provides a comparison of the performance indices. It can be clearly observed that the proposed predefined time scheme, SPSMC, gives lower numerical values for ISV and IAE, compared with the no predefined time schemes, FSMC and TSMC. In consequence, the proposed SPSMC scheme outperforms the other two schemes in terms of ISV and IAE.

Table 1: Comparison of the performance indices

	$ISV_u \times 10^8$	IAE_{e_1}	IAE_{e_2}
SPSMC	1.8834	20.3221	14.9090
FSMC	2.0384	25.3626	18.0809
TSMC	2.2759	34.8989	22.7365

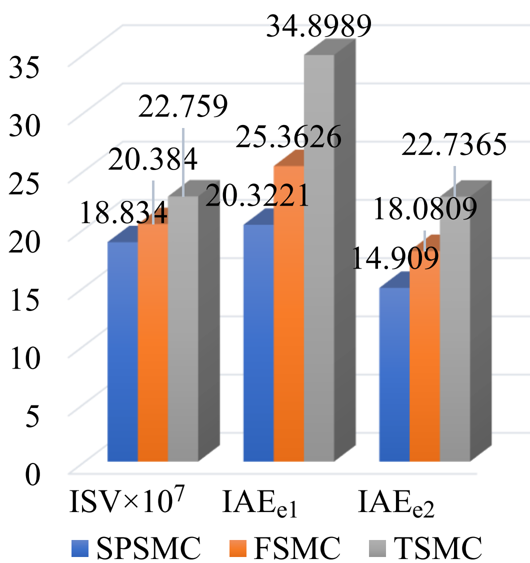

Figure 5: Comparison of the performance indices

4.2. Synchronization of two chaotic systems of Genesio Tesi and Coullet

In this section, two three-order chaotic nonlinear systems Genesio Tesi and Coullet gave in [35, 36], are synchronized with predefined convergence time. The mathematical model of Genesio Tesi is given as follows

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= x_3,\\ \dot{x}_3 &= -0.45x_3 - 1.1x_2 - 1x_1 + x_1^2.\end{aligned} \tag{24}$$

The Genesio Tesi system is considered as the master system. The mathematical model of the Coullet system is given in Eq. (25) as the slave system, as follows

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= y_3, \\ \dot{y}_3 &= -0.45y_3 - 1.1y_2 + 0.8y_1 - y_1^3 + u. \end{aligned} \tag{25}$$

The initial conditions for the two systems to be in a chaotic state are as $x_1(0) = 0.22$, $x_2(0) = 0.21$, $x_3(0) = 0.61$, $y_1(0) = 0.1$, $y_2(0) = 0.41$, $y_3(0) = 0.31$. Tracking errors for synchronization of the two systems Eqs. (24) and (25) are defined as

$$\begin{aligned} e_1 &= y_1 - x_1, \\ e_2 &= y_2 - x_2, \\ e_3 &= y_3 - x_3. \end{aligned} \tag{26}$$

By defining sliding surfaces and control input as Eq. (27), the two systems of Eqs. (24) and (25) synchronize as strongly predefined time.

$$\begin{aligned} s_1 &= \dot{e}_1 + \Phi_{q0}(e_1; T_{c0}), \\ s_2 &= \dot{s}_1 + \Phi_{q_1}(s_1; T_{c_1}), \\ u &= -\left(-0.45y_3 - 1.1y_2 + 0.8y_1 - y_1^3\right) + \left(-0.45x_3 - 1.1x_2 - 1x_1 + x_1^2\right) \\ &\quad - \ddot{\Phi}_{q0}(e_1; T_{c0}) - \dot{\Phi}_{q_1}(s_1; T_{c_1}) - \frac{1}{2^{(\gamma+1)/2} T_{c_f} q_f} e^{(V^{q_f})} sig^{\gamma}(s_2). \end{aligned} \tag{27}$$

Also, the control parameters are chosen as $q_0 = 0.9$, $q_1 = 0.9$, $q_f = 0.1$, $T_{c_0} = 0.5$, $T_{c_1} = 0.5$, $T_{c_f} = 1$, $\gamma = 1 - 2q_f$. Figs. 6 to 8 represent time responses

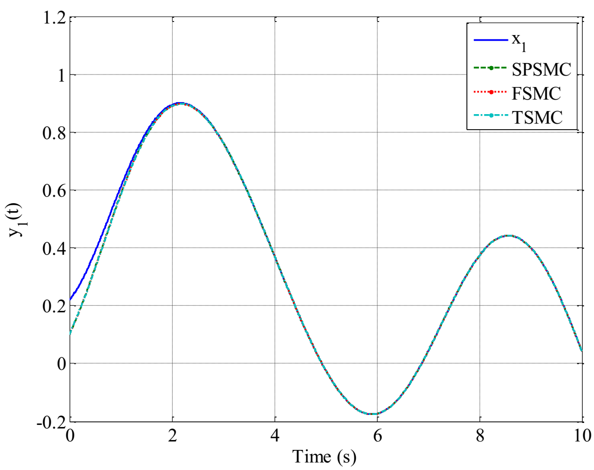

Figure 6: Time responses of $y_1(t)$, $x_1(t)$

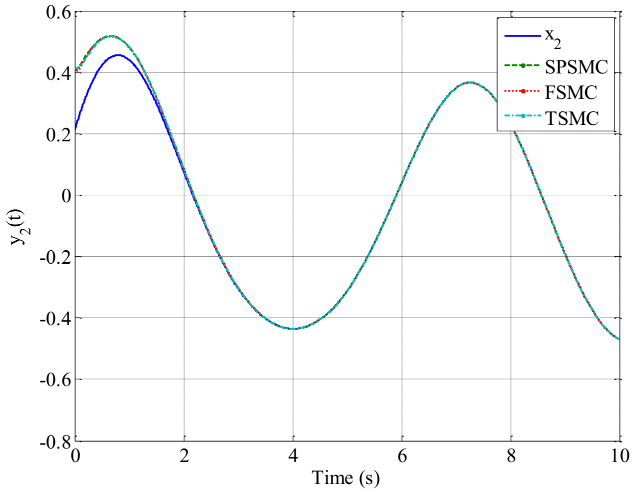

Figure 7: Time responses of $y_2(t)$, $x_2(t)$

of the system states of the master system (24) and the slave system (25) after applying the proposed controller and the other two controllers. Fig. 9 displays the time responses of the synchronization errors after applying the proposed controller and the other two controllers. Fig. 10 represents time responses of the designed control input effect, after applying the proposed controller and the other two controllers. It can be observed from Figs. 6 to 10 that the time responses of the proposed predefined time scheme, SPSMC, is more satisfactory with a better tracking performance than the no predefined time scheme, FSMC and TSMC.

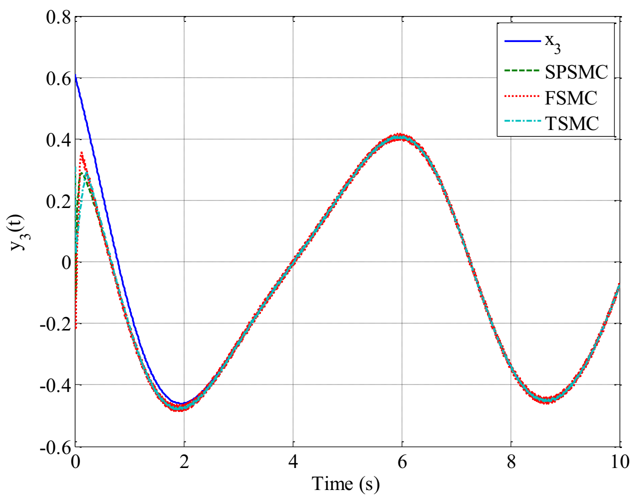

Figure 8: Time responses of $y_3(t)$, $x_3(t)$

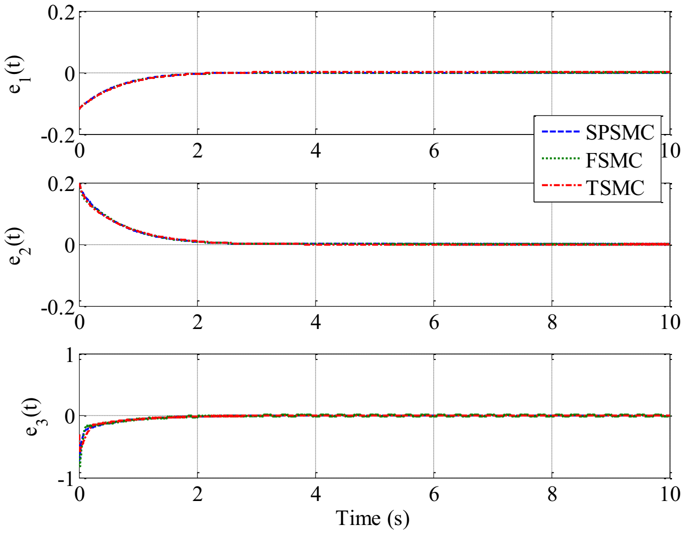

Figure 9: Time responses of $e_1(t)$, $e_2(t)$ and $e_3(t)$

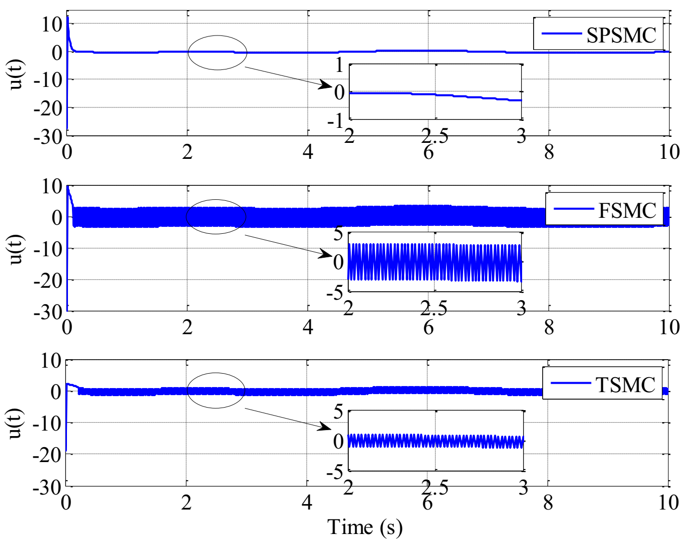

Figure 10: Time responses of u

Table 2 and Figs. 11 and 12 provide a comparison of the performance indices. It can be clearly seen that the proposed predefined time scheme, SPSMC, gives lower numerical values for ISV and IAE, compared with the no predefined time schemes, FSMC and TSMC. Therefore, the proposed SPSMC scheme outperforms the other two schemes in terms of ISV and IAE.

Table 2: Comparison of the performance indices

	ISV_u	IAE_{e_1}	IAE_{e_2}	IAE_{e_3}
SPSMC	158.1327	3.8099	4.2245	4.9800
FSMC	1018.5	3.8248	4.2250	5.4647
TSMC	310.5559	3.8348	4.2287	5.1033

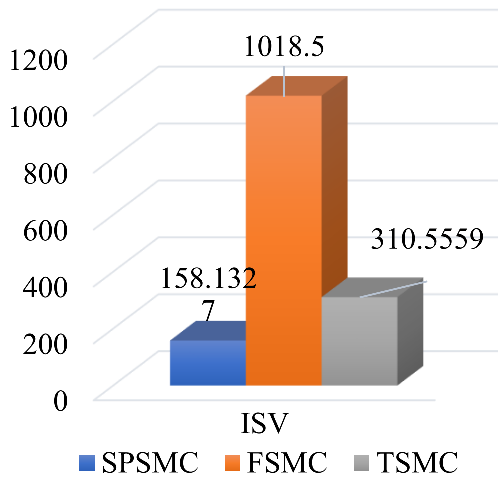

Figure 11: Comparison of the performance indices

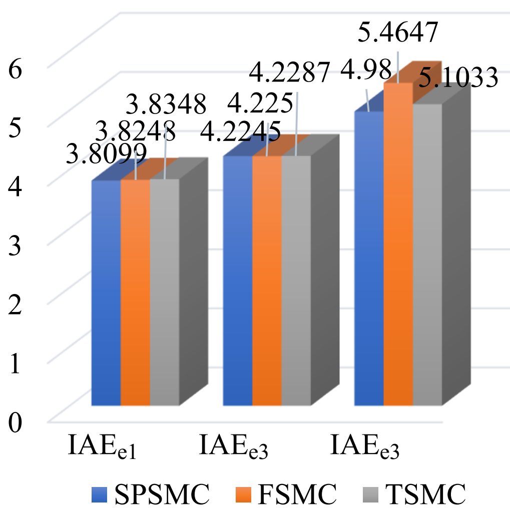

Figure 12: Comparison of the performance indices

5. Conclusions

In this paper, a new predefined time controller is designed using the SPSMC method for a class of cascade high-order nonlinear systems. Subsequently, the simulation results for two simulation examples demonstrate the correctness and effectiveness of our designs. Also, two no predefined time existing methods, FSMC and TSMC are simulated for the two examples. The considered simulation examples are in the form of cascade nonlinear systems with different control goals. The first numerical example is trajectory tracking of the second-order ship course system. The simulation results of the SPSMC method show a precise convergence of the system states to the desired tracking trajectories, once the designed control input applied to the system (see Figs. 1 to 3). The second numerical example is the synchronization of two chaotic third-order nonlinear systems, Genesio Tesi and Coullet. The simulation results of the SPSMC method for the second example demonstrate accurate synchronization of the master system, Genesio Tesi, to the slave system, Coullet, after applying the proposed controller (see Figs. 6 to 9). Furthermore, the proposed predefined time method, SPSMC, outperforms the no predefined time methods, FSMC and TSMC. The controller proposed in this paper can be used to control a range of physical systems because the considered mathematical model for cascade systems can describe many real systems. Further work can be done on robust control of such systems and optimization of control parameters.

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