Int. J. Appl. Math. Comput. Sci., 2016, Vol. 26, No. 3, 521–531
DOI: 10.1515/amcs-2016-0036

CONTROLLABILITY CRITERIA FOR TIME–DELAY FRACTIONAL SYSTEMS WITH A RETARDED STATE

BEATA SIKORA [a]

[a] Institute of Mathematics
Silesian University of Technology, Kaszubska 23, 44-100 Gliwice, Poland
e-mail: beata.sikora@polsl.pl

The paper is concerned with time-delay linear fractional systems with multiple delays in the state. A formula for the solution of the discussed systems is presented and derived using the Laplace transform. Definitions of relative controllability with and without constraints for linear fractional systems with delays in the state are formulated. Relative controllability, both with and without constraints imposed on control values, is discussed. Various types of necessary and sufficient conditions for relative controllability and relative U-controllability are established and proved. Numerical examples illustrate the obtained theoretical results.

Keywords: fractional dynamical systems, controllability, delays in the state, constraints, pseudo-transition matrix, Caputo derivative.

1. Introduction

Fractional-order derivatives are a generalization of classical integer-order ones. Mathematical modeling of systems and processes with the use of fractional-order derivatives leads to fractional differential equations. Fractional differential equations occur in mathematical models of, among other things, mechanical, biological, chemical and medical phenomena. It has become apparent that fractional-order models reflect the behavior of many real-life processes more accurately than integer-order ones. For more details concerning fractional calculus and its practical applications refer to the monographs of Oldham and Spanier (1974), Miller and Ross (1993), Samko *et al.* (1993), Podlubny (1999), Kilbas *et al.* (2006), Sabatier *et al.* (2007) or Monje *et al.* (2010).

Numerous mathematical models describe dynamical systems with delays in control, or both the state and control. Therefore, studying the properties of systems with delays is especially important.

The controllability of dynamical systems plays a crucial role in their analysis. In recent years, various controllability problems for different types of fractional-order dynamical systems have been considered in many publications and monographs. The controllability of deterministic fractional dynamical systems without delays was studied, among others, by Klamka (2010; 2011), Klamka *et al.* (2014) or Babiarz *et al.* (2016) for discrete-time fractional systems, Kaczorek (2011) as well as Kaczorek and Rogowski (2015) for positive fractional linear systems, both discrete- and continuous-time, and Chen *et al.* (2006), Chikriy and Matichin (2008), Sakthivel *et al.* (2011), Wang and Zhou (2012), Balachandran and Kokila (2012; 2013) or Balachandran *et al.* (2012b) for continuous time fractional systems.

The controllability of fractional systems with delays in control was analyzed by Trzasko (2008), Kaczorek (2011), Balachandran *et al.*, (2012c; 2012a), Wei (2012) as well as Kaczorek and Rogowski (2015). The controllability of fractional systems with delays in the state was discussed by Zhang *et al.* (2013) and Busłowicz (2014). However, there are only few works concerning the controllability of time-delay fractional systems with retarded state. It should also be noted that the majority of papers on controllability of fractional systems address controllability issues for unconstrained controls. Constrained controllability of integer order systems with delays was discussed, among others, by Sikora (2003; 2005), Klamka (2008; 2009), or Sikora and Klamka (2012). Works on controllability of linear fractional systems with bounded inputs include those by Kaczorek (2014a; 2014b) for fractional positive discrete-time linear systems and fractional positive continuous-time linear systems, respectively, Sikora (2016) for fractional continuous-time linear systems, as well as Pawłuszewicz and Mozyrska (2013) for h-difference linear systems with two fractional orders. It should be stressed that, in practice, control (an input function) is not completely unlimited, but is usually constrained in various ways.

The aim of the paper is to give new controllability criteria (necessary and sufficient conditions) for continuous-time linear fractional systems with delays in the state. The controllability criteria both for unconstrained and constrained controls are formulated and proved. Theoretical results presented in the paper can be applied, among other things, to chemical solution control systems. For example, the cascade connection of two fully filled mixers can be described by a system of fractional equations with one delay in the state.

The paper is organized as follows. In Section 2 we recall some preliminary definitions and formulas. In Section 3 we present the mathematical model of linear fractional dynamical systems with multiple delays in the state. We formulate and prove the existence theorem for the solution of the discussed systems. Section 4 contains the main results of the paper. First, we formulate definitions for relative controllability and relative U-controllability of systems. Next, the main results of the paper are presented, i.e., the criteria (necessary and sufficient conditions) for relative controllability of the examined time-delay fractional systems with a retarded state. The proofs of the theorems are given. In Section 5 the theoretical results are illustrated by numerical examples. Finally, some concluding remarks are included in Section 6.

2. Preliminaries

Mathematical models containing fractional differential equations turn out to better describe some phenomena previously modeled by integer-order differential equations. Different fractional derivatives have been defined in fractional calculus. In this paper we use the Caputo fractional derivative. This is due to the fact that in the Caputo approach the initial conditions for fractional differential equations take on the same form as for integer-order differential equations (Podlubny, 1999). The definition is the following.

Definition 1. The *Caputo fractional derivative* of the order α ($n < \alpha < n+1$, $n \in \mathbb{N}$) for a given function $f\colon \mathbb{R}^+ \to \mathbb{R}$ is the function

$$^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha+1)} \int_0^t \frac{f^{(n+1)}(\tau)}{(t-\tau)^{\alpha-n}}\,\mathrm{d}\tau,$$

where Γ is the gamma function.

It is obvious that as $\alpha \to n$ the Caputo derivative tends to the n-th order conventional derivative of the function f, e.g., $\lim_{\alpha\to n} {}^{C}D^{\alpha} f(t) = f^{(n)}(t)$.

As has been mentioned, the Caputo fractional derivative allows traditional initial conditions to be considered in the formulation of a mathematical model of a dynamical system.

In the theory of fractional calculus, an important role is that of the Mittag-Leffler function. Below we recall the definition (Podlubny, 1999).

Definition 2. The *Mittag-Leffler function* is a special function defined by the following series:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

for $z \in \mathbb{C}$ and $\alpha, \beta > 0$.

For real positive α and β, the series in Definition 2 converges for all values of the argument z.

Based on the definition of the Mittag-Leffler function, for $\alpha > 0$ and an arbitrary n-th order square matrix A, we can give the formula for a *pseudo-transition matrix* $\Phi_0(t)$ of the linear fractional system $^{C}D^{\alpha}x(t) = A(t)x(t)$ (Monje *et al.*, 2010):

$$\Phi_0(t) = E_{\alpha,1}(At^{\alpha}) = \sum_{k=0}^{\infty} \frac{A^k t^{\alpha k}}{\Gamma(k\alpha+1)},$$

and then we set

$$\Phi(t) = t^{\alpha-1} E_{\alpha,\alpha}(At^{\alpha}) = t^{\alpha-1} \sum_{k=0}^{\infty} \frac{A^k t^{\alpha k}}{\Gamma((k+1)\alpha)}.$$

For $\alpha = 1$ we obtain the classical transition matrix of ordinary differential equations,

$$\Phi_0(t) = \sum_{k=0}^{\infty} \frac{A^k t^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = e^{At}.$$

Therefore the pseudo-transition matrix $\Phi_0(t)$ is also called the matrix α-exponential function and is denoted by $\Phi_0(t) = e_{\alpha}^{At}$ (Kilbas *et al.*, 2006). It is convergent in the space of the n-th order matrices with real elements.

For the purposes of further calculations there are some formulas for the Laplace and the inverse Laplace transforms. For $\alpha \in (0,1)$, the following formulas hold:

$$\mathcal{L}[^{C}D^{\alpha} f(t)] = s^{\alpha}\mathcal{L}[f(t)] - s^{\alpha-1} f(0)$$

(see, e.g., Zhang *et al.*, 2013) and

$$\mathcal{L}^{-1}[s^{\alpha-1}(s^{\alpha}I - A)^{-1}] = \Phi_0(t),$$

$$\mathcal{L}^{-1}[(s^{\alpha}I - A)^{-1}] = \Phi(t)$$

(see, e.g., Monje *et al.*, 2010).

3. Mathematical model

We consider time-delay linear fractional dynamical systems with multiple, constant delays in the state described by a fractional differential equation with a retarded argument of the following form:

$$ {}^{C}D^{\alpha}x(t) = \sum_{i=0}^{M} A_i\, x(t-h_i) + B\, u(t) \tag{1}$$

for $t \geq 0$ and $0 < \alpha < 1$, where

- $x(t) \in \mathbb{R}^n$ is the state vector,
- $u \in L^2_{\text{loc}}([0,\infty), \mathbb{R}^m)$ is the control,
- A_i are $(n \times n)$-dimensional matrices with real elements for $i = 0, 1, \dots, M$,
- B is an $(n \times m)$-dimensional matrix with real elements,
- $h_i \colon [0,T] \to \mathbb{R}$ for $i = 1, 2 \dots, M$ are constant delays in the state such that the following inequalities hold:

$$0 = h_0 < h_1 < \ldots < h_i < \ldots < h_{M-1} < h_M.$$

Let $z_0 = (x(0), x_0) \in \mathbb{R}^n \times L^2([-h_M, 0), \mathbb{R}^n)$ be the initial conditions, where $x(0) \in (\mathbb{R})^n$ is the state vector at time $t = 0$ and x_0 is a function given on $[-h_M, 0]$. The Hilbert space $\mathbb{R}^n \times L^2([-h_M, 0), \mathbb{R}^n)$ with the scalar product defined as

$$\begin{aligned}&\langle (x(t), x_t), (y(t), y_t) \rangle\\ &\quad= \sum_{i=1}^{n} x_i(t) y_i(t) + \int_{-h_M}^{0} \langle x_t(\tau), y_t(\tau) \rangle_{\mathbb{R}^n}\, \mathrm{d}\tau\end{aligned}$$

is denoted by $M^2([-h_M, 0], \mathbb{R}^n)$.

The pair $z_0 = (x(0), x_0) \in M^2([-h_M, 0), \mathbb{R}^n)$, where $x(0)$ is the state vector at time $t = 0$ and $x_0 = x_0(\tau)$ for $\tau \in [-h_M, 0)$ is the segment of the trajectory of the length h_M defined on $[t - h_M, t)$, is called the *initial complete state* of the fractional system (1) for $t \geq 0$.

Any control $u \in L^2_{\text{loc}}([0, \infty), \mathbb{R}^m)$ is called an *admissible control* for the fractional system (1).

Theorem 1. *For given initial conditions $z_0 = (x(0), x_0)$ and an admissible control u, for every $t \geq 0$ there exists a unique solution $x(t) = x(t, z_0, u)$ of the fractional equation (1) of the form*

$$\begin{aligned}x(t) &= \Phi_0(t)x(0) + \int_0^t \Phi(t-\tau) \sum_{i=1}^{M} A_i x(\tau - h_i)\, \mathrm{d}\tau\\ &\quad+ \int_0^t \Phi(t-\tau) B u(\tau)\, \mathrm{d}\tau,\end{aligned} \tag{2}$$

where

$$\Phi_0(t) = \sum_{k=0}^{\infty} \frac{A_0^k t^{\alpha k}}{\Gamma(k\alpha + 1)},$$

$$\Phi(t) = t^{\alpha-1} \sum_{k=0}^{\infty} \frac{A_0^k t^{\alpha k}}{\Gamma((k+1)\alpha)}.$$

Proof. Applying the Laplace transform to both the sides of the fractional differential equation (1), for $t \geq 0$ we have

$$\begin{aligned}&s^{\alpha}\mathcal{L}[x(t)] - s^{\alpha-1}x(0)\\ &\quad= A_0\mathcal{L}[x(t)] + \mathcal{L}\Big[\sum_{i=1}^{M} A_i x(t-h_i) + Bu(t)\Big].\end{aligned}$$

Hence

$$\begin{aligned}&(s^{\alpha}I - A_0)\mathcal{L}[x(t)]\\ &\quad= s^{\alpha-1}x(0) + \mathcal{L}\Big[\sum_{i=1}^{M} A_i x(t-h_i) + Bu(t)\Big],\end{aligned}$$

and thus

$$\begin{aligned}\mathcal{L}[x(t)] &= (s^{\alpha}I - A_0)^{-1}s^{\alpha-1}x(0)\\ &\quad+ (s^{\alpha}I - A_0)^{-1}\mathcal{L}\Big[\sum_{i=1}^{M} A_i x(t-h_i) + Bu(t)\Big]\\ &= \mathcal{L}[\Phi_0(t)x(0)]\\ &\quad+ \mathcal{L}\big[\Phi(t)\big]\mathcal{L}\Big[\sum_{i=1}^{M} A_i x(t-h_i) + Bu(t)\Big].\end{aligned}$$

Now we apply the convolution theorem for the Laplace transform and obtain

$$\begin{aligned}\mathcal{L}[x(t)] &= \mathcal{L}[\Phi_0(t)x(0)]\\ &\quad+ \mathcal{L}\Big[\int_0^t \Phi(t-\tau)\Big(\sum_{i=1}^{M} A_i x(\tau - h_i)\\ &\quad+ Bu(\tau)\Big)\, \mathrm{d}\tau\Big].\end{aligned}$$

Using the inverse Laplace transform on both the sides of the above equation, we have

$$\begin{aligned}x(t) &= \Phi_0(t)x(0) + \int_0^t \Phi(t-\tau)\\ &\quad\times \Big(\sum_{i=1}^{M} A_i x(\tau - h_i) + Bu(\tau)\Big)\, \mathrm{d}\tau\\ &= \Phi_0(t)x(0) + \int_0^t \Phi(t-\tau) \sum_{i=1}^{M} A_i x(\tau - h_i)\, \mathrm{d}\tau\\ &\quad+ \int_0^t \Phi(t-\tau) B u(\tau)\, \mathrm{d}\tau.\end{aligned}$$

∎

4. Main results

In this section we formulate and prove some theorems that establish the criteria (necessary and sufficient conditions) of relative controllability and relative U-controllability for the time-delay fractional system (1).

Before we formulate some controllability criteria for the time-delay fractional system (1), in this section we define relative controllability and relative U-controllability for the system (1) on the time interval $[0, T]$.

Definition 3. The time-delay fractional system (1) is called *relatively controllable* on $[0, T]$ from the initial complete state $z_0 = (x(0), x_0)$ into a set $S \subset \mathbb{R}^n$ if, for every $\widetilde{x} \in S$, there exists an admissible control $\widetilde{u} \in L^2([0, T], \mathbb{R}^m)$ such that

$$x(T, z_0, \widetilde{u}) = \widetilde{x}.$$

Definition 4. The time-delay fractional system (1) is called *relatively U-controllable* on $[0, T]$ from the initial complete state $z_0 = (x(0), x_0)$ into a set $S \subset \mathbb{R}^n$ if, for every $\widehat{x} \in S$, there exists an admissible control $\widehat{u} \in L^2([0, T], U)$, $U \subset \mathbb{R}^m$, such that

$$x(T, z_0, \widehat{u}) = \widehat{x}.$$

If $S = \{0\}$, then the system is called relatively null controllable or relatively null U-controllable. If $S = \mathbb{R}^n$, the system is called relatively controllable or relatively U-controllable, respectively.

The theorem below formulates the necessary and sufficient conditions for relative controllability of the retarded system (1).

Theorem 2. *The fractional system (1) is relatively controllable on the time interval $[0, T]$ if and only if the $(n \times n)$-dimensional Gramian matrix*

$$W(0, T) = \int_0^T \Phi(T - \tau) B B' \Phi'(T - \tau) \, \mathrm{d}\tau$$

is nonsingular, where $'$ denotes the matrix transpose and $\Phi'(t) = E_{\alpha,\alpha}(A_0' t^\alpha)$.

Proof. We prove the sufficient condition first. Suppose that $W(0, T)$ is nonsingular. It follows that there exists the inverse matrix $W^{-1}(0, T)$. For any initial complete state $z_0 \in \mathbb{R}^n \times L^2([-h_M, 0), \mathbb{R}^n)$ we can take the following control function:

$$\widetilde{u}(t) = B' \Phi'(T - t) W^{-1}(0, T) \Big[\widetilde{x} - \Phi_0(t) x(0) - \int_0^T \Phi(T - s) \sum_{i=1}^{M} A_i x(s - h_i) \, \mathrm{d}s \Big]$$

for $\widetilde{x} \in \mathbb{R}^n$. From Theorem 1 it follows that

$$\begin{aligned} x(T) = x(T, z_0, \widetilde{u}) &= \Phi_0(T) x(0) \\ &\quad + \int_0^T \Phi(T - \tau) \sum_{i=1}^{M} A_i x(\tau - h_i) \, \mathrm{d}\tau \\ &\quad + \int_0^T \Phi(T - \tau) B \widetilde{u}(\tau) \, \mathrm{d}\tau. \end{aligned}$$

After substitution we obtain

$$\begin{aligned} & x(T, z_0, \widetilde{u}) \\ &= \Phi_0(T) x(0) + \int_0^T \Phi(T - \tau) \sum_{i=1}^{M} A_i x(\tau - h_i) \, \mathrm{d}\tau \\ &\quad + \int_0^T \Phi(T - \tau) B B' \Phi'(T - t) W^{-1}(0, T) \\ &\quad \times \Big[\widetilde{x} - \Phi_0(T) x(0) - \int_0^T \Phi(T - s) \\ &\quad \times \sum_{i=1}^{M} A_i x(s - h_i) \, \mathrm{d}s \Big] \, \mathrm{d}\tau = \widetilde{x}. \end{aligned}$$

By Definition 3, the system (1) is relatively controllable on $[0, T]$.

We prove the necessary condition by contradiction. Suppose that the system (1) is relatively controllable, but the Gramian matrix $W(0, T)$ is singular. Then there exists a vector $\widetilde{x} \neq 0$ such that

$$\begin{aligned} & \widetilde{x}' W(0, T) \widetilde{x} = 0 \\ &\qquad = \int_0^T \widetilde{x}' \Phi(T - \tau) B B' \Phi'(T - \tau) \widetilde{x} \, \mathrm{d}\tau. \end{aligned}$$

Thus, for $t \in [0, T]$, we have

$$\widetilde{x}' \Phi(T - t) B = 0. \tag{3}$$

Since the system is controllable, it can be driven from the initial state z_0 to an arbitrary state $x(T) \in \mathbb{R}^n$. Hence there exists a control $u_0(t)$ that drives the initial state z_0 to zero. This means that

$$\begin{aligned} x(T) &= x(T, z_0, u_0) \\ &= \Phi_0(T) x(0) + \int_0^T \Phi(T - \tau) \sum_{i=1}^{M} A_i x(\tau - h_i) \, \mathrm{d}\tau \\ &\quad + \int_0^T \Phi(T - \tau) B u_0(\tau) \, \mathrm{d}\tau = 0. \end{aligned}$$

Moreover, there exists a control $\widetilde{u}(t)$ that drives the initial state z_0 to the state $\widetilde{x}$, so

$$\begin{aligned}\widetilde{x} &= x(T, z_0, \widetilde{u})\\ &= \Phi_0(T)x(0) + \int_0^T \Phi(T-\tau)\sum_{i=1}^{M} A_i x(\tau - h_i)\,\mathrm{d}\tau\\ &\quad + \int_0^T \Phi(T-\tau)B\widetilde{u}(\tau)\,\mathrm{d}\tau.\end{aligned}$$

Combining the above two solutions, we have

$$\widetilde{x} - \int_0^T \Phi(T-\tau)B[\widetilde{u}(\tau) - u_0(\tau)]\,\mathrm{d}\tau = 0.$$

Multiplying both the sides of the equality by $\widetilde{x}'$, and using (3), it follows that $\widetilde{x}'\widetilde{x} = 0$. Thus $\widetilde{x} = 0$, which contradicts the assumption. Therefore the Gramian matrix $W(0,T)$ is nonsingular. ■

Now we rewrite the solution (2) in the form containing the so-called free solution of the system (1). Let $\Phi_f(t)$ be defined as

$$\Phi_f(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} I + \sum_{i=1}^{M}\int_0^{t-h_i} \Phi(\tau)A_i\,\mathrm{d}\tau \qquad (4)$$

for $t > 0$, and assume that $\Phi_f(0) = I$ and $\Phi_f(t) = 0$ for $t < 0$. Then

$$\begin{aligned}x(t, z_0, 0) &= \Phi_0(t)x(0)\\ &\quad + \sum_{i=1}^{M}\int_{-h_i}^{0} \Phi_f(t-\tau-h_i)A_i x_0(\tau)\,\mathrm{d}\tau\end{aligned}$$

is the free solution that depends only on the initial complete state $z_0 = (x(0), x_0)$. Therefore the solution (2) takes the form

$$x(t) = x(t, z_0, 0) + \int_0^t \Phi_f(t-\tau)Bu(\tau)\,\mathrm{d}\tau.$$

Applying the methods presented by Manitius (1974) for integer-order systems with one delay in the state, we state the following theorem that formulates a new criterion for relative controllability of the system (1).

Theorem 3. *The fractional system (1) is relatively controllable on the time interval $[0,T]$ if and only if the relation*

$$a'\Phi_f(T-t)B = 0$$

for $a \in \mathbb{R}^n$ and $t \in [0,T]$ implies $a = 0$.

Proof. Consider the linear mapping

$$F\colon L^2([0,T], \mathbb{R}^m) \to \mathbb{R}^n$$

given by

$$F(u) = \int_0^T \Phi_f(T-\tau)Bu(\tau)\,\mathrm{d}\tau.$$

Then F is a continuous linear operator from the Hilbert space $L^2([0,T],\mathbb{R}^m)$ to the Hilbert space $\mathbb{R}^n$. Let $\operatorname{im} F$ denote the range (image) of F. Since the range of linear mapping is a subset of its codomain, we have

$$\operatorname{im} F \subset \mathbb{R}^n,$$

and the orthogonal complement of the linear subspace $\operatorname{im} F$ satisfies the relation

$$(\operatorname{im} F)^{\perp} = \ker \overline{F},$$

where $\overline{F}$ is the adjoint of F and $\ker \overline{F}$ denotes the null space (kernel) of $\overline{F}$.

Since the controllability condition becomes

$$(\operatorname{im} F)^{\perp} = \{0\},$$

we obtain

$$\ker \overline{F} = 0.$$

However, for any $a \in \mathbb{R}^n$ and $u \in L^2([0,T],\mathbb{R}^m)$, the scalar products in the given spaces are equal, that is,

$$\langle a, F(u)\rangle_{\mathbb{R}^n} = \langle \overline{F}(a), u\rangle_{L^2([0,T],\mathbb{R}^m)}.$$

Therefore, we have

$$\begin{aligned}&\Big\langle a, \int_0^T \Phi_f(T-\tau)Bu(\tau)\,\mathrm{d}\tau\Big\rangle_{\mathbb{R}^n}\\ &\qquad = \int_0^T \big[a'\Phi_f(T-\tau)B\big]u(\tau)\,\mathrm{d}\tau.\end{aligned}$$

Thus $\overline{F}(a) = a'\Phi_f(T-t)B$ for $t \in [0,T]$. It follows that $\ker \overline{F}$ is a set of $a \in \mathbb{R}^n$ such that $a'\Phi_f(T-t)B = 0$ almost everywhere in $[0,T]$. But we have obtained above that $\ker \overline{F}$ consists of zero only ($\ker \overline{F} = 0$), which proves the theorem. ■

In order to formulate the next criterion, for $t \in [0,\infty)$, we define recursively the following matrices:

$$Q_k(t) = \sum_{i=0}^{M} A_i Q_{k-1}(t-h_i), \quad k = 1, 2, \dots, \qquad (5)$$

$$Q_0(t) = \begin{cases} B & \text{for } t = 0,\\ 0 & \text{for } t \neq 0\end{cases}$$

and the set

$$\widehat{Q}_n(T) = \{Q_0(t), Q_1(t), \dots, Q_{n-1}(t), \quad t \in [0,T]\}$$

for $t = h_i, 2h_i, 3h_i, \dots, i = 0, 1, 2, \dots M$.

Let rank $\widehat{Q}_n(T)$ mean the rank of the block matrix composed of all matrices from the set $\widehat{Q}_n(T)$.

The theorem below is the necessary and sufficient condition for relative controllability of the system (1) that is based on the matrices $A_0, \dots, A_M$ and B. This, easy to use, algebraic criterion is similar to the commonly known Kalman rank condition. The criterion is a generalization of the controllability condition formulated for integer-order systems with delays. Details can be found in the work of Klamka (1991).

Theorem 4. *The fractional system (1) is relatively controllable on the time interval* $[0,T]$ *if and only if*

$$\operatorname{rank}\widehat{Q}_n(T)=n.$$

Remark 1. It is worth noticing that for $T\le h_1$ we obtain

$$\widehat{Q}_n(T)=\{B,A_0B,A_0^2B,\dots,A_0^{n-1}B\},$$

and the condition $\operatorname{rank}\widehat{Q}_n(T)=n$ is reduced to well-known controllability criteria for fractional systems without delays (see, e.g., Monje *et al.*, 2010)

$$\operatorname{rank}[B\ \ A_0B\ \ A_0^2B\ \ \dots\ \ A_0^{n-1}B]=n.$$

Now we impose constraints on control values. Let $U\subset\mathbb{R}^m$ be a nonempty set and $S\subset\mathbb{R}^n$ have the form

$$S=\{x\in\mathbb{R}^n\colon Lx=c\},\tag{6}$$

where L is a $(p\times n)$-matrix and $c\in\mathbb{R}^p$ is a given vector. If $L=I_n$ (identity matrix) and $c=0$, we obtain $S=\{0\}$.

In much the some way as for integer-order dynamical systems (Klamka, 1991), we can formulate a definition of the *attainable set* for the dynamical system (1).

Definition 5. The set

$$K([0,t],z_0)=\Big\{x\in\mathbb{R}^n: x(t)=x(t,z_0,0)+\int_0^t\Phi_f(t-\tau)Bu(\tau)\,\mathrm{d}\tau\Big\}$$

is called an *attainable set* from the initial complete state $z_0=(x(0),x_0)$ for the time-delay fractional system (1). $K([0,t],z_0)$ is also called the set of reachable states.

The next theorem gives a new criterion for relative U-controllability of the fractional system (1) when U is a convex and compact subset of $\mathbb{R}^m$ containing the origin.

Theorem 5. *Let* $U\subset\mathbb{R}^m$ *be a convex and compact set containing the origin in its interior and let the system (1) be of commensurate order. If*

$$\operatorname{rank}\widehat{Q}_n(T)=n,$$

$|\arg(\lambda_i)|>\alpha\pi/2$, $1\le i\le n$, *where* λ_i *are the eigenvalues of matrix* A, *and the characteristic equation* $\det(s^\alpha I-A)=0$ *has no purely imaginary roots, then the fractional system (1) is relatively null* U*-controllable on* $[0,T]$.

Proof. Suppose $U\subset\mathbb{R}^m$ is a convex and compact set containing the origin in its interior. If $\operatorname{rank}\widehat{Q}_n(T)=n$, then the fractional system (1) is relatively controllable on $[0,T]$ based on Theorem 4. Moreover, if the system is of a commensurate order and the eigenvalues of matrix A satisfy the condition $|\arg(\lambda_i)|>\alpha\pi/2$, $1\le i\le n$, and the characteristic equation $\det(s^\alpha I-A)=0$ has no purely imaginary roots, then the fractional system with delays in the state (1) is asymptotically stable (Deng *et al.*, 2007; Busłowicz, 2012). We shall show that it can be driven to the origin in a finite time.

Let $\Omega=\{z_0\in\mathbb{R}^n\times L^2([-h_M,0),\mathbb{R}^n)\colon \exists_{u\in U}\ x(T,z_0,u)=0,\ T\in(0,\infty)\}$ denote the domain of relative null U-controllability for the system (1) in time $T>0$. We can see that $0\in\Omega$ (0 means here the pair $(0,0)\in\mathbb{R}^n\times L^2([-h_M,0),\mathbb{R}^n)$ because with zero initial conditions, owing to the stability assumption, $x=0$ is the solution of Eqn. (1) for the admissible control $u=0$.

Since the attainable set $K([0,T],z_0)$ of the fractional system (1) is a convex and closed subset of the space $\mathbb{R}^n$, $0\in\mathbb{R}^n$ lies in the interior of the attainable set from the zero initial complete state of the fractional system (1) with constraints, i.e., $0\in\operatorname{int}K([0,T],z_0)$ for every $T>0$.

We will show that $0\in\operatorname{int}\Omega$. Let us assume that $0\notin\operatorname{int}\Omega$. Then there exists a sequence of initial complete states $\{z_{n0}\}_{n\in\mathbb{N}}$ convergent to zero, and for every z_{n0}, $z_{n0}\notin\operatorname{int}\Omega$, that is, $z_{n0}\neq 0$. Hence

$$\begin{aligned}0&\neq x(T,z_{n0},u)\\&=\Phi_0(T)x(0)\\&\quad+\int_0^T\Phi(T-\tau)\sum_{i=1}^M A_ix(\tau-h_i)\,\mathrm{d}\tau\\&\quad+\int_0^T\Phi(T-\tau)Bu(\tau)\,\mathrm{d}\tau\end{aligned}$$

for every $T>0$ and $u\in U$.

Let us establish a sequence of final states $\{x_n(T,z_{n0},u)\}_{n\in\mathbb{N}}$ corresponding to the initial complete states sequence $\{z_{n0}\}_{n\in\mathbb{N}}$, with $u=0$. The elements of this sequence are nonzero elements of the attainable set $K([0,T],z_0)$ that approach zero as $n\to\infty$. This implies $0\notin\operatorname{int}K([0,T],z_0)$ for every $T>0$. But this contradicts the assumption that $0\in\operatorname{int}K([0,T],z_0)$ for every $T>0$ and any initial complete state z_0. Therefore $0\in\operatorname{int}\Omega$.

Since the set Ω contains 0 in its interior, it also contains a neighborhood of $0=(0,0)\in\mathbb{R}^n\times L^2([-h_M,0],\mathbb{R}^n)$. Let $z_0=(x(0),x_0)\neq(0,0)$ be an initial complete state of the fractional system (1). Using the null control $u(t)=0$, the solution $x(T,z_0,0)$ satisfies the conditions

$$\lim_{t\to\infty}x(t,z_0,0)=0\quad\text{and}\quad x(T,z_0,0)\in N(0)$$

for some finite $T \in (0, \infty)$, where $N(0)$ is a sufficiently small neighborhood of $0 \in \mathbb{R}^n$. Then, the instantaneous state $x(T, z_0, 0)$ can be driven to $0 \in \mathbb{R}^n$ in a finite time, so that the fractional system (1) is relatively null U-controllable. ■

In order to formulate the next criteria of relative U-controllability of the fractional system (1), we introduce a scalar function $J : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}$, related to the attainable set $K([0, T], z_0)$ of the system (1) and defined as follows:

$$\begin{aligned} J(z_0, T, a) &= a' Lx(t, z_0, 0) \\ &\quad + \int_0^T \sup \Big\{ a' L \Phi_f(T - \tau) B u(\tau), \\ &\qquad u \in L^2([0, T], U) \Big\} \, \mathrm{d}\tau - a' c, \end{aligned} \tag{7}$$

where $a \in \mathbb{R}^p$ is any vector and a' is its transpose. The function J is called a *supporting function* of the attainable set. Applications of the supporting function for integer-order systems can be found in the works of Klamka (1991), Sikora (2003) as well as Sikora and Klamka (2012).

Theorem 6. *Let U be a compact set and $E \subset \mathbb{R}^p$ be any set containing the origin as an interior point. Then the fractional dynamical system (1) with delays in the state is relatively U-controllable from the complete state $z_0 \in \mathbb{R}^n \times L^2([-h_M, 0], \mathbb{R}^n)$ into the set S of the form (6) if and only if, for some $T \in [0, \infty)$,*

$$J(z_0, T, a) \geq 0 \quad \textit{for every } a \in E,$$

where $J(z_0, T, a)$ is defined by (7).

Proof. By Definition 5, the attainable set $K([0, T], z_0)$ for (1) is

$$\begin{aligned} K([0, T], z_0) = \Big\{ x \in \mathbb{R}^n : x(t) = x(t, z_0, 0) \\ + \int_0^t \Phi_f(t - \tau) B u(\tau) \, \mathrm{d}\tau \Big\}. \end{aligned}$$

This set is convex and compact. The convexity follows from the work of Chukwu (1979). The compactness follows from the fact that the set $L^2([0, T], U)$ of admissible controls is weakly compact in $L^2([0, T], \mathbb{R}^m)$ (cf. Lee and Marcus, 1967), and the pseudo-transition matrix $\Phi_0(t)$ is convergent in the matrix space, which implies the convergence of $\Phi_f(t)$. Indeed, every sequence of points $x_1(T), x_2(T), \ldots, x_k(T), \ldots$ in $K([0, T], z_0)$ has a subsequence convergent to some $\bar{x}(T) \in K([0, T], z_0)$. There exists a subsequence of controls $u_{k_i} \in L^2([0, T], U)$ weakly convergent to some control $\bar{u}$ such that

$$\begin{aligned} \lim_{k_i \to \infty} \int_0^T \Phi_f(t - \tau) B u_{k_i}(\tau) \, \mathrm{d}\tau \\ = \int_0^T \Phi_f(T - \tau) B \bar{u}(\tau) \, \mathrm{d}\tau. \end{aligned}$$

Let $\bar{x}(t)$ be the solution corresponding to $\bar{u}(t)$. Then in $[0, T]$ we have

$$\begin{aligned} \bar{x}(t) &= x(t, z_0, 0) + \int_0^t \Phi_f(t - \tau) B \bar{u}(\tau) \, \mathrm{d}\tau \\ &= \lim_{k_i \to \infty} x_{k_i}(t) \end{aligned}$$

and

$$\lim_{k_i \to \infty} x_{k_i}(T) = \bar{x}(T) \in K([0, T], z_0),$$

which proves the compactness.

It follows that the set $\bar{K}([0, T], z_0)$ of the form

$$\bar{K}([0, T], z_0) = \{ y \in \mathbb{R}^p : y = Lx, x \in K([0, T], z_0) \}$$

is also convex and compact. An initial complete state z_0 can be driven to the set S in time $T > 0$ if and only if the vector c and the set $\bar{K}([0, T], z_0)$ cannot be strictly separated by a hyperplane, that is, if

$$a' c \leq \sup \{ a' \bar{x} : \bar{x} \in \bar{K}([0, T], z_0) \}$$

for all vectors $a \in \mathbb{R}^p$.

Since $\bar{x} \in \bar{K}([0, T], z_0)$, we have $\bar{x} = Lx$ for any $x \in K([0, T], z_0)\}$, that is,

$$\bar{x} = L \Big(x(t, z_0, 0) + \int_0^t \Phi_f(t - \tau) B u(\tau) \, \mathrm{d}\tau \Big).$$

Therefore the above inequality can be equivalently written in the following form:

$$\begin{aligned} a' Lx(t, z_0, 0) + \int_0^T \sup \Big\{ a' L \Phi_f(T - \tau) B u(\tau), \\ u \in L^2([0, T], U) \Big\} \, \mathrm{d}\tau - a' c \geq 0. \end{aligned}$$

Interchanging integration and the supremum operation, we conclude that $c \in \bar{K}_U([0, T], z_0)$ if and only if $J(z_0, T, a) \geq 0$ for all $a \in \mathbb{R}^p$. Moreover, we can show that

$$kJ(z_0, T, a) = J(z_0, T, ka) \quad \text{for every } k \geq 0.$$

Therefore, for vectors $a \in E$, the proof is complete. ■

Corollary 1. *Let $U \subset \mathbb{R}^m$ be a compact set and $E \subset \mathbb{R}^n$ be any set containing the origin as an interior point. Then the fractional dynamical system (1) is relatively null U-controllable $z_0 \in \mathbb{R}^n \times L^2([-h_M, 0], \mathbb{R}^n)$ if and only if for some $T \in [0, \infty)$*

$$J(z_0, T, a) \geq 0 \quad \textit{for every } a \in E.$$

Proof. The corollary is an immediate consequence of Theorem 4 for $S = \{0\}$, i.e., for $L = I_n$ and $c = 0$. E is then a subset of $\mathbb{R}^n$. ■

Theorem 7. *If U is a compact set containing the origin, then the fractional system (1) is relatively null U-controllable if and only if the equality*

$$\int_0^{+\infty} \sup\{a'\Phi_f(T-\tau)Bu(\tau)\colon u \in L^2_{loc}([0,\infty),U)\}\,\mathrm{d}\tau = +\infty \quad (8)$$

holds for every nonzero vector $a \in \mathbb{R}^n$ and $T > 0$.

Proof. Let us prove the necessary condition by contradiction. Assume that the retarded fractional system (1) is relatively null U-controllable and the condition (8) is not satisfied. Then there exists a constant k, $0 < k < +\infty$, and a nonzero vector $\widetilde{a} \in \mathbb{R}^n$ such that, for all $T \geq 0$,

$$\int_0^{T} \sup\{\widetilde{a}'\Phi_f(T-\tau)Bu(\tau)\colon u \in L^2([0,T],U)\}\,\mathrm{d}\tau < k.$$

For $S = \{0\}$ the supporting function takes the form

$$\begin{aligned} &J(z_0,T,a) \\ &= a'x(t,z_0,0) \\ &\quad + \int_0^T \sup\{a'\Phi_f(T-\tau)Bu(\tau)\colon u \in L^2([0,T],U)\}\,\mathrm{d}\tau. \end{aligned}$$

We put

$$x(t,\widetilde{z}_0,0) = -2k\widetilde{a}$$

for some initial conditions $\widetilde{z}_0$.

For $\widetilde{a} \neq 0$, also $x(t,\widetilde{z}_0,0) \neq 0$ and we have

$$\begin{aligned} &J(\widetilde{z}_0,T,\widetilde{a}) \\ &= \widetilde{a}'x(t,\widetilde{z}_0,0) \\ &\quad + \int_0^T \sup\{\widetilde{a}'\Phi_f(T-\tau)Bu(\tau) : u \in L^2([0,T],U)\}\,\mathrm{d}\tau \\ &= -2k \\ &\quad + \int_0^T \sup\{\widetilde{a}'\Phi_f(T-\tau)Bu(\tau) : u \in L^2([0,T],U)\}\,\mathrm{d}\tau \\ &< -k. \end{aligned}$$

Hence, for all $T \geq 0$, we obtain $J(\widetilde{z}_0,T,\widetilde{a}) < 0$, which implies that the retarded system (1) is not relatively null U-controllable from the initial complete conditions $\widetilde{z}_0$ (see Corollary 1). This contradicts the assumption that (1) is relative null U-controllable. In this way, the necessary condition is proved.

The sufficient condition will be also proved by contradiction. Assume that (8) holds and the retarded fractional system (1) is not relatively null U-controllable. Thus these are initial conditions $\widehat{z}_0$ from which the system cannot be driven into zero. Therefore, for some $T > 0$, there exists $\widehat{a} \neq 0$ such that the following inequality is satisfied:

$$\begin{aligned} &\widehat{a}'x(t,\widehat{z}_0,0) \\ &\quad + \int_0^T \sup\{\widehat{a}'\Phi_f(T-\tau)Bu(\tau) : u \in L^2([0,T],U)\}\,\mathrm{d}\tau < 0. \end{aligned}$$

Applying the Schwarz inequality (see Rolewicz, 1987) we have

$$\begin{aligned} &\int_0^T \sup\{\widehat{a}'\Phi_f(T-\tau)Bu(\tau)\colon u \in L^2([0,T],U)\}\,\mathrm{d}\tau \\ &\qquad \leq ||\widehat{a}'||\,||x(t,\widehat{z}_0,0)||, \end{aligned}$$

which contradicts the assumption that (8) is true. This completes the proof. ■

The next theorem follows from Theorem 7 and provides the controllability criterion for nonnegative constraints.

Corollary 2. *If $U = \mathbb{R}^m_+ \cup \{0\}$, then the fractional system (1) is relatively null U-controllable if and only if the equality*

$$\int_0^{+\infty} \sup\{a'\Phi_f(T-\tau)Bu(\tau)\colon u \in L^2_{loc}([0,\infty),U)\}\mathrm{d}\tau = +\infty$$

holds for every nonzero vector $a \in \mathbb{R}^n$ and $T > 0$.

Proof. The corollary follows immediately from Theorem 7, since the first orthant in the m-dimensional Euclidean space $\mathbb{R}^m$ includes a compact subset containing $0 \in \mathbb{R}^m$. ■

Nonnegative or positive controls are especially important in practical applications. Some controllability criteria for continuous-time linear fractional systems with positive controls are presented by Kaczorek (2014a; 2014b). However they concern positive systems. Here we consider arbitrary systems, which means that other parameters of the systems do not have to be positive. Continuous-time positive linear systems are also discussed by Zhao *et al.* (2013; 2014) in the case of one time-delay and without delays, respectively.

5. Examples

The numerical examples below illustrate the theoretical results presented in the paper.

Example 1. Consider a linear fractional system described by the following fractional differential equation:

$$ {}^{C}D^{\frac{1}{2}}x(t) = A_0x(t) + A_1x(t-1) + A_2x(t-2) + Bu(t), \quad (9) $$

for $t \in [0,3]$ with the initial conditions $z(0) = \{x(0), z_0\}$, where

$$ A_0 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}, $$

$$ A_2 = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. $$

We have $n = 2, M = 3, h_0 = 0, h_1 = 1, h_2 = 3$. In order to verify whether the system (9) is relatively controllable on $[0,3]$, we apply Theorem 2. We will show that the matrix $W(0,3)$ is nonsingular. This means that the matrix has to be full rank, i.e., $\operatorname{rank} W(0,3) = 2$. Thus we have to show that

$$ \operatorname{rank} \int_0^3 \Phi(3-\tau)BB'\Phi'(3-\tau)\,\mathrm{d}\tau = 2. $$

Using the Cayley–Hamilton method (see Monje *et al.*, 2010) we calculate

$$ \begin{aligned} \Phi(t) &= t^{-\frac{1}{2}} \sum_{k=0}^{1} \frac{A_0^k\, t^{\frac{k}{2}}}{\Gamma(\frac{1}{2}k+1)} \\ &= t^{-\frac{1}{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{t^0}{\sqrt{\pi}} + t^{-\frac{1}{2}} \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \frac{t^{\frac{1}{2}}}{1} \\ &= \begin{bmatrix} \frac{1}{\sqrt{\pi}} t^{-\frac{1}{2}} & 2 \\ 0 & \frac{1}{\sqrt{\pi}} t^{-\frac{1}{2}} \end{bmatrix} \end{aligned} $$

and

$$ \begin{aligned} \Phi'(t) &= \sum_{k=0}^{1} \frac{(A_0')^k\, t^{\frac{k}{2}}}{\Gamma(\frac{1}{2}k+1)} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{t^0}{\sqrt{\pi}} + \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \frac{t^{\frac{1}{2}}}{1} \\ &= \begin{bmatrix} \frac{1}{\sqrt{\pi}} & 0 \\ 2t^{\frac{1}{2}} & \frac{1}{\sqrt{\pi}} \end{bmatrix}. \end{aligned} $$

It follows that

$$ \begin{aligned} W(0,3) = \int_0^3 &\begin{bmatrix} \frac{1}{\sqrt{\pi}}(3-\tau)^{-\frac{1}{2}} & 2 \\ 0 & \frac{1}{\sqrt{\pi}}(3-\tau)^{-\frac{1}{2}} \end{bmatrix} \\ &\times \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{\pi}} & 0 \\ 2(3-\tau)^{\frac{1}{2}} & \frac{1}{\sqrt{\pi}} \end{bmatrix} \mathrm{d}\tau, \end{aligned} $$

since

$$ BB' = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. $$

Multiplying the matrices under the integral and integrating each element of the result matrix, we obtain

$$ W(0,3) = \begin{bmatrix} \frac{2\sqrt{3}}{\pi} - \frac{12\sqrt{\pi}}{\pi} + 4\sqrt{3} & \frac{2}{\pi}(\sqrt{3} - 3\sqrt{\pi}) \\ \frac{2}{\pi}(\sqrt{3} - 3\sqrt{\pi}) & \frac{2\sqrt{3}}{\pi} \end{bmatrix}. $$

We see that $\operatorname{rank} W(0,3) = 2$, which implies the relative controllability of the fractional system (9) on $[0,3]$. ♦

Example 2. Let a fractional system with two delays in the state be described by the equation

$$ {}^{C}D^{\frac{1}{2}}x(t) = A_0x(t) + A_1x(t-1) + A_2x(t-2) + Bu(t), \quad (10) $$

on any interval $[0,T]$, for

$$ A_0 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}, $$

$$ A_2 = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}. $$

We apply Theorem 6 to verify relative controllability of the system (10) on the interval $[0,T]$. Since $n = 2$, we have to show that $\operatorname{rank} \widehat{Q}_2(T) = 2$ for each $T > 0$.

Let us find all matrices $Q_k(t)$ belonging to the set $\widehat{Q}_2(T)$, defined by the formula (5). We have

$$ Q_0(0) = B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, $$

$$ Q_0(t) = 0 $$

for $t \neq 0$. Next, since

$$ \begin{aligned} Q_1(t) &= \sum_{i=0}^{2} A_i Q_0(t-h_i) \\ &= A_0Q_0(t) + A_1Q_0(t-h_1) + A_2Q_0(t-h_2), \end{aligned} $$

for $t = h_i, 2h_i, 3h_i, \dots$ and $i = 0,1,2$, we calculate

$$ Q_1(0) = A_0B = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, $$

$$ Q_1(h_1) = A_1Q_0(0) = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, $$

$$ Q_1(h_2) = A_2Q_0(0) = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}, $$

$$ Q_1(2h_1) = A_2Q_0(0) = Q_1(h_2), $$

since $2h_1 = h_2$. Other matrices $Q_1(jh_i)$, $j = 2,3,\dots$, are equal to zero. Therefore

$$ \widehat{Q}_2(T) = \{Q_0(0), Q_1(0), Q_1(h_1), Q_1(h_2)\}, $$

and finally

$$ \operatorname{rank} \widehat{Q}_2(T) = \operatorname{rank} \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 2 & 2 & -3 \end{bmatrix} = 2. $$

Based on Theorem 6, we conclude that the delayed fractional system (10) is relatively controllable on any interval $[0,T]$. ♦

6. Concluding remarks

Relative controllability and relative constrained controllability of linear fractional systems with delays in the state were discussed in the paper. Constraints imposed on the delay values were considered. The formula for a solution of the discussed systems was derived with the use of the Laplace transform (Theorem 1). Definitions of relative controllability for unconstrained as well as for constrained controls were formulated. The contribution of the paper consists of several new necessary and sufficient conditions for relative controllability (Theorems 2–4) and relative U-controllability (Theorem 5, 6, Corollary 1, Theorem 7, Corollary 2) for time-delay fractional systems described by Eqn. (1), which were established and proved in detail. Numerical examples were presented to illustrate how to verify relative controllability of the discussed systems with the use of the established criteria. The presented theoretical results can be extended to semilinear fractional systems with retarded controls.

Acknowledgment

This research was supported by the Polish National Science Centre (NCN) grant PBU-5/RAu-1/2015 (decision DEC-2014/13/B/ST7/00755).

References

Babiarz, A., Grzejszczak, T., Łegowski, A. and Niezabitowski, M. (2016). Controllability of discrete-time switched fractional order systems, *Proceedings of the 12th World Congress on Intelligent Control and Automation, Guilin, China*, pp. 1754–1757.

Balachandran, K. and Kokila, J. (2012). On the controllability of fractional dynamical systems, *International Journal and Applied Mathematics and Computer Science* **22**(3): 523–531, DOI: 10.2478/v10006-012-0039-0.

Balachandran, K. and Kokila, J. (2013). Controllability of fractional dynamical systems with prescribed controls, *IET Control Theory and Applications* **7**(9): 1242–1248.

Balachandran, K., Kokila, J. and Trujillo, J. (2012a). Relative controllability of fractional dynamical systems with multiple delays in control, *Computers and Mathematics with Appllications* **64**(10): 3037–3045.

Balachandran, K., Park, J. and Trujillo, J. (2012b). Controllability of nonlinear fractional dynamical systems, *Nonlinear Analysis* **75**(4): 1919–1926.

Balachandran, K., Zhou, Y. and Kokila, J. (2012c). Relative controllability of fractional dynamical systems with delays in control, *Communications in Nonlinear Science and Numerical Simulation* **17**(9): 3508–3520.

Busłowicz, M. (2012). Stability analysis of continuous-time linear systems consisting of n subsystems with different fractional orders, *Bulletin of the Polish Academy of Sciences: Technical Sciences* **60**(2): 270–284.

Busłowicz, M. (2014). Controllability, reachability and minimum energy control of fractional discrete-time linear systems with multiple delays in state, *Bulletin of the Polish Academy of Sciences: Technical Sciences* **62**(2): 233–239.

Chen, Y., Ahn, H. and Xue, D. (2006). Robust controllability of interval fractional order linear time invariant systems, *Signal Processes* **86**(10): 2794–2802.

Chikriy, A. and Matichin, I. (2008). Presentation of solutions of linear systems with fractional derivatives in the sense of Riemann–Liouville, Caputo and Miller–Ross, *Journal of Automation and Information Science* **40**(6): 1–11.

Chukwu, E. (1979). Euclidean controllability of linear delay systems with limited controls, *IEEE Transactions on Automatic Control* **24**(5): 798–800.

Deng, W., Li, C. and Lu, J. (2007). Stability analysis of linear fractional differential systems with multiple time delays, *Nonlinear Dynamics* **48**: 409–416.

Kaczorek, T. (2011). *Selected Problems of Fractional Systems Theory*, Lecture Notes in Control and Information Science, Vol. 411, Springer-Verlag, Berlin/Heidelberg.

Kaczorek, T. (2014a). An extension of Klamka's method of minimum energy control to fractional positive discrete-time linear systems with bounded inputs, *Bulletin of the Polish Academy of Sciences: Technical Sciences* **62**(2): 227–231.

Kaczorek, T. (2014b). Minimum energy control of fractional positive continuous-time linear systems with bounded inputs, *International Journal of Applied Mathematics and Computer Science* **24**(2): 335–340, DOI: 10.2478/amcs-2014-0025.

Kaczorek, T. and Rogowski, K. (2015). *Fractional Linear Systems and Electrical Circuits*, Studies in Systems, Decision and Control, Vol. 13, Springer International Publishing, Cham.

Kilbas, A., Srivastava, H. and Trujillo, J. (2006). *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Vol. 204, Elsevier, Amsterdam.

Klamka, J. (1991). *Controllability of Dynamical Systems*, Kluwer Academic Publishers, Dordrecht.

Klamka, J. (2008). Constrained controllability of semilinear systems with delayed controls, *Bulletin of the Polish Academy of Sciences: Technical Sciences* **56**(4): 333–337.

Klamka, J. (2009). Constrained controllability of semilinear systems with delays, *Nonlinear Dynamics* **56**(1): 169–177.

Klamka, J. (2010). Controllability and minimum energy control problem of fractional discrete-time systems, *in* D. Balenau (Ed.), *New Trends Nanotechology and Fractional Calculus Applications*, Springer, Dordrecht, pp. 503–509.

Klamka, J. (2011). Local controllability of fractional discrete-time semilinear systems, *Acta Mechanica at Automatica* **5**(2): 55–58.

Klamka, J., Czornik, A., Niezabitowski, M. and Babiarz, A. (2014). Controllability and minimum energy control of linear fractional discrete-time infinite-dimensional systems, *Proceedings of the 11th IEEE International Conference on Control and Automation, Taichung, Taiwan*, pp. 1210–1214.

Lee, E. and Marcus, L. (1967). *Foundations of Optimal Control Theory*, John Wiley and Sons, New York, NY.

Manitius, A. (1974). Optimal control of hereditary systems, *in* J.W. Weil (Ed.), *Control Theory and Topics in Functional Analysis*, Vol. 3, International Centre for Theoretical Physics, Trieste, pp. 43–178.

Miller, K. and Ross, B. (1993). *An Introduction to the Fractional Calculus and Fractional Differential Calculus*, John Wiley and Sons, New York, NY.

Monje, A., Chen, Y., Viagre, B., Xue, D. and Feliu, V. (2010). *Fractional-Order Systems and Control: Fundamentals and Applications*, Springer-Verlag, London.

Oldham, K. and Spanier, J. (1974). *The Fractional Calculus*, Academic Press, New York, NY.

Pawłuszewicz, E. and Mozyrska, D. (2013). Constrained controllability of h-difference linear systems with two fractional orders, *in* W. Mitkowski *et al.* (Ed.), *Advances in the Theory and Applications of Non-integer Order Systems*, Lecture Notes in Electrical Engineering, Vol. 257, Springer International Publishing, Cham, pp. 67–75.

Podlubny, I. (1999). *Fractional Differential Equations*, Academic Press, San Diego, CA.

Rolewicz, S. (1987). *Functional Analysis and Control Theory: Linear Systems*, Mathematics and Its Applications, Springer, Dordrecht.

Sabatier, J., Agrawal, O. and Tenreiro Machado, J. (Eds.) (2007). *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer-Verlag, Dordrecht.

Sakthivel, R., Ren, Y. and Mahmudov, N. (2011). On the approximate controllability of semilinear fractional differential systems, *Computers and Mathematics with Applications* **62**(3): 1451–1459.

Samko, S., Kilbas, A. and Marichev, O. (1993). *Fractional Integrals and Derivatives: Theory and Applications*, Gordan and Breach Science Publishers, Philadelphia, PA.

Sikora, B. (2003). On the constrained controllability of dynamical systems with multiple delays in the state, *International Journal of Applied Mathematics and Computer Science* **13**(4): 469–479.

Sikora, B. (2005). On constrained controllability of dynamical systems with multiple delays in control, *Applicationes Mathematicae* **32**(1): 87–101.

Sikora, B. (2016). Controllability of time-delay fractional systems with and without constraints, *IET Control Theory and Applications* **10**(3): 320–327.

Sikora, B. and Klamka, J. (2012). On constrained stochastic controllability of dynamical systems with multiple delays in control, *Bulletin of the Polish Academy of Sciences: Technical Sciences* **60**(12): 301–305.

Trzasko, B. (2008). Reachability and controllability of positive fractional discrete-time systems with delay, *Journal of Automation Mobile Robotics and Intelligent Systems* **2**(3): 43–47.

Wang, J. and Zhou, Y. (2012). Complete controllability of fractional evolution systems, *Communications in Nonlinear Science and Numerical Simulation* **17**(11): 4346–4355.

Wei, J. (2012). The controllability of fractional control systems with control delay, *Computers and Mathematics with Applications* **64**(10): 3153–3159.

Zhang, H., Cao, J. and Jiang, W. (2013). Controllability criteria for linear fractional differential systems with state delay and impulses, *Journal of Applied Mathematics*, Article ID: 146010.

Zhao, X., Liu, X., Yin, S. and Li, H. (2013). Stability of a class of switched positive linear time-delay systems, *International Journal of Robust and Nonlinear Control* **23**(5): 578–589.

Zhao, X., Liu, X., Yin, S. and Li, H. (2014). Improved results on stability of continuous-time switched positive linear systems, *Automatica* **50**(2): 614–621.

Beata Sikora was born in Poland in 1969. She received an M.Sc. degree in applied mathematics from the University of Silesia in 1995 and a Ph.D. degree in control engineering from the Silesian University of Technology in 2002. She is a member of the Polish Mathematical Society. Her scientific interests are control theory and mathematical modeling of dynamical systems, in particular, constrained and unconstrained controllability of dynamical systems with delays. She is concerned with both integer and fractional-order systems. Moreover, her current interests include data analysis, especially the analysis of data coming from monitoring systems, and machine learning methods application for natural hazards assessment. She is an author or coauthor of about 20 scientific papers and 3 university textbooks.

Received: 16 December 2015
Revised: 7 May 2016
Accepted: 10 May 2016