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# LOWER DENSITY OPERATORS. $\Phi_f$ VERSUS $\Phi_d$

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**Abstract.** Using the new method of the construction of lower density operator introduced in the earlier paper of the first two authors, we study how much the new operator can be different from the classical one. The aim of this paper is to show that if $f$ is a good adjusted measure-preserving bijection then the lower density operator generated by $f$ can be really different from the classical density operator.

**Keywords:** lower density operator, measure-preserving bijection.

**Mathematics Subject Classification:** 54C60, 26E25, 28D05.

## 1. INTRODUCTION

The density topology plays essential role in the study of approximate continuity, approximate differentiation ([1,3,4]), and in the potential theory ([6]). In this paper we study a new method of constructing a large class of lower density operators generated by measure-preserving bijections, which where introduced by the first authors ([5]).

Let $\mathcal{S}$ be a $\sigma$-algebra of subsets of $X$ and let $\mathcal{I} \subset \mathcal{S}$ be a $\sigma$-ideal. For $A, B \in \mathcal{S}$, if the symmetric difference $A \Delta B$ belongs to $\mathcal{I}$, then we will write $A \sim B$.

Let $\Phi : \mathcal{S} \to \mathcal{S}$ be a *a lower density operator* on $\mathcal{S}$, i.e. for each $A, B \in \mathcal{S}$, the following conditions are fulfilled:

1. $\Phi(\emptyset) = \emptyset, \quad \Phi(X) = X$;
2. if $A \sim B$, then $\Phi(A) = \Phi(B)$;
3. $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$;
4. $\Phi(A) \sim A$.

If $X = \mathbb{R}$, $\mathcal{S}$ is a $\sigma$-algebra $\mathcal{L}$ of Lebesgue measurable sets and $\mathcal{I}$ - a $\sigma$-ideal $\mathcal{N}$ of nullsets, then the classical density operator $\Phi_d$ assigns to each $A \in \mathcal{L}$ the set $\Phi_d(A)$ of all density points of $A$, of the form

$$\Phi_d(A) = \Big\{ x \in \mathbb{R} : \lim_{h \to 0} \frac{\lambda([x-h, x+h] \cap A)}{2h} = 1 \Big\},$$

 

where $\lambda$ stands for Lebesgue measure on $\mathbb{R}$. It is nearly obvious that

$$\mathrm{Int}(A) \subset \Phi_d(A) \subset \overline{A} \tag{1.1}$$

for each measurable set $A$, where $\mathrm{Int}(A)$ and $\overline{A}$ denote the interior and closure of $A$, respectively.

In [5] authors concentrate on lower density operators for which the analogue of (1.1) is not fulfilled for some $A$ in $\mathcal{S}$. They prove that if $f : \mathbb{R} \to \mathbb{R}$ is a measure-preserving bijection, i.e. $\lambda(f(A)) = \lambda(A) = \lambda(f^{-1}(A))$ for each $A \in \mathcal{L}$, then the operator $\Phi_f : \mathcal{L} \to \mathcal{L}$ defined as follows:

$$\Phi_f(A) = f^{-1}\left(\Phi_d(f(A))\right) \tag{1.2}$$

for each $A \in \mathcal{L}$, is a lower density operator. Using this method there are given two examples of lower density operators for which the first or the second inclusion from (1.1) does not hold.

In this note we focus on comparison of the classical density operator $\Phi_d$ with the operator $\Phi_f$ generated by measure-preserving bijection $f$.

Obviously, if $\Phi$ is an arbitrary lower density operator defined on a family of Lebesgue measurable sets, then from the Lebesgue Density Theorem follows that the symmetric difference of $\Phi(A)$ and $\Phi_d(A)$ must be a set of measure zero for an arbitrary measurable $A$. However, the symmetric difference between $\Phi(A)$ and $\Phi_d(A)$ can be a large set in the sense of cardinality, as well as the Baire category.

The main result is Theorem 2.2 saying that the difference $\Phi_f(A) \setminus \Phi_d(A)$ can be a set of cardinality continuum. Here we use as $A$ some subset of a symmetric Cantor-type set $C \subset [0, 1]$, where Lebesgue measure of $C$ equals $1/2$, and $f$ is a suitably adjusted measure-preserving bijection on [0,1]. We prove also that if $g = f^{-1}$ and $A = (0, 1/2)$, then $\Phi_d(A) \setminus \Phi_g(A)$ is a set of the second category, so is big in the Baire category sense (Theorem 2.6).

In the last part we prove that the equalities $\Phi_f(A) = \Phi_d(A)$ and $\Phi_f(A + a) = \Phi_f(A) + a$ hold for arbitrary measurable $A$ and $a \in \mathbb{R}$ if and only if $f(x) = x + c$ or $f(x) = -x + c$, where $c$ is an arbitrary real number.

## 2. $\Phi_f$ VERSUS $\Phi_d$

It is easy to see that $\Phi_f \Delta \Phi_d$ can be denumerable. Let $\mathbb{Z}$ denote the set of integers. Put

$$f(x) = \begin{cases} x + (-1)^{\lfloor x \rfloor} & \text{for } x \notin \mathbb{Z}, \\ x & \text{for } x \in \mathbb{Z} \end{cases}$$

and

$$A = \bigcup_{k \in \mathbb{Z}} \left[ \left(2k + \frac{1}{2}, 2k + 1\right) \cup \left(2k + 1, 2k + \frac{3}{2}\right) \right].$$

Then

$$f(A) = \bigcup_{k \in \mathbb{Z}} \left[ \left(2k - \frac{1}{2}, 2k\right) \cup \left(2k, 2k + \frac{1}{2}\right) \right],$$

so

$$\Phi_d\left(f\left(A\right)\right)=f(A)\cup\bigcup_{k\in\mathbb{Z}}\{2k\}$$

and

$$\Phi_f(A)=f^{-1}\left(\Phi_d\left(f(A)\right)\right)=A\cup\bigcup_{k\in\mathbb{Z}}\{2k\}.$$

Simultaneously, $\Phi_d(A)=A\cup\bigcup_{k\in\mathbb{Z}}\{2k+1\}$, so

$$\Phi_d(A)\bigtriangleup\Phi_f(A)=\mathbb{Z}.$$

The proof that this symmetric difference can be a set of cardinality continuum is more complicated. For this purpose we need some auxiliary lemma.

**Lemma 2.1.** *If* $f:(0,1)\to(0,1)$ *is a bijection such that*

$$\lambda((0,b))=\lambda(f((0,b)))=\lambda\left(f^{-1}((0,b))\right)$$

*for an arbitrary* $b\in(0,1]$, *then* $f$ *is a measure-preserving bijection.*

*Proof.* First we will prove that for arbitrary $a\in[0,1)$

$$\lambda((a,1))=\lambda\left(f((a,1))\right)=\lambda\left(f^{-1}((a,1))\right). \tag{2.1}$$

Let $a\in[0,1)$. From our assumption we obtain

$$\lambda\left(f((a,1))\right)=1-\lambda\left(f((0,a))\right)=1-a=\lambda((a,1))$$

and

$$\lambda\left(f^{-1}((a,1))\right)=1-\lambda\left(f^{-1}((0,a))\right)=1-a=\lambda((a,1)).$$

Hence for arbitrary $a\in[0,1)$, $b\in(0,1]$, $a<b$

$$\lambda((a,b))=\lambda(f((a,b)))=\lambda\left(f^{-1}((a,b))\right). \tag{2.2}$$

If $A\subset(0,1)$ is open or closed, then from (2.2)

$$\lambda(A)=\lambda(f(A))=\lambda\left(f^{-1}(A)\right).$$

Now let $E$ be a measurable subset of $(0,1)$. There exists a decreasing sequence $\{G_n\}_{n\in\mathbb{N}}$ of open sets and an increasing sequence $\{F_n\}_{n\in\mathbb{N}}$ of closed sets such that

$$\bigcup_{n=1}^{\infty}F_n\subset E\subset\bigcap_{n=1}^{\infty}G_n$$

and

$$\lambda(F_n)\nearrow\lambda(E)\swarrow\lambda(G_n).$$

Hence
$$\bigcup_{n=1}^{\infty} f(F_n) \subset f(E) \subset \bigcap_{n=1}^{\infty} f(G_n) \tag{2.3}$$
and
$$\lambda(f(F_n)) \nearrow \lambda(E) \swarrow \lambda(f(G_n)),$$
so
$$\lambda\left(\bigcup_{n=1}^{\infty} f(F_n)\right) = \lambda(E) = \lambda\left(\bigcap_{n=1}^{\infty} f(G_n)\right). \tag{2.4}$$
Consequently, from (2.3) and (2.4), $\lambda(f(E)) = \lambda(E)$.

Analogously,
$$\bigcup_{n=1}^{\infty} f^{-1}(F_n) \subset f^{-1}(E) \subset \bigcap_{n=1}^{\infty} f^{-1}(G_n)$$
and
$$\lambda(f^{-1}(F_n)) \nearrow \lambda(E) \swarrow \lambda(f^{-1}(G_n)),$$
so
$$\lambda\left(f^{-1}(E)\right) = \lambda(E),$$
i.e. $f$ is a measure-preserving bijection. □

**Theorem 2.2.** *There exists a measure-preserving bijection $f : \mathbb{R} \to \mathbb{R}$ and a measurable set $A \subset \mathbb{R}$ such that*
$$card\,(\Phi_f(A) \setminus \Phi_d(A)) = \mathfrak{c}.$$

*Proof.* In the first step we construct a symmetric Cantor-type set $C \subset [0,1]$ such that $\lambda(C) = \frac{1}{2}$ in the following way: remove from $[0,1]$ the open interval $I_1^1$ with center $1/2$ and length $1/4$. Designate the remaining closed intervals $J_1^1$ and $J_2^1$. From $J_1^1$ and $J_2^1$ remove the concentric open intervals $I_1^2$ and $I_2^2$ of length $1/4^2$.

At the $k$-th stage of the construction there remain $2^k$ closed intervals $J_1^k, \ldots, J_{2^k}^k$ of equal length. From each $J_i^k$ remove the concentric open interval $I_i^{k+1}$ of length $1/4^{k+1}$, $i = 1, \ldots, 2^k$, and so on. Put
$$C = \bigcap_{k=1}^{\infty} \bigcup_{i=1}^{2^k} J_i^k.$$
Then $C$ is a symmetric perfect nowhere dense set and $\lambda(C) = \frac{1}{2}$.

From the classical Lebesgue Density Theorem it follows that $\lambda(C \setminus \Phi_d(C)) = 0$. We will show that card$\,(C \setminus \Phi_d(C)) = \mathfrak{c}$. It suffices to prove that $C \setminus \Phi_d(C)$ includes a non-empty perfect set.

Let us start with $I_1^1$. There exists $k_1 \in \mathbb{N}$ and $j_1^1 \in \{1, \ldots, 2^{k_1}\}$ such that consecutive closed intervals $J_{j_1^1}^{k_1}$ and $J_{j_1^1+1}^{k_1}$ contiguous to $I_1^1$ from both sides are shorter than $I_1^1$. Denote them by $J_{j_1^1,l}^{k_1}$ and $J_{j_1^1,r}^{k_1}$, respectively. Observe that if $x \in J_{j_1^1,l}^{k_1}$, then
$$\frac{\lambda\left(C \cap \left[x, \sup I_1^1\right]\right)}{\sup I_1^1 - x} < \frac{1}{2}$$

and if $x \in J^{k_1}_{j^1_1,r}$, then

$$\frac{\lambda\left(C \cap \left[\inf I^1_1, x\right]\right)}{x - \inf I^1_1} < \frac{1}{2}.$$

Put

$$J_1 = J^{k_1}_{j^1_1,l} \cup J^{k_1}_{j^1_1,r}.$$

If $I^{k_1+1}_{j^1_1}$ and $I^{k_1+1}_{j^1_1+1}$ are open intervals removed in $(k_1+1)$-th stage from $J^{k_1}_{j^1_1,l}$ and $J^{k_1}_{j^1_1,r}$, respectively, then there exists $k_2 > k_1$ and $j^2_1 \in \{1, \dots, 2^{k_2}\}$ such that consecutive closed intervals $J^{k_2}_{j^2_1}$ and $J^{k_2}_{j^2_1+1}$ contiguous to $I^{k_1+1}_{j^1_1}$ are shorter that $I^{k_1+1}_{j^1_1}$ and there exists $j^2_2 \in \{1, \dots, 2^{k_2}\}$ such that consecutive closed intervals $J^{k_2}_{j^2_2}$ and $J^{k_2}_{j^2_2+1}$ contiguous to $I^{k_1+1}_{j^1_1+1}$ are shorter that $I^{k_1+1}_{j^1_1+1}$. Denote the intervals $J^{k_2}_{j^2_1}$, $J^{k_2}_{j^2_1+1}$, $J^{k_2}_{j^2_2}$, $J^{k_2}_{j^2_2+1}$ by $J^{k_2}_{j^2_1,l}$, $J^{k_2}_{j^2_1,r}$, $J^{k_2}_{j^2_2,l}$, $J^{k_2}_{j^2_2,r}$, respectively. If $x \in J^{k_2}_{j^2_1,l}$ and $z = \sup I^{k_1+1}_{j^1_1}$, then

$$\frac{\lambda\left(C \cap [x, z]\right)}{z - x} < \frac{1}{2}$$

and if $x \in J^{k_2}_{j^2_2,l}$ and $z = \sup I^{k_1+1}_{j^1_1+1}$, then

$$\frac{\lambda\left(C \cap [x, z]\right)}{z - x} < \frac{1}{2}.$$

Similarly, if $x \in J^{k_2}_{j^2_1,r}$ and $z = \inf I^{k_1+1}_{j^1_1}$, then

$$\frac{\lambda\left(C \cap [z, x]\right)}{x - z} < \frac{1}{2}$$

and if $x \in J^{k_2}_{j^2_2,r}$ and $z = \inf I^{k_1+1}_{j^1_1+1}$, then

$$\frac{\lambda\left(C \cap [z, x]\right)}{x - z} < \frac{1}{2}.$$

Put

$$J_2 = J^{k_2}_{j^2_1,l} \cup J^{k_2}_{j^2_1,r} \cup J^{k_2}_{j^2_2,l} \cup J^{k_2}_{j^2_2,r}.$$

Suppose that we have chosen closed intervals $J^{k_m}_{j^m_i}$, $i \in \{1, \dots, 2^m\}$, and $J_m = \bigcup_{i=1}^{2^m} J^{k_m}_{j^m_i}$.

Consider open intervals $I^{k_m+1}_{j^m_i}$, for $i \in \{1, \dots, 2^m\}$ removed in $(k_m+1)$-th stage from $J^{k_m}_{j^m_i}$, respectively. There exists $k_{m+1} > k_m$ such that consecutive closed intervals remaining in the $k_{m+1}$-th stage contiguous to $I^{k_m+1}_{j^m_i}$ are shorter than $I^{k_m+1}_{j^m_i}$, for $i \in \{1, \dots, 2^m\}$. We will denote the pair of such intervals by $J^{k_m+1}_{j^{m+1}_i,l}$ and $J^{k_m+1}_{j^{m+1}_i,r}$, $i \in \{1, \dots, 2^m\}$. Put

$$J_{m+1} = \bigcup_{i=1}^{2^m} \left(J^{k_m+1}_{j^{m+1}_i,l} \cup J^{k_m+1}_{j^{m+1}_i,r}\right).$$

Observe that if $x \in J^{k_m+1}_{j_i^{m+1},l}$ and $z = \sup I^{k_m+1}_{j_i^m}$ then

$$\frac{\lambda\left(C \cap [x,z]\right)}{x-z} < \frac{1}{2}$$

and if $x \in J^{k_m+1}_{j_i^{m+1},r}$ and $z = \inf I^{k_m+1}_{j_i^m}$ then

$$\frac{\lambda\left(C \cap [z,x]\right)}{x-z} < \frac{1}{2}$$

for $i \in \{1,\dots,2^m\}$.

Put

$$E = \bigcap_{m=1}^{\infty} J_m.$$

Clearly, $E$ is perfect, nowhere dense, $E \subset C$ and if $x \in E$ then $x \notin \Phi_d(C)$. Obviously $\operatorname{card}(E) = \mathfrak{c}$. Consequently,

$$\operatorname{card}(C \setminus \Phi_d(C)) = \mathfrak{c}. \tag{2.5}$$

Now we will construct a measure-preserving bijection $f : \mathbb{R} \to \mathbb{R}$. It will be convenient to denote by $I_1$ the open interval $I_1^1$ removed in the first stage of the construction, $I_2$ and $I_3$ - open intervals $I_1^2$ and $I_2^2$ removed in the second stage, $I_{2^{k-1}}, I_{2^{k-1}+1}, \dots, I_{2^k-1}$ - open intervals removed in the $k$-th stage.

Let $D$ be the set of the left-hand ends of all intervals $I_n$, $n \in \mathbb{N}$, contiguous to $C$,

$$C_0 = C \setminus (D \cup \{0,1\})$$

and $H_n = I_n \cup \{\inf I_n\}$ for $n \in \mathbb{N}$.

Put

$$f_1(x) = \lambda(C \cap (0,x))$$

for $x \in (0,1)$. Then $f_1$ is strictly increasing on $C_0$, constant on intervals $I_n$, $n \in \mathbb{N}$, and continuous, so $f_1((0,1)) = (0,1/2)$. Hence $f_1 : C_0 \to (0,1/2)$ is a bijection.

Put

$$f_2(x) = \frac{1}{2} + \sum_{i=1}^{n-1} \lambda(I_i) + x - \inf I_n$$

for $x \in H_n$, $n \in \mathbb{N}$ (we assume that $\sum_{i=1}^{0} \lambda(I_i) = 0$). Then $f_2 : \bigcup_{n=1}^{\infty} H_n \to [1/2,1)$ is a bijection, too.

At last

$$f(x) = \begin{cases} f_1(x) & \text{for } x \in C_0, \\ f_2(x) & \text{for } x \in \bigcup_{n=1}^{\infty} H_n, \\ x & \text{for } x \notin (0,1). \end{cases}$$

Clearly, $f : \mathbb{R} \to \mathbb{R}$ is a bijection. Now we will prove that $f$ is a measure-preserving transformation. It suffices to prove that $\lambda(A) = \lambda(f(A)) = \lambda\left(f^{-1}(A)\right)$ for each measurable $A \subset (0,1)$.

Let $b \in (0,1]$. First we will show that

$$\lambda((0,b)) = \lambda(f((0,b))). \tag{2.6}$$

We have

$$(0,b) = [(0,b) \cap C_0] \cup \left[(0,b) \cup \bigcup_{n=1}^{\infty} H_n\right],$$

and

$$\lambda\left[(0,b) \cap C_0\right] = f_1(b).$$

Simultaneously,

$$f\left[(0,b) \cap C_0\right] = f_1\left[(0,b) \cap C_0\right] = (0, f_1(b)),$$

so

$$\lambda\left(f\left[(0,b) \cap C_0\right]\right) = f_1(b) = \lambda\left[(0,b) \cap C_0\right]. \tag{2.7}$$

On the other hand,

$$\lambda[H_n \cap (0,b)] = \lambda[f_2(H_n \cap (0,b))]$$

for each $n \in \mathbb{N}$, as $f_2$ is a translation on each $H_n$. Hence

$$\begin{aligned}\lambda\Big(f[(0,b) \cap \bigcup_{n=1}^{\infty} H_n]\Big) &= \sum_{n=1}^{\infty} \lambda\left[f_2(H_n \cap (0,b))\right] \\ &= \sum_{n=1}^{\infty} \lambda\left[H_n \cap (0,b)\right] = \lambda\left[(0,b) \cap \bigcup_{n=1}^{\infty} H_n\right].\end{aligned} \tag{2.8}$$

Finally, (2.7) and (2.8) implies (2.6).

Now we will prove that

$$\lambda((0,b)) = \lambda(f^{-1}((0,b))) \tag{2.9}$$

for $b \in (0,1]$.

Let $b \in (0,1/2]$. Then

$$\lambda(f^{-1}((0,b))) = \lambda(C_0 \cap f^{-1}((0,b))) = \lambda(C_0 \cap (0, f_1^{-1}(b))).$$

Put $x = f_1^{-1}(b)$. Hence

$$\lambda(C_0 \cap (0, f_1^{-1}(b))) = \lambda(C_0 \cap (0,x)) = f_1(x) = b,$$

which gives (2.9) for $b \in (0,1/2]$.

Now let $b \in (1/2,1]$. We have

$$f^{-1}((0,b)) = C_0 \cup \left[\bigcup_{n=1}^{\infty} H_n \cap \left[\frac{1}{2}, b\right)\right],$$

so we get

$$\lambda(f^{-1}((0,b))) = \lambda(C_0) + \lambda\left(\bigcup_{n=1}^{\infty} H_n \cap \left[\frac{1}{2}, b\right)\right) = \frac{1}{2} + b - \frac{1}{2} = \lambda((0,b)).$$

Consequently, the equality (2.9) holds for each $b \in (0,1]$. Using Lemma 2.1 we obtain that $f$ is a measure-preserving bijection.

Finally we will prove that

$$\operatorname{card}\left(\Phi_f(C_0) \setminus \Phi_d(C_0)\right) = \mathfrak{c}. \tag{2.10}$$

Obviously, $C \setminus C_0$ is denumerable, so $\Phi_d(C_0) = \Phi_d(C)$ and using (2.5) we obtain

$$\operatorname{card}\left(C_0 \setminus \Phi_d(C_0)\right) = \mathfrak{c}.$$

On the other hand, $f(C_0) = (0, 1/2) = \Phi_d(f(C_0))$, so $\Phi_f(C_0) = C_0$, which proves (2.10). □

In [2] A. Denjoy proved that the set of points of porosity of a nonempty closed nowhere dense set $F \subset \mathbb{R}$ is residual in $F$ (compare also [9], Proposition 2.7). Hence, as each point of porosity of $F$ is not a density point of $F$, we obtain

**Theorem 2.3.** *If $C \subset [0,1]$ is a symmetric Cantor set such that $\lambda(C) = 1/2$, then $\Phi_d(C)$ is of the first category in $C$.*

**Corollary 2.4.** *$\Phi_d(C)$ is of the first category in $C_0$.*

Now let $f$ be a function from Theorem 2.2.

**Theorem 2.5.** *The set $f(\Phi_d(C))$ is of the first category in $(0, 1/2)$.*

*Proof.* From Corollary 2.4 it follows that $\Phi_d(C) \subset \bigcup_{n=1}^{\infty} F_n$, where $F_n$, for $n \in \mathbb{N}$, are closed subsets of $C$ nowhere dense in $C_0$. Let $n \in \mathbb{N}$. Observe that $f(F_n \cap C_0)$ is nowhere dense in $(0, 1/2)$. Indeed, if $(a, b) \subset (0, 1/2)$, $a < b$, then there exist $x_1, x_2 \in C_0$ such that $f(x_1) = \lambda(C_0 \cap (0, x_1)) = a$ and $f(x_2) = \lambda(C_0 \cap (0, x_2)) = b$. Hence $(x_1, x_2) \cap C_0 \neq \emptyset$ as $\lambda(C_0 \cap (x_1, x_2)) > 0$. Then there exists an interval $(c, d) \subset (x_1, x_2)$ such that $C_0 \cap (c, d) \neq \emptyset$ and $C_0 \cap (c, d) \cap F_n = \emptyset$, so $(f(c), f(d)) \cap f(F_n \cap C_0) = \emptyset$. Clearly, the interval $(f(c), f(d))$ is non-degenerated, as $\lambda((f(c), f(d))) = \lambda((c, d)) > 0$, and $(f(c), f(d)) \subset (a, b)$. Consequently, $f(F_n \cap C_0)$ is nowhere dense in $(0, 1/2)$, for $n \in \mathbb{N}$. □

Now we will prove that the difference between $\Phi_d(A)$ and $\Phi_g(A)$ can be a set which is big in the category sense, for some measure-preserving bijection $g : \mathbb{R} \to \mathbb{R}$.

**Theorem 2.6.** *There exists a measure-preserving bijection $g : \mathbb{R} \to \mathbb{R}$ and a measurable set $A \subset \mathbb{R}$ such that $\Phi_d(A) \setminus \Phi_g(A)$ is of the second category.*

*Proof.* Let $f$ be a function from Theorem 2.2. Put $g = f^{-1}$ and $A = (0, 1/2)$. We have $g(A) = C_0$, $\Phi_d(g(A)) = \Phi_d(C_0) = \Phi_d(C)$ and $\Phi_g(A) = g^{-1}(\Phi_d(C)) = f(\Phi_d(C))$. From the previous theorem $\Phi_g(A)$ is of the first category in $(0, 1/2)$, while $\Phi_d(A) = (0, 1/2)$, so $\Phi_d(A) \setminus \Phi_g(A)$ is residual on $(0, 1/2)$. □

It is natural to ask when $\Phi_f(A) = \Phi_d(A)$ for each $A \in \mathcal{L}$. We will prove that this equality holds if and only if $f(x) = x + c$ or $f(x) = -x + c$ where $c$ is an arbitrary real number.

**Theorem 2.7.** *If $f(x) = x + c$ or $f(x) = -x + c$, where $c \in \mathbb{R}$ is arbitrary, then $\Phi_f(A) = \Phi_d(A)$ for each measurable $A \subset \mathbb{R}$.*

*Proof.* Let $f(x) = x + c$, where $c \in \mathbb{R}$. Then for arbitrary $A \in \mathcal{L}$

$$\Phi_f(A) = f^{-1}\left(\Phi_d\left(f(A)\right)\right) = f^{-1}\left(\Phi_d(A) + c\right) = \Phi_d(A) + c - c = \Phi_d(A).$$

In the second case the proof is analogous. □

Let $\tau_d$ denote the classical density topology on the real line (comp. [7,8]), i.e.

$$\tau_d = \{A \in \mathcal{L} : A \subset \Phi_d(A)\}.$$

**Theorem 2.8.** *If $\Phi_f(A) = \Phi_d(A)$ for each $A \in \mathcal{L}$, then $f(x) = x + c$ or $f(x) = -x + c$ for some $c \in \mathbb{R}$.*

*Proof.* From the assumption it follows that

$$f^{-1}\left(\Phi_d\left(f(A)\right)\right) = \Phi_d(A)$$

for each $A \in \mathcal{L}$, so

$$\Phi_d\left(f(A)\right) = f\left(\Phi_d(A)\right).$$

Hence if $A \in \tau_d$ then $f(A) \in \tau_d$, as $A \subset \Phi_d(A)$ implies $f(A) \subset f\left(\Phi_d(A)\right) = \Phi_d\left(f(A)\right)$.

Obviously, $\mathbb{R} = (-\infty, x_0) \cup \{x_0\} \cup (x_0, \infty)$ for each $x_0 \in \mathbb{R}$, so

$$f\left((-\infty, x_0)\right) \cup f\left((x_0, \infty)\right) = \mathbb{R} \setminus \{f(x_0)\} \tag{2.11}$$

and $f\left((-\infty, x_0)\right)$, $f\left((x_0, \infty)\right)$ are open in $\tau_d$.

Observe that

$$\Phi_d\left(f\left((-\infty, x_0)\right)\right) \cup \Phi_d\left(f\left((x_0, \infty)\right)\right) = \mathbb{R} \setminus \{f(x_0)\}.$$

The inclusion " $\supset$ " is obvious. On the other hand, if

$$f(x_0) \in \Phi_d\left(f\left((-\infty, x_0)\right)\right) \cup \Phi_d\left(f\left((x_0, \infty)\right)\right),$$

then $\mathbb{R}$ is a union of two disjoint non-empty sets from $\tau_d$ - a contradiction, as the families of sets which are connected in the density topology and in the natural topology on the real line are equal (compare with [4, Theorem 3]).

Now we will show that

$$f\left((-\infty, x_0)\right) = (-\infty, f(x_0)) \text{ and } f\left((x_0, \infty)\right) = (f(x_0), \infty) \tag{2.12}$$

or

$$f\left((-\infty, x_0)\right) = (f(x_0), \infty) \text{ and } f\left((x_0, \infty)\right) = (-\infty, f(x_0)). \tag{2.13}$$

Suppose that this is not the case. Then there exist two points $y_1 \in f((-\infty, x_0))$ and $y_2 \in f((x_0, \infty))$ such that $y_1, y_2 \in (-\infty, f(x_0))$ or $y_1, y_2 \in (f(x_0), \infty)$. Consider the first case. If $(z_1, z_2)$ is an open interval such that $(z_1, z_2) \subset (-\infty, f(x_0))$ and $y_1, y_2 \in (z_1, z_2)$ then $(z_1, z_2) \cap f((-\infty, x_0)) \in \tau_d$ and $(z_1, z_2) \cap f((x_0, \infty)) \in \tau_d$. Since $(z_1, z_2) \subset \mathbb{R} \setminus \{f(x_0)\}$, from (2.11) we obtain

$$(z_1, z_2) = (z_1, z_2) \cap f((-\infty, x_0)) \cup (z_1, z_2) \cap f((x_0, \infty))$$

and both summands are non-empty, so $(z_1, z_2)$ is not connected on $\tau_d$ – a contradiction. The proof in the second case is analogous.

Now from the arbitrariness of $x_0 \in \mathbb{R}$, using (2.12) or (2.13) we will prove that $f$ is monotone. Suppose that (2.12) holds. Let $x_1 < x_2$, $x_1, x_2 \in \mathbb{R}$. Then

$$(-\infty, f(x_1)) = f((-\infty, x_1)) \subset f((-\infty, x_2)) = (-\infty, f(x_2)),$$

so $f(x_1) < f(x_2)$, i.e. $f$ is increasing.

Analogously, $f$ is decreasing if (2.13) holds. Since $f$ is a monotone bijection, it must be continuous.

Now we will prove that $f$ must be a linear function of the form $f(x) = x + c$ or $f(x) = -x + c$, for some $c \in \mathbb{R}$.

Assume that $f$ is increasing. Fix $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$. Then $f((x_1, x_2)) = (f(x_1), f(x_2))$, so $f(x_2) - f(x_1) = x_2 - x_1$, as $f$ preserves measure. Hence

$$f(x_2) = x_2 + (f(x_1) - x_1).$$

From the arbitrariness of $x_1$ and $x_2$ we obtain

$$f(x) = x + c$$

for all $x \in \mathbb{R}$. If $f$ is decreasing the proof is analogous. □

Now we will prove that $\Phi_f(A + a) = \Phi_f(A) + a$ for each $A \in \mathcal{L}$ and $a \in \mathbb{R}$ if and only if $f$ is linear function of the form $f(x) = x + c$ or $f(x) = -x + c$, where $c \in \mathbb{R}$ is arbitrary. Obviously, such equality holds for the classical density operator, as the Lebesgue measure is invariant with respect to translation.

**Theorem 2.9.** *If $f(x) = x + c$ or $f(x) = -x + c$, where $c \in \mathbb{R}$ is arbitrary, then*

$$\Phi_f(A + a) = \Phi_f(A) + a$$

*for each $A \in \mathcal{L}$ and $a \in \mathbb{R}$.*

*Proof.* Let $A \in \mathcal{L}$ and $a \in \mathbb{R}$. According to Theorem 2.7, if $f(x) = x + c$ then

$$\Phi_f(A + a) = \Phi_d(A + a) = \Phi_d(A) + a = \Phi_f(A) + a.$$

In the second case the proof is analogous. □

**Theorem 2.10.** *If $\Phi_f(A+a) = \Phi_f(A)+a$ for each $A \in \mathcal{L}$ and $a \in \mathbb{R}$ then $f(x) = x+c$ or $f(x) = -x+c$ for some $c \in \mathbb{R}$.*

*Proof.* If $A_1, A_2 \in \mathcal{L}$ and $A_1 \subset A_2$ then $\Phi_f(A_1) \subset \Phi_f(A_2)$, as $\Phi_f$ is a lower density operator. In particular, if $A = (-\infty, x_0)$ for some $x_0 \in \mathbb{R}$ and $a > 0$ then $A \subset A + a$, so

$$\Phi_f(A) \subset \Phi_f(A+a) = \Phi_f(A) + a. \tag{2.14}$$

We will prove that if $A = (-\infty, x_0)$ then $B = \Phi_f(A) = \Phi_f((-\infty, x_0))$ is also a left half-line. Suppose that it is not the case. Then there exist two points $y_1, y_2 \in \mathbb{R}$, $y_2 < y_1$ such that $y_1 \in B$ and $y_2 \notin B$. If $a = y_1 - y_2$ then $y_2 + a = y_1 \in B$. Simultaneously $y_2 + a \notin B + a$ - a contradiction, as from (2.14)

$$B = \Phi_f(A) \subset \Phi_f(A+a) = B + a.$$

Hence $\Phi_f(A)$ is a left half-line (open or closed). Similarly one can prove that $\Phi_f((x_0, \infty))$ is a right-half-line (open or closed).

As $\Phi_f$ is a lower density operator, we have

$$\lambda\left(\Phi_f((-\infty, x_0))\Delta(-\infty, x_0)\right) = 0,$$

so $\Phi_f((-\infty, x_0)) = (-\infty, x_0)$ or $\Phi_f((-\infty, x_0)) = (-\infty, x_0]$ as both sets are left half-lines. Similarly we prove that $\Phi_f((x_0, \infty)) = (x_0, \infty)$ or $\Phi_f((x_0, \infty)) = [x_0, \infty)$.

Suppose that $\Phi_f((-\infty, x_0)) = (-\infty, x_0]$. From the third property of a lower density operator $\Phi_f((x_0, \infty)) = (x_0, \infty)$ and

$$\Phi_f((-\infty, x_0)) \cup \Phi_f((x_0, \infty)) = \mathbb{R}.$$

Hence $f\left(\Phi_f((-\infty, x_0)) \cup \Phi_f((x_0, \infty))\right) = \mathbb{R}$. On the other hand for arbitrary $C \in \mathcal{L}$

$$f(\Phi_f(C)) = \Phi_d(f(C)), \tag{2.15}$$

so

$$\Phi_d(f((-\infty, x_0))) \cup \Phi_d(f((x_0, \infty))) = \mathbb{R},$$

a contradiction, as $\mathbb{R}$ is connected relative to the density topology $\tau_d$.

Similarly one can proceed under the assumption $\Phi_f((x_0, \infty)) = [x_0, \infty)$.

Consequently, $\Phi_f((-\infty, x_0)) = (-\infty, x_0)$ and $\Phi_f((x_0, \infty)) = (x_0, \infty)$.

Hence, using (2.15),

$$\Phi_d(f((-\infty, x_0))) = f\left(\Phi_f((-\infty, x_0))\right) = f((-\infty, x_0))$$

and

$$\Phi_d(f((x_0, \infty))) = f\left(\Phi_f((x_0, \infty))\right) = f((x_0, \infty)),$$

so $f((-\infty, x_0)) \in \tau_d$ and $f((x_0, \infty)) \in \tau_d$. Clearly,

$$f((-\infty, x_0)) \cup f((x_0, \infty)) = \mathbb{R} \setminus \{f(x_0)\}.$$

Now we will prove that $f((-\infty, x_0)) = (-\infty, f(x_0))$ or $f((-\infty, x_0)) = (f(x_0), \infty)$. Suppose that this is not the case. Then there exist two points $x_1, x_2 \in \mathbb{R}$, $x_1 < x_0 < x_2$ such that $f(x_1)$ and $f(x_2)$ are in the same half-line. To fix ideas suppose that $f(x_1), f(x_2) \in (-\infty, f(x_0))$. If $(z_1, z_2)$ is an open interval such that $f(x_1), f(x_2) \in (z_1, z_2)$ and $(z_1, z_2) \subset (-\infty, f(x_0))$ then $\emptyset \neq (z_1, z_2) \cap f((-\infty, x_0)) \in \tau_d$ and $\emptyset \neq (z_1, z_2) \cap f((x_0, \infty)) \in \tau_d$. As $f(x_0) \notin (z_1, z_2)$, $(z_1, z_2)$ is not connected in $\tau_d$ – a contradiction.

In the second case the proof is analogous. Similarly as in the proof of Theorem 2.8 we obtain that $f$ is monotone and continuous. Simultaneously, as $f$ preserves measure, $f(x) = x + c$ if $f$ is increasing and $f(x) = -x + c$ in the other case. □

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REFERENCES

[1] A.M. Bruckner, *Differentiation of Real Functions*, Lecture Notes in Mathematics, vol. 659, Springer-Verlag, Berlin, 1978.

[2] A. Denjoy, *Leçons sur le Calcul des Coefficients d'une Série Trigonométrique. Tome I. La Différentiation Seconde Mixte et Son Application aux Séries Trigonométriques*, Gauthier-Villars, Paris, 1941.

[3] C. Goffman, C. Neugebauer, T. Nishiura, *Density topology and approximate continuity*, Duke Math. J. **28** (1961), 497–505.

[4] C. Goffman, D. Waterman, *Approximately continuous transformations*, Proc. AMS **12** (1961), 116–121.

[5] G. Ivanova, E. Wagner-Bojakowska, *On lower density operators*, submitted.

[6] J. Lukeš, J. Malý, L. Zajiček, *Fine Topology Methods in Real Analysis and Potential Theory*, Lecture Notes in Mathematics, vol. 1189, Springer-Verlag, 1986.

[7] J.C. Oxtoby, *Measure and Category*, Springer-Verlag, New York, 1971.

[8] W. Wilczyński, *Density topologies*, [in:] E. Pap (ed.), *Handbook of Measure Theory*, vol. I and II, Amsterdam, North-Holland, 2002, 675–702.

[9] L. Zajíček, *Porosity and $\sigma$-porosity*, Real Anal. Exchange **13** (1987/88), 314–350.

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