Opuscula Math. 36, no. 2 (2016), 215–238
http://dx.doi.org/10.7494/OpMath.2016.36.2.215 Opuscula Mathematica

SOLUTIONS OF FRACTIONAL NABLA DIFFERENCE EQUATIONS – EXISTENCE AND UNIQUENESS

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Communicated by Josef Diblík

Abstract. In this article, we discuss existence, uniqueness and dependency of solutions of nonlinear fractional nabla difference equations in a Banach space equipped with a suitable norm, using the contractive mapping approach. As an application of the established results we present and analyse a few examples.

Keywords: nabla difference, exponential function, fixed point, existence, uniqueness, continuous dependence.

Mathematics Subject Classification: 34A08, 39A10, 39A99.

1. INTRODUCTION

Discrete fractional calculus is one of the most recent and interesting branches of mathematics. In 1988, Gray and Zhang initiated the nabla approach of discrete fractional calculus while Miller and Ross used the delta approach for the first time in 1989. Abdeljawad, Anastassiou, Atici, Cermak, Chen, Eloe, Hein and many other mathematicians have developed the qualitative theory of fractional nabla difference equations. A series of research articles on this topic appeared recently [1–3, 5–8, 11, 12, 14–16, 22, 23].

The first and foremost step in a qualitative study of fractional nabla difference equations is to establish sufficient conditions on existence and uniqueness of its solutions. But it is not obvious, unlike the theory of delta difference equations. For example, consider a nonautonomous nabla difference equation together with an initial condition of the form

$$\nabla u = f(t, u), \quad t \in \mathbb{N}_1, \tag{1.1}$$

$$u(0) = c, \tag{1.2}$$

where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, $\mathbb{N}_1 = \mathbb{N}_0 \setminus \{0\}$, $u : \mathbb{N}_0 \to \mathbb{R}$, $f : \mathbb{N}_0 \times \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$. We know that u is a solution of the initial value problem (1.1)–(1.2) if and only if it has the following representation

$$u(t) = c + \sum_{s=1}^{t} f(s, u(s)), \quad t \in \mathbb{N}_0. \tag{1.3}$$

Since f appears on the right hand side of equation (1.3), existence of u is not trivial. It depends on the nature of f.

Similarly in this paper, we impose a few restrictions on $\mathbf{f}$ to obtain the existence, uniqueness and continuous dependence of solutions of initial value problems associated with nonlinear fractional nabla difference equations of the form

$$\nabla^{\alpha}_{-1}\mathbf{u} = \mathbf{f}(t, \mathbf{u}), \quad 0 < \alpha < 1, \quad t \in \mathbb{N}_1, \tag{1.4}$$

$$\left(\nabla^{-(1-\alpha)}_{-1}\mathbf{u}\right)(t)\Big|_{t=0} = \mathbf{u}(0) = \mathbf{c}, \tag{1.5}$$

and

$$\nabla^{\alpha}_{0*}\mathbf{u} = \mathbf{f}(t, \mathbf{u}), \quad 0 < \alpha < 1, \quad t \in \mathbb{N}_1, \tag{1.6}$$

$$\mathbf{u}(0) = \mathbf{c}, \tag{1.7}$$

where ∇^{α}_{-1} and ∇^{α}_{0*} are Riemann-Liouville and Caputo type fractional difference operators respectively, $\mathbf{u}$ is an n-vector whose components are functions of the variable t, $\mathbf{c}$ is a constant n-vector and $\mathbf{f}(t, \mathbf{u})$ is an n-vector whose components are functions of the variable t and the n-vector $\mathbf{u}$.

We present a few examples to illustrate the applicability of the main results. In particular, we also investigate the existence and uniqueness of solutions of logistic, prey-predator and SIR epidemic models in a discrete fractional nabla perspective.

2. PRELIMINARIES

This section is organized as follows. Subsection 2.1 contains preliminaries on discrete fractional calculus. In Subsection 2.2 we construct novel Banach spaces.

2.1. DISCRETE FRACTIONAL CALCULUS

Throughout, we shall use the following notations, definitions and known results of discrete fractional calculus. For any $a \in \mathbb{R}$, $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$. Let $u : \mathbb{N}_a \to \mathbb{R}$. For all t_1, $t_2 \in \mathbb{N}_a$ and $t_1 > t_2$,

$$\sum_{t=t_1}^{t_2} u(t) = 0 \quad \text{and} \quad \prod_{t=t_1}^{t_2} u(t) = 1,$$

i.e. empty sums and products are taken to be 0 and 1, respectively.

Definition 2.1. The gamma function is a generalization of the factorial function $t!$, where the factorial is only applicable for $t \in \mathbb{N}_0$. The gamma function can be used for any real number. For any $t \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$,

$$\Gamma(t) = \int_0^\infty e^{-s} s^{t-1} ds, \quad t > 0,$$
$$\Gamma(t+1) = t\,\Gamma(t).$$

Definition 2.2. For any α, $t \in \mathbb{R}$, the α rising function is defined by

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \quad 0^{\overline{\alpha}} = 0.$$

We observe the following properties of gamma and rising factorial functions.

Lemma 2.3 ([19])**.** *Assume the following factorial functions are well defined.*

1. $t^{\overline{\alpha}}(t+\alpha)^{\overline{\beta}} = t^{\overline{\alpha+\beta}}$.
2. *If* $t \le r$ *then* $t^{\overline{\alpha}} \le r^{\overline{\alpha}}$.
3. *If* $\alpha < t \le r$ *then* $r^{\overline{-\alpha}} \le t^{\overline{-\alpha}}$.

Lemma 2.4 ([20])**.** *For any a, $b \in \mathbb{R}$, the quotient expansion of two gamma functions at infinity is given by*

$$\frac{\Gamma(t+a)}{\Gamma(t+b)} = t^{a-b}\Big[1 + O\Big(\frac{1}{t}\Big)\Big], \quad |t| \to \infty.$$

Definition 2.5. Let $u : \mathbb{N}_a \to \mathbb{R}$, $\alpha \in \mathbb{R}$ and choose $N \in \mathbb{N}_1$ such that $N-1 < \alpha < N$.

1. (Nabla Difference, [4]) The first order backward difference or nabla difference of u is defined by
$$(\nabla u)(t) = u(t) - u(t-1), \quad t \in \mathbb{N}_{a+1},$$
and the N^{th}-order nabla difference of u is defined recursively by
$$\big(\nabla^N u\big)(t) = \big(\nabla\big(\nabla^{N-1} u\big)\big)(t), \quad t \in \mathbb{N}_{a+N}.$$
In addition, we take ∇^0 as the identity operator.
2. (Fractional Nabla Sum, [15]) The α^{th}-order fractional nabla sum of u is given by
$$\big(\nabla_a^{-\alpha} u\big)(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{\alpha-1}} u(s), \quad t \in \mathbb{N}_a,$$
where $\rho(s) = s-1$. Also, we define the trivial sum by $\big(\nabla_a^{-0} u\big)(t) = u(t)$ for $t \in \mathbb{N}_a$.
3. (Riemann-Liouville Fractional Nabla Difference, [15]) The α^{th}-order Riemann-Liouville type fractional nabla difference of u is given by
$$\big(\nabla_a^{\alpha} u\big)(t) = \big(\nabla^N\big(\nabla_a^{-(N-\alpha)} u\big)\big)(t), \quad t \in \mathbb{N}_{a+N}.$$
For $\alpha = 0$, we set $\big(\nabla_a^0 u\big)(t) = u(t)$, $t \in \mathbb{N}_a$.

4. (Caputo Fractional Nabla Difference, [1]) The α^{th}-order Caputo type Fractional Nabla Difference of u is given by

$$\big(\nabla_{a*}^{\alpha}u\big)(t) = \big(\nabla_a^{-(N-\alpha)}\big(\nabla^N u\big)\big)(t), \quad t \in \mathbb{N}_{a+N}.$$

For $\alpha = 0$, we set $\big(\nabla_{a*}^{0}u\big)(t) = u(t)$, $t \in \mathbb{N}_a$.

The unified definition for fractional nabla sum and differences is as follows.

Definition 2.6 ([1, 15])**.** Let $u : \mathbb{N}_a \to \mathbb{R}$, $\alpha \in \mathbb{R}$ and choose $N \in \mathbb{N}_1$ such that $N - 1 < \alpha < N$.

1. The α^{th}-order fractional nabla sum of u is given by

$$\big(\nabla_a^{-\alpha}u\big)(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} u(s), \quad t \in \mathbb{N}_a.$$

2. The α^{th}-order Riemann-Liouville type fractional nabla difference of u is given by

$$\big(\nabla_a^{\alpha}u\big)(t) = \frac{1}{\Gamma(-\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{-\alpha-1}} u(s), \quad t \in \mathbb{N}_{a+N}.$$

3. The α^{th}-order Caputo type fractional nabla difference of u is given by

$$\begin{aligned}\big(\nabla_{a*}^{\alpha}u\big)(t) = {} & \frac{1}{\Gamma(-\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{-\alpha-1}} u(s) \\ & - \sum_{k=0}^{N-1} \frac{(t-a)^{\overline{k-\alpha}}}{\Gamma(k-\alpha+1)} \big(\nabla^k u\big)(a), \quad t \in \mathbb{N}_{a+N}.\end{aligned}$$

Theorem 2.7 (Power Rule, [2])**.** *Let $\alpha > 0$ and $\mu > -1$. Then the following conditions hold:*

1. $\nabla(t-a)^{\overline{\mu}} = \mu(t-a)^{\overline{\mu-1}}$, $t \in \mathbb{N}_{a+1}$,
2. $\nabla^N(t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-N+1)}(t-a)^{\overline{\mu-N}}$, $t \in \mathbb{N}_{a+N}$,
3. $\nabla_a^{-\alpha}(t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(t-a)^{\overline{\mu+\alpha}}$, $t \in \mathbb{N}_a$,
4. $\nabla_a^{\alpha}(t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}(t-a)^{\overline{\mu-\alpha}}$, $t \in \mathbb{N}_{a+N}$,
5. $\nabla_{a*}^{\alpha}(t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}(t-a)^{\overline{\mu-\alpha}}$, $t \in \mathbb{N}_{a+N}$.

2.2. CONSTRUCTION OF BANACH SPACES

In this subsection, we construct certain norms and establish Banach spaces with respect to these norms.

Definition 2.8. $\mathbb{R}^n$ is the space of all ordered n-tuples of real numbers. Clearly, $\mathbb{R}^n$ is a Banach space with respect to the Euclidean norm $\|\cdot\|$.

Definition 2.9.

$$\mathbf{l}^\infty = \mathbf{l}^\infty(\mathbb{R}^n) = \{\mathbf{u} : \mathbf{u} = \{\mathbf{u}(t)\}_{t\in\mathbb{N}_0}, \mathbf{u}(t) \in \mathbb{R}^n \text{ with } \|\mathbf{u}\|_\infty < \infty\}$$

denotes the Banach space comprising bounded sequences of vectors with respect to the supremum norm $\|\cdot\|_\infty$ defined by

$$\|\mathbf{u}\|_\infty = \sup_{t\in\mathbb{N}_0} \|\mathbf{u}(t)\|.$$

An open ball with radius r centered on the null vector sequence in $\mathbf{l}^\infty$ is defined by

$$B_r^\infty(\mathbf{0}) = \{\mathbf{u} : \|\mathbf{u}\|_\infty < r\} \subseteq \mathbf{l}^\infty.$$

Atsushi Nagai defined in [22] the one parameter discrete Mittag-Leffler function of fractional nabla calculus as follows.

Definition 2.10. The one parameter discrete Mittag-Leffler function of fractional nabla calculus is defined by

$$F_\alpha(\lambda, t) = \sum_{k=0}^{\infty} \lambda^k \frac{t^{\overline{\alpha k}}}{\Gamma(\alpha k + 1)},$$

where $0 < \alpha < 1$, $|\lambda| < 1$ and $t \geq 0$.

We observe the following properties of a discrete Mittag-Leffler function.

Lemma 2.11. *Let* $0 < \lambda < 1$. *Then*

1. $F_\alpha(\lambda, 0) = 1$,
2. $F_\alpha(\lambda, t)$ *is monotonically increasing on* $[0, \infty)$,
3. $F_\alpha(\lambda, t) \to \infty$ *as* $t \to \infty$,
4. $F_\alpha(\lambda, t) : [0, \infty) \to [1, \infty)$,
5. $\nabla_0^{-\alpha} F_\alpha(\lambda, t) = \frac{1}{\lambda}\Big[F_\alpha(\lambda, t) - 1\Big]$.

Proof. (1) Consider

$$F_\alpha(\lambda, t) = 1 + \sum_{k=1}^{\infty} \lambda^k \frac{t^{\overline{\alpha k}}}{\Gamma(\alpha k + 1)} = 1.$$

(2) Let $t, s \in \mathbb{N}_0$ such that $t \geq s$. Then $t - s = T \in \mathbb{N}_0$. Now consider

$$\begin{aligned}
&F_\alpha(\lambda, t) - F_\alpha(\lambda, s) \\
&= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+1)}\Big[t^{\overline{\alpha k}} - s^{\overline{\alpha k}}\Big] = \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+1)}\Big[\frac{\Gamma(t+\alpha k)}{\Gamma(t)} - \frac{\Gamma(s+\alpha k)}{\Gamma(s)}\Big] \\
&= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+1)}\Big[\frac{\Gamma(s+T+\alpha k)}{\Gamma(s+T)} - \frac{\Gamma(s+\alpha k)}{\Gamma(s)}\Big] \\
&= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+1)}\frac{\Gamma(s+\alpha k)}{\Gamma(s)}\Big[\Big(\frac{s+T-1+\alpha k}{s+T-1}\Big)\Big(\frac{s+T-2+\alpha k}{s+T-2}\Big)\dots\Big(\frac{s+\alpha k}{s}\Big) - 1\Big] \\
&= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+1)}\frac{\Gamma(s+\alpha k)}{\Gamma(s)}\Big[\Big(1+\frac{\alpha k}{s+T-1}\Big)\Big(1+\frac{\alpha k}{s+T-2}\Big)\dots\Big(1+\frac{\alpha k}{s}\Big) - 1\Big] \\
&\geq 0.
\end{aligned}$$

Thus, we have $F_\alpha(\lambda, s) \leq F_\alpha(\lambda, t)$ whenever $s \leq t$.

(3) As $t \to \infty$, from Lemma 2.4, we have

$$\begin{aligned}
F_\alpha(\lambda, t) &= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+1)}\frac{\Gamma(t+\alpha k)}{\Gamma(t)} \\
&= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+1)} t^{\alpha k}\Big[1+O\Big(\frac{1}{t}\Big)\Big] = \sum_{k=0}^{\infty} \frac{(\lambda t^\alpha)^k}{\Gamma(\alpha k+1)}\Big[1+O\Big(\frac{1}{t}\Big)\Big] \to \infty.
\end{aligned}$$

The proof of (4) is obvious from (1), (2) and (3).

(5) Consider

$$\begin{aligned}
\nabla_0^{-\alpha} F_\alpha(\lambda, t) &= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+1)} \nabla_0^{-\alpha} t^{\overline{\alpha k}} \\
&= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+1)} \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1)+1)} t^{\overline{\alpha(k+1)}} \\
&= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha(k+1)+1)} t^{\overline{\alpha(k+1)}} = \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{\Gamma(\alpha k+1)} t^{\overline{\alpha k}} = \frac{1}{\lambda}\Big[F_\alpha(\lambda, t) - 1\Big].
\end{aligned}$$

□

Definition 2.12. $\mathbf{l}_h^\infty$ denotes the Banach space comprising of bounded sequences of real numbers with respect to the weighted supremum norm defined by

$h : [0, \infty) \to [1, \infty)$, $h(0) = 1$, $h(t) \to \infty$ monotonically, and for $\mathbf{u} = \{\mathbf{u}(t)\}_{t \in \mathbb{N}_0} \in \mathbf{l}_h^\infty$,

$$\|\mathbf{u}\|_h = \sup_{t \in \mathbb{N}_0} \frac{\|\mathbf{u}(t)\|}{h(t)}.$$

An open ball with radius r centered on the null vector sequence in $\mathbf{l}_h^\infty$ is defined by

$$B_r^h(\mathbf{0}) = \{\mathbf{u} : \|\mathbf{u}\|_h < r\} \subseteq \mathbf{l}_h^\infty.$$

We use $h(t) = F_\alpha(\lambda, t)$ with $0 < \lambda < 1$ throughout Section 3.

3. MAIN RESULTS

Existence of solutions of differential equations can be formulated well in terms of fixed points of mappings. Motivated by this fact, we have discussed:

1. the existence and the uniqueness of solutions on $\mathbb{N}_{0,m}$ ([19]),
2. the existence and the uniqueness of solutions on $\mathbb{N}_0$ ([18]),
3. the existence of asymptotically stable solutions on $\mathbb{N}_0$ ([17])

for the initial value problems (1.4)–(1.5) and (1.6)–(1.7) using fixed point theorems based on the ideas of topological degree (Krasnoselskii and Schauder fixed point theorems) via the supremum norm in normed spaces. But, the conditions established in [17–19] are applicable only for a particular class of functions. The following three examples illustrates this fact.

1. Consider a scalar initial value problem

$$\nabla_{-1}^{0.5} u = (0.25)(t+1)^{\overline{-0.75}}[u^2 + 1], \quad \big(\nabla_{-1}^{-0.5} u\big)(t)\Big|_{t=0} = u(0) = c, \quad t \in \mathbb{N}_1.$$

2. Consider a scalar initial value problem

$$\nabla_{0*}^{0.5} u = (0.5)\sin u, \quad u(0) = c, \quad t \in \mathbb{N}_1.$$

3. Consider a discrete fractional order logistic equation together with an initial condition of the form

$$\nabla_{0*}^{\alpha} u = ru[1-u], \quad u(0) = c, \quad t \in \mathbb{N}_1.$$

In Example 1, we cannot achieve global existence of solutions using the sufficient conditions established in [18]. Similarly, the results formulated in [18] are not applicable for the existence of solutions in Examples 2 and 3. In fact, we need localized conditions for Examples 1 and 3 that guarantee the existence of solutions which will be discussed in Section 3.4.

In this section, we establish some new results that guarantee the global / local existence and uniqueness of solutions to the initial value problems (1.4)–(1.5) and (1.6)–(1.7). Banach's fixed point theorem will be the main tool to be used via the supremum and weighted supremum norms in normed spaces.

3.1. CONTRACTION MAPPING

Banach's fixed point theorem (also known as the contraction mapping principle) has been widely used as an important tool to determine the existence and uniqueness of solutions of initial value problems defined on complete metric spaces.

Definition 3.1 (Contraction Mapping)**.** Let (X, ρ) be a complete metric space and $P : X \to X$. The map P is said to be contractive if there exists a positive constant $a < 1$ such that for each pair x, $y \in X$ we have

$$\rho(Px, Py) \leq a\rho(x, y).$$

The constant a is called the contraction constant of P.

Theorem 3.2 (Banach Fixed Point Theorem)**.** *Let (X, ρ) be a complete metric space and let $P : X \to X$ be contractive. Then P has a unique fixed point $u \in X$, that is,*

$$Pu = u.$$

Consider the initial value problems (1.4)–(1.5) and (1.6)–(1.7). Then, $\mathbf{u}(t)$ is a solution of the initial value problem (1.4)–(1.5) if and only if

$$\mathbf{u}(t) = \frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\mathbf{c} + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s, \mathbf{u}(s)), \quad t \in \mathbb{N}_0. \tag{3.1}$$

Similarly, $\mathbf{u}(t)$ is a solution of the initial value problem (1.6)–(1.7) if and only if

$$\mathbf{u}(t) = \mathbf{c} + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s, \mathbf{u}(s)), \quad t \in \mathbb{N}_0. \tag{3.2}$$

Define the operators

$$(P\mathbf{u})(t) = \frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\mathbf{c} + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s, \mathbf{u}(s)), \quad t \in \mathbb{N}_0, \tag{3.3}$$

$$(P'\mathbf{u})(t) = \mathbf{c} + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\mathbf{f}(s, \mathbf{u}(s)), \quad t \in \mathbb{N}_0. \tag{3.4}$$

It is evident from (3.1)–(3.4) that $\mathbf{u}$ is a fixed point of P if and only if $\mathbf{u}$ is a solution of (1.4)–(1.5) and $\mathbf{u}$ is a fixed point of P' if and only if $\mathbf{u}$ is a solution of (1.6)–(1.7).

3.2. ASSUMPTIONS

We make the following assumptions on $\mathbf{f}$ defined on $\mathbb{N}_0 \times \mathbb{R}^n$.

(C) $\mathbf{f}$ is continuous with respect to its second argument.

(L1) There exists a nonnegative function $a(t)$ defined on $\mathbb{N}_0$ such that, for all $(t, \mathbf{u})$, $(t, \mathbf{v}) \in \mathbb{N}_0 \times \mathbb{R}^n$,

$$\|\mathbf{f}(t, \mathbf{u}) - \mathbf{f}(t, \mathbf{v})\| \leq a(t)\|\mathbf{u} - \mathbf{v}\|.$$

(A1) Assume that

$$\sup_{t\in\mathbb{N}_0} \Big[\frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t - \rho(s))^{\overline{\alpha-1}} a(s)\Big] = L < 1.$$

(A2) Assume that

$$\sup_{t\in\mathbb{N}_0} \Big[\frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t - \rho(s))^{\overline{\alpha-1}} \|\mathbf{f}(s, \mathbf{0})\|\Big] = \Lambda < \infty.$$

(L2) $\mathbf{f}$ is Lipschitz continuous with respect to its second argument, i.e. there exists a constant $K \in [0, \lambda)$ such that, for all $(t, \mathbf{u})$, $(t, \mathbf{v}) \in \mathbb{N}_0 \times \mathbb{R}^n$,

$$\|\mathbf{f}(t, \mathbf{u}) - \mathbf{f}(t, \mathbf{v})\| \leq K\|\mathbf{u} - \mathbf{v}\|. \tag{3.5}$$

(B1) Assume that

$$\|\mathbf{f}\|_h = \sup_{t\in\mathbb{N}_0} \frac{\|\mathbf{f}(t, \mathbf{0})\|}{h(t)} = \Omega < \infty.$$

Remark 3.3. In general, the Lipschitz constant K in (3.5) is independent of the arguments of $\mathbf{f}$ but may depend on its domain. The following theorem (see [4]) is useful for identifying if a given function satisfies the Lipschitz continuity in a given domain.

Theorem 3.4. *Let $M > 0$ be an arbitrary constant. Consider a function $\mathbf{f}$ defined on $\mathbb{N}_0 \times \mathbb{R}^n$ or on a region of the type*

$$D = \{(t, \mathbf{u}) : \|\mathbf{u}(t)\| \leq M\} \subseteq \mathbb{N}_0 \times \mathbb{R}^n.$$

If $\mathbf{f}$ is continuously differentiable with respect to the second variable on $\mathbb{N}_0 \times \mathbb{R}^n$ (or D), and suppose there exists a constant $K > 0$ such that, for all $(t, \mathbf{u}) = (t, u_1, u_2, \ldots, u_n)$ $\in \mathbb{N}_0 \times \mathbb{R}^n$ (or D),

$$\Big\|\frac{\partial \mathbf{f}(t, \mathbf{u})}{\partial u_i}\Big\| \leq K, \quad (i = 1, 2, \ldots, n),$$

then $\mathbf{f}$ is Lipschitz continuous with respect to its second argument on $\mathbb{N}_0 \times \mathbb{R}^n$ (or D) with Lipschitz constant K.

3.3. GLOBAL EXISTENCE AND UNIQUENESS

Theorem 3.5 (The nonautonomous case)**.** *Let* (C), (L1), (A1) *and* (A2) *hold. Then there exists a unique solution of the initial value problems* (1.4)–(1.5) *and* (1.6)–(1.7) *in* $\mathbf{l}^\infty$.

Proof. We use Banach's fixed point theorem (Theorem 3.2) to establish global existence and uniqueness of solutions of the initial value problems (1.4)–(1.5) and (1.6)–(1.7) in $\mathbf{l}^\infty$. We know that $\mathbf{l}^\infty$ is a complete metric space with respect to the sup-metric defined by

$$\rho(\mathbf{u}, \mathbf{v}) = \sup_{t \in \mathbb{N}_0} \|\mathbf{u}(t) - \mathbf{v}(t)\|,$$

for each pair $\mathbf{u}$, $\mathbf{v} \in \mathbf{l}^\infty$. First, we prove that P' maps $\mathbf{l}^\infty$ into $\mathbf{l}^\infty$. Let $\mathbf{u} \in \mathbf{l}^\infty$. Using Lemma 2.3, Theorem 2.7, (L1), (A1) and (A2), we have

$$\begin{aligned}
\|(P'\mathbf{u})(t)\| &\leq \|\mathbf{c}\| + \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t - \rho(s))^{\overline{\alpha-1}} \|\mathbf{f}(s, \mathbf{u}(s))\| \\
&= \|\mathbf{c}\| + \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t - \rho(s))^{\overline{\alpha-1}} \|\mathbf{f}(s, \mathbf{u}(s)) - \mathbf{f}(s, \mathbf{0}) + \mathbf{f}(s, \mathbf{0})\| \\
&\leq \|\mathbf{c}\| + \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t - \rho(s))^{\overline{\alpha-1}} \|\mathbf{f}(s, \mathbf{u}(s)) - \mathbf{f}(s, \mathbf{0})\| \\
&\quad + \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t - \rho(s))^{\overline{\alpha-1}} \|\mathbf{f}(s, \mathbf{0})\| \\
&\leq \|\mathbf{c}\| + \|\mathbf{u}\|_\infty \Big[\frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t - \rho(s))^{\overline{\alpha-1}} a(s) \Big] + \Lambda \\
&\leq \|\mathbf{c}\| + L\|\mathbf{u}\|_\infty + \Lambda < \infty, \quad t \in \mathbb{N}_0,
\end{aligned}$$

implies $P'\mathbf{u} \in \mathbf{l}^\infty$. For all $\mathbf{u}$, $\mathbf{v} \in \mathbf{l}^\infty$, using Lemma 2.3, Theorem 2.7, (L1) and (A1), we have

$$\begin{aligned}
\|(P'\mathbf{u})(t) - (P'\mathbf{v})(t)\| &\leq \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t - \rho(s))^{\overline{\alpha-1}} \|\mathbf{f}(s, \mathbf{u}(s)) - \mathbf{f}(s, \mathbf{v}(s))\| \\
&\leq \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t - \rho(s))^{\overline{\alpha-1}} a(s) \|\mathbf{u}(s) - \mathbf{v}(s)\| \\
&\leq L\|\mathbf{u} - \mathbf{v}\|_\infty, \quad t \in \mathbb{N}_0,
\end{aligned}$$

implies $\|P'\mathbf{u} - P'\mathbf{v}\|_\infty \leq L\|\mathbf{u} - \mathbf{v}\|_\infty$. Since $L < 1$, P' is contractive. Hence, by Theorem 3.2, P' has a unique fixed point $\mathbf{u} \in \mathbf{l}^\infty$. Similarly, we can prove that P has a unique fixed point $\mathbf{u} \in \mathbf{l}^\infty$. Hence the proof. □

Corollary 3.6 (The autonomous case)**.** *If* $\mathbf{f}$ *is constant with respect to its first argument and let* (C), (L1), (A1) *and* (A2) *hold, then there exists a unique solution of the initial value problems* (1.4)–(1.5) *and* (1.6)–(1.7) *in* $\mathbf{l}^\infty$*.*

The following example demonstrates the applicability of Theorem 3.5.

Example 3.7. Consider the scalar initial value problem

$$\nabla_{-1}^{0.5}u = (0.25)(t+1)^{\overline{-0.75}}\sin u, \quad t \in \mathbb{N}_1, \tag{3.6}$$

$$\big(\nabla_{-1}^{-0.5}u\big)(t)\Big|_{t=0} = u(0) = c. \tag{3.7}$$

We have $f(t,u) = (0.25)(t+1)^{\overline{-0.75}}\sin u$ is continuous with respect to its second argument. For any u, $v \in \mathbb{R}$ and for all $t \in \mathbb{N}_0$,

$$|f(t,u) - f(t,v)| \leq (0.25)(t+1)^{\overline{-0.75}}|u-v|.$$

Here $a(t) = (0.25)(t+1)^{\overline{-0.75}}$ is a nonnegative function defined on $\mathbb{N}_0$ and

$$\begin{aligned}
L &= \sup_{t\in\mathbb{N}_0}\Big[\frac{(0.25)}{\Gamma(0.5)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{0.5-1}}(s+1)^{\overline{-0.75}}\Big] \\
&= \sup_{t\in\mathbb{N}_0}\Big[\frac{(0.25)}{\Gamma(0.5)}\sum_{s=0}^{t}(t-\rho(s))^{\overline{0.5-1}}(s+1)^{\overline{-0.75}} - \frac{(0.25)}{\Gamma(0.5)}(t+1)^{\overline{0.5-1}}(1)^{\overline{-0.75}}\Big] \\
&= \sup_{t\in\mathbb{N}_0}\Big[(0.25)\nabla_{-1}^{-0.5}(t+1)^{\overline{-0.75}} - (0.25)\frac{\Gamma(0.25)}{\Gamma(0.5)}(t+1)^{\overline{-0.5}}\Big] \\
&= \sup_{t\in\mathbb{N}_0}\Big[(0.25)\frac{\Gamma(0.25)}{\Gamma(0.75)}(t+1)^{\overline{-0.25}}\Big] - (0.25)\frac{\Gamma(0.25)}{\Gamma(0.5)}\inf_{t\in\mathbb{N}_0}\Big[(t+1)^{\overline{-0.5}}\Big] \\
&= (0.25)\frac{\Gamma(0.25)}{\Gamma(0.75)}(1)^{\overline{-0.25}} - 0 = (0.25)\Gamma(0.25) < 1.
\end{aligned}$$

Thus, all the assumptions of Theorem 3.5 hold and hence the initial value problem (3.6)–(3.7) has a unique solution in l^∞.

Remark 3.8. In Theorem 3.5, conditions (L1), (A1) and (A2) are seem to be artificial in nature or motivated towards some special set of functions. Consequently, these conditions are not applicable for all classes of fractional nabla difference equations. To fix this problem, we use a different norm to establish global existence and uniqueness of solutions of initial value problems (1.4)–(1.5) and (1.6)–(1.7).

Theorem 3.9. *Assume that conditions* (L2) *and* (B1) *hold. Then there exists a unique solution for the initial value problems* (1.4)–(1.5) *and* (1.6)–(1.7) *in* $\mathbf{l}_h^\infty$.

Proof. We use the contraction mapping principle (Theorem 3.2) to establish global existence and uniqueness of solutions of the initial value problems (1.4)–(1.5) and (1.6)–(1.7) in $\mathbf{l}_h^\infty$. We know that $\mathbf{l}_h^\infty$ is a complete metric space with respect to the weighted sup-metric defined by

$$\rho(\mathbf{u},\mathbf{v}) = \sup_{t\in\mathbb{N}_0}\frac{1}{h(t)}\|\mathbf{u}(t) - \mathbf{v}(t)\|,$$

for each pair $\mathbf{u}$, $\mathbf{v} \in \mathbf{l}_h^\infty$. First, we prove that P' maps $\mathbf{l}_h^\infty$ into $\mathbf{l}_h^\infty$. Let $\mathbf{u} \in \mathbf{l}_h^\infty$. Using Lemma 2.3, Theorem 2.7, Lemma 2.11, (L2) and (B1), we have

$$
\begin{aligned}
\|P'\mathbf{u}\|_h &= \sup_{t\in\mathbb{N}_0} \Big[\frac{\|(P'\mathbf{u})(t)\|}{h(t)}\Big] \\
&\leq \sup_{t\in\mathbb{N}_0} \frac{1}{h(t)}\Big[\|\mathbf{c}\| + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\|\mathbf{f}(s,\mathbf{u}(s))\|\Big] \\
&= \sup_{t\in\mathbb{N}_0} \frac{1}{h(t)}\Big[\|\mathbf{c}\| + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\|\mathbf{f}(s,\mathbf{u}(s)) - \mathbf{f}(s,\mathbf{0}) + \mathbf{f}(s,\mathbf{0})\|\Big] \\
&\leq \sup_{t\in\mathbb{N}_0} \frac{1}{h(t)}\Big[\|\mathbf{c}\| + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\|\mathbf{f}(s,\mathbf{u}(s)) - \mathbf{f}(s,\mathbf{0})\| \\
&\quad + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\|\mathbf{f}(s,\mathbf{0})\|\Big] \\
&= \sup_{t\in\mathbb{N}_0} \frac{1}{h(t)}\|\mathbf{c}\| + \sup_{t\in\mathbb{N}_0} \frac{1}{h(t)}\Big[\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\|\mathbf{f}(s,\mathbf{u}(s)) - \mathbf{f}(s,\mathbf{0})\|\Big] \\
&\quad + \sup_{t\in\mathbb{N}_0} \frac{1}{h(t)}\Big[\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\|\mathbf{f}(s,\mathbf{0})\|\Big] \\
&\leq \|\mathbf{c}\|_h + K\sup_{t\in\mathbb{N}_0} \frac{1}{h(t)}\Big[\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}h(s)\frac{\|\mathbf{u}(s)\|}{h(s)}\Big] \\
&\quad + \sup_{t\in\mathbb{N}_0} \frac{1}{h(t)}\Big[\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}h(s)\frac{\|\mathbf{f}(s,\mathbf{0})\|}{h(s)}\Big] \\
&\leq \|\mathbf{c}\|_h + \big[K\|\mathbf{u}\|_h + \|\mathbf{f}\|_h\big]\sup_{t\in\mathbb{N}_0} \frac{1}{h(t)}\Big[\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}h(s)\Big] \\
&= \|\mathbf{c}\|_h + \big[K\|\mathbf{u}\|_h + \|\mathbf{f}\|_h\big]\sup_{t\in\mathbb{N}_0} \frac{1}{h(t)}\nabla_0^{-\alpha}[h(t)] \\
&= \|\mathbf{c}\|_h + \big[K\|\mathbf{u}\|_h + \|\mathbf{f}\|_h\big]\sup_{t\in\mathbb{N}_0} \frac{1}{\lambda h(t)}\Big[h(t) - 1\Big] \\
&= \|\mathbf{c}\|_h + \frac{1}{\lambda}\big[K\|\mathbf{u}\|_h + \|\mathbf{f}\|_h\big]\sup_{t\in\mathbb{N}_0}\Big[1 - \frac{1}{h(t)}\Big] \\
&= \|\mathbf{c}\|_h + \frac{1}{\lambda}\big[K\|\mathbf{u}\|_h + \|\mathbf{f}\|_h\big] < \infty,
\end{aligned}
$$

which implies that $P'\mathbf{u} \in \mathbf{l}_h^\infty$. For all $\mathbf{u}$, $\mathbf{v} \in \mathbf{l}_h^\infty$, using Lemma 2.3, Theorem 2.7, Lemma 2.11 and (L2), we have

$$\begin{aligned}
\|P'\mathbf{u} - P'\mathbf{v}\|_h &= \sup_{t\in\mathbb{N}_0} \Big[\frac{\|(P'\mathbf{u})(t) - (P'\mathbf{v})(t)\|}{h(t)}\Big] \\
&\le \sup_{t\in\mathbb{N}_0} \frac{1}{h(t)}\Big[\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\|\mathbf{f}(s,\mathbf{u}(s)) - \mathbf{f}(s,\mathbf{v}(s))\|\Big] \\
&\le K \sup_{t\in\mathbb{N}_0} \frac{1}{h(t)}\Big[\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\|\mathbf{u}(s) - \mathbf{v}(s)\|\Big] \\
&= K \sup_{t\in\mathbb{N}_0} \frac{1}{h(t)}\Big[\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}h(s)\frac{\|\mathbf{u}(s) - \mathbf{v}(s)\|}{h(s)}\Big] \\
&\le K\|\mathbf{u}-\mathbf{v}\|_h \sup_{t\in\mathbb{N}_0} \frac{1}{h(t)}\Big[\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}h(s)\Big] \\
&= K\|\mathbf{u}-\mathbf{v}\|_h \sup_{t\in\mathbb{N}_0} \frac{1}{h(t)}\nabla_0^{-\alpha}[h(t)] \\
&= K\|\mathbf{u}-\mathbf{v}\|_h \sup_{t\in\mathbb{N}_0} \frac{1}{\lambda h(t)}\Big[h(t) - 1\Big] \\
&= \frac{K}{\lambda}\|\mathbf{u}-\mathbf{v}\|_h \sup_{t\in\mathbb{N}_0}\Big[1 - \frac{1}{h(t)}\Big] = \frac{K}{\lambda}\|\mathbf{u}-\mathbf{v}\|_h.
\end{aligned}$$

Since $K < \lambda$, P' is contractive. Hence, by Theorem 3.2, there is one and only one point $\mathbf{u} \in \mathbf{l}_h^\infty$ with $P'\mathbf{u} = \mathbf{u}$. Similarly, we can prove that there is one and only one point $\mathbf{u} \in \mathbf{l}_h^\infty$ with $P\mathbf{u} = \mathbf{u}$. Hence the proof. □

Corollary 3.10 (The autonomous case)**.** *If* $\mathbf{f}$ *is constant with respect to its first argument and let* (L2) *and* (B1) *hold, then there exists a unique solution of the initial value problems* (1.4)–(1.5) *and* (1.6)–(1.7) *in* $\mathbf{l}_h^\infty$*.*

The following examples demonstrate the applicability of Corollary 3.10.

Example 3.11. Let $a \in \mathbb{R}$. Consider the scalar initial value problem

$$\nabla_{-1}^{\alpha} u = au, \quad 0 < \alpha < 1, \quad t \in \mathbb{N}_1, \tag{3.8}$$

$$\big(\nabla_{-1}^{-(1-\alpha)} u\big)(t)\Big|_{t=0} = u(0) = c. \tag{3.9}$$

We have $f(u) = au$ is continuous. For any u, $v \in \mathbb{R}$,

$$|f(u) - f(v)| \le |a||u - v|.$$

Here $K = |a| \ge 0$. We choose a suitable $\lambda \in (0,1)$ such that $K < \lambda < 1$. Thus, all the assumptions of Corollary 3.10 hold and hence the initial value problem (3.8)–(3.9) has a unique solution in l_h^∞.

Remark 3.12. Atici and Eloe obtained in [8] the unique solution of (3.8)–(3.9) as the exponential function defined by

$$\widehat{e}_{\alpha,\alpha}(a, t^{\overline{\alpha}}) = (1-a)\sum_{k=0}^{\infty} \frac{a^k (t+1)^{\overline{(k+1)\alpha-1}}}{\Gamma((k+1)\alpha)}, \quad |a| < 1,$$

which justifies our established results.

Example 3.13. Let $a \in \mathbb{R}$. Consider the scalar initial value problem

$$\nabla^{\alpha}_{0*} u = au, \quad 0 < \alpha < 1, \quad t \in \mathbb{N}_1, \tag{3.10}$$

$$u(0) = c. \tag{3.11}$$

We have $f(u) = au$ is continuous. For any u, $v \in \mathbb{R}$,

$$|f(u) - f(v)| \leq |a||u - v|.$$

Here $K = |a| \geq 0$. We choose a suitable $\lambda \in (0, 1)$ such that $K < \lambda < 1$. Thus, all the assumptions of Corollary 3.10 hold and hence the initial value problem (3.10)–(3.11) has a unique solution in l^{∞}_{h}.

Remark 3.14. Atsushi Nagai has obtained in [22] the unique solution of (3.10)–(3.11) as the one parameter discrete Mittag-Leffler function $F_{\alpha}(a, t)$ which justifies our established results.

Example 3.15. Consider the scalar initial value problem

$$\nabla^{0.5}_{0*} u = (0.5) \sin u, \quad t \in \mathbb{N}_1, \tag{3.12}$$

$$u(0) = c. \tag{3.13}$$

We have $f(u) = (0.5) \sin u$ is continuous. For any u, $v \in \mathbb{R}$,

$$|f(u) - f(v)| \leq (0.5)|u - v|.$$

Here $K = 0.5 > 0$. We choose a suitable $\lambda \in (0, 1)$ such that $K < \lambda$. For example, $\lambda = 0.75$. Thus, all the assumptions of Corollary 3.10 hold and hence the initial value problem (3.12)–(3.13) has a unique solution in l^{∞}_{h}.

3.4. DEPENDENCE OF SOLUTIONS

In (1.4)–(1.5) and (1.6)–(1.7), the initial value $\mathbf{c}$ may be subject to some errors either by necessity or for convenience. Hence, it is important to know how the solution changes when the initial conditions are slightly altered. We shall discuss this question quantitatively in the following theorems.

Theorem 3.16. *Assume that conditions* (C), (L1), (A1) *and* (A2) *hold. Suppose* $\mathbf{u}$ *and* $\mathbf{v}$ *are the solutions of the initial value problems*

$$\nabla^{\alpha}_{-1}\mathbf{u} = \mathbf{f}(t, \mathbf{u}), \quad \left(\nabla^{-(1-\alpha)}_{-1}\mathbf{u}\right)(t)\Big|_{t=0} = \mathbf{u}(0) = \mathbf{c}, \quad t \in \mathbb{N}_1, \tag{3.14}$$

$$\nabla^{\alpha}_{-1}\mathbf{v} = \mathbf{f}(t, \mathbf{v}), \quad \left(\nabla^{-(1-\alpha)}_{-1}\mathbf{v}\right)(t)\Big|_{t=0} = \mathbf{v}(0) = \mathbf{d}, \quad t \in \mathbb{N}_1, \tag{3.15}$$

respectively, where $0 < \alpha < 1$. *Then*

$$\|\mathbf{u} - \mathbf{v}\|_\infty = O(\|\mathbf{c} - \mathbf{d}\|_\infty).$$

Theorem 3.17. *Assume that conditions* (C), (L1), (A1) *and* (A2) *hold. Suppose* $\mathbf{u}$ *and* $\mathbf{v}$ *are the solutions of the initial value problems*

$$\nabla_{0*}^{\alpha}\mathbf{u} = \mathbf{f}(t, \mathbf{u}), \quad \mathbf{u}(0) = \mathbf{c}, \quad t \in \mathbb{N}_1, \tag{3.16}$$

$$\nabla_{0*}^{\alpha}\mathbf{v} = \mathbf{f}(t, \mathbf{v}), \quad \mathbf{v}(0) = \mathbf{d}, \quad t \in \mathbb{N}_1, \tag{3.17}$$

respectively, where $0 < \alpha < 1$. *Then*

$$\|\mathbf{u} - \mathbf{v}\|_\infty = O(\|\mathbf{c} - \mathbf{d}\|_\infty).$$

Theorem 3.18. *Assume that conditions* (L2) *and* (B1) *hold. Suppose* $\mathbf{u}$ *and* $\mathbf{v}$ *are the solutions of the initial value problems* (3.14) *and* (3.15), *respectively. Then*

$$\|\mathbf{u} - \mathbf{v}\|_h = O(\|\mathbf{c} - \mathbf{d}\|_h).$$

Proof. Consider

$$\begin{aligned}
\|\mathbf{u} - \mathbf{v}\|_h &= \sup_{t\in\mathbb{N}_0} \frac{\|\mathbf{u}(t) - \mathbf{v}(t)\|}{h(t)} \\
&\le \sup_{t\in\mathbb{N}_0} \frac{1}{h(t)} \Big[\frac{(t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} \|\mathbf{c} - \mathbf{d}\| \\
&\quad + \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t - \rho(s))^{\overline{\alpha-1}} \|\mathbf{f}(s, \mathbf{u}(s)) - \mathbf{f}(s, \mathbf{v}(s))\| \Big] \\
&\le \sup_{t\in\mathbb{N}_0} \frac{1}{h(t)} \Big[\frac{(1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} \|\mathbf{c} - \mathbf{d}\| + \frac{K}{\Gamma(\alpha)} \sum_{s=1}^{t} (t - \rho(s))^{\overline{\alpha-1}} \|\mathbf{u}(s) - \mathbf{v}(s)\| \Big] \\
&= \sup_{t\in\mathbb{N}_0} \frac{1}{h(t)} \|\mathbf{c} - \mathbf{d}\| + K \sup_{t\in\mathbb{N}_0} \frac{1}{h(t)} \nabla_0^{-\alpha} \|\mathbf{u}(t) - \mathbf{v}(t)\| \\
&= \sup_{t\in\mathbb{N}_0} \frac{1}{h(t)} \|\mathbf{c} - \mathbf{d}\| + K \sup_{t\in\mathbb{N}_0} \frac{1}{h(t)} \nabla_0^{-\alpha} \Big(h(t) \frac{\|\mathbf{u}(t) - \mathbf{v}(t)\|}{h(t)} \Big) \\
&\le \sup_{t\in\mathbb{N}_0} \frac{1}{h(t)} \|\mathbf{c} - \mathbf{d}\| + K \|\mathbf{u} - \mathbf{v}\|_h \sup_{t\in\mathbb{N}_0} \frac{1}{h(t)} \nabla_0^{-\alpha} [h(t)] \\
&= \|\mathbf{c} - \mathbf{d}\|_h + \frac{K}{\lambda} \|\mathbf{u} - \mathbf{v}\|_h.
\end{aligned}$$

Then, we have the relation

$$\|\mathbf{u} - \mathbf{v}\|_h \le \frac{\lambda}{(\lambda - K)} \|\mathbf{c} - \mathbf{d}\|_h$$

which implies

$$\|\mathbf{u} - \mathbf{v}\|_h = O(\|\mathbf{c} - \mathbf{d}\|_h). \qquad \square$$

Theorem 3.19. *Assume that conditions* (L2) *and* (B1) *hold. Suppose* $\mathbf{u}$ *and* $\mathbf{v}$ *are the solutions of the initial value problems* (3.16) *and* (3.17), *respectively. Then*

$$\|\mathbf{u}-\mathbf{v}\|_h = O(\|\mathbf{c}-\mathbf{d}\|_h).$$

3.5. LOCAL EXISTENCE AND UNIQUENESS

The Banach principle introduced the idea of a unique fixed point of a contractive map in metric spaces. However, not all maps are contractive for an entire space but they may be contractive within a small subset usually considered as a ball in a metric space. Such maps are called locally contractive maps. So locally contractive maps can be utilized for having fixed points within a ball in a metric space, there exists a local version of the Banach theorem presented as the following theorem.

Theorem 3.20. *Let* (X,ρ) *be a complete metric space containing an open ball having center* x_0 *and radius* r. *Let* $P : B_r(x_0) \to X$ *be a contractive map with a positive number* $a < 1$ *as the contraction constant. If*

$$\rho(Px_0, x_0) < (1-a)r,$$

then P *has a unique fixed point in* $B_r(x_0)$.

Theorem 3.21 (The nonautonomous case)**.** *Assume that conditions* (C), (A1),

(L1') *there exists a nonnegative function* $a(t)$ *defined on* $\mathbb{N}_0$ *such that, for all* $(t,\mathbf{u}), (t,\mathbf{v}) \in D$,

$$\|\mathbf{f}(t,\mathbf{u}) - \mathbf{f}(t,\mathbf{v})\| \le a(t)\|\mathbf{u}-\mathbf{v}\|, \tag{3.18}$$

and
(A2')

$$\sup_{t\in\mathbb{N}_0}\Big[\frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\|\mathbf{f}(s,\mathbf{0})\|\Big] < \frac{M}{1-L}, \tag{3.19}$$

hold. Let $p > 0$ *and define a set*

$$B_p^{\infty}(\mathbf{0}) = \Big\{\mathbf{u} : \|\mathbf{u}\|_{\infty} < p\Big\} \subseteq \mathbf{l}^{\infty},$$

where

$$p = \frac{M}{(1-L)^2}.$$

Then there exists a unique bounded solution of the initial value problems (1.4)–(1.5) *and* (1.6)–(1.7) *in* $B_p^{\infty}(\mathbf{0})$.

Proof. We use Theorem 3.20 to establish local existence and uniqueness of solutions of the initial value problems (1.4)–(1.5) and (1.6)–(1.7). Clearly, P' maps $B_p^{\infty}(\mathbf{0})$ into $\mathbf{l}^{\infty}$. We have already proved that P' is contractive with contraction constant $L < 1$. Now consider

$$\|P'\mathbf{0}-\mathbf{0}\| \le \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\|\mathbf{f}(s,\mathbf{0})\| < \frac{M}{1-L} = (1-L)p,$$

implies $\|P'\mathbf{0} - \mathbf{0}\|_\infty < (1-L)p$. Hence by Theorem 3.20, P' has a unique fixed point $\mathbf{u} \in B_p^\infty(\mathbf{0})$. Similarly, we can prove that P has a unique fixed point $\mathbf{u} \in B_p^\infty(\mathbf{0})$. $\square$

Corollary 3.22 (The autonomous case)**.** *If* $\mathbf{f}$ *is constant with respect to its first argument and let* (C)*,* (L1')*,* (A1) *and* (A2') *hold, then there exists a unique bounded solution of the initial value problems* (1.4)–(1.5) *and* (1.6)–(1.7) *in* $B_p^\infty(\mathbf{0})$.

The following example demonstrates the applicability of Theorem 3.21.

Example 3.23. Consider the scalar initial value problem

$$\nabla_{-1}^{0.5} u = (0.25)(t+1)^{\overline{-0.75}}[u^2+1], \quad t \in \mathbb{N}_1, \tag{3.20}$$

$$\left(\nabla_{-1}^{-0.5} u\right)(t)\Big|_{t=0} = u(0) = c. \tag{3.21}$$

We have $f(t,u) = (0.25)(t+1)^{\overline{-0.75}}[u^2+1]$ is continuous with respect to its second argument. Let $M > 0$ be an arbitrary constant. For (t,u), $(t,v) \in D$, consider

$$\begin{aligned} |f(t,u) - f(t,v)| &= (0.25)(t+1)^{\overline{-0.75}}|u^2 - v^2| \\ &= (0.25)(t+1)^{\overline{-0.75}}|u-v||u+v| \\ &\le (0.25)(t+1)^{\overline{-0.75}}|u-v|[|u|+|v|] \\ &\le (0.5)M(t+1)^{\overline{-0.75}}|u-v|, \end{aligned}$$

so that (3.18) holds with $a(t) = (0.5)M(t+1)^{\overline{-0.75}}$ is a nonnegative function defined on $\mathbb{N}_0$. Now consider

$$\begin{aligned} L &= \sup_{t \in \mathbb{N}_0} \Big[\frac{(0.5)M}{\Gamma(0.5)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{0.5-1}}(s+1)^{\overline{-0.75}}\Big] \\ &= (0.5)M\Gamma(0.25). \end{aligned}$$

We choose $M = 0.5 < \frac{1}{(0.5)\Gamma(0.25)}$ so that $L < 1$ and thus (A1) holds. Also,

$$\begin{aligned} &\sup_{t \in \mathbb{N}_0} \Big[\frac{1}{\Gamma(0.5)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{0.5-1}}|f(s,0)|\Big] \\ &= (0.25) \sup_{t \in \mathbb{N}_0} \Big[\frac{1}{\Gamma(0.5)} \sum_{s=1}^{t} (t-\rho(s))^{\overline{0.5-1}}(s+1)^{\overline{-0.75}}\Big] \\ &= (0.25)\Gamma(0.25) < \frac{M}{1-L}, \end{aligned}$$

and thus (3.19) holds. Thus, all the assumptions of Theorem 3.21 are satisfied and hence the initial value problem (3.20)–(3.21) has a unique bounded solution in an open ball having center 0 and radius $p = 57$.

Remark 3.24. The problem in the previous example does not satisfy (L1) in the global sense and thus Theorem 3.5 does not apply.

Theorem 3.25. *Assume that the following conditions hold:*

(L2') $\mathbf{f}$ *is Lipschitz continuous with respect to its second argument on* D *with Lipschitz constant* K, *i.e. there exists a nonnegative constant* $K \in [0, \lambda)$ *such that, for all* $(t, \mathbf{u}), (t, \mathbf{v}) \in D$,

$$\|\mathbf{f}(t, \mathbf{u}) - \mathbf{f}(t, \mathbf{v})\| \leq K\|\mathbf{u} - \mathbf{v}\|, \tag{3.22}$$

(B1')

$$\|\mathbf{f}\|_h = \sup_{t \in \mathbb{N}_0} \frac{\|\mathbf{f}(t, \mathbf{0})\|}{h(t)} < \frac{M}{\lambda - K}. \tag{3.23}$$

Let $q > 0$ *and define a set*

$$B_q^h(\mathbf{0}) = \Big\{\mathbf{u} : \|\mathbf{u}\|_h < q\Big\} \subseteq \mathbf{l}_h^\infty,$$

where

$$q = \frac{M}{(\lambda - K)^2}.$$

Then there exists a unique bounded solution for the initial value problems (1.4)–(1.5) *and* (1.6)–(1.7) *in* $B_q^h(\mathbf{0})$.

Proof. We use Theorem 3.20 to establish the local existence and uniqueness of solutions of the initial value problems (1.4)–(1.5) and (1.6)–(1.7). Clearly P' maps $B_q^h(\mathbf{0})$ into $\mathbf{l}_h^\infty$. We have already proved that P' is contractive with contraction constant $\frac{K}{\lambda} < 1$. Now consider

$$\begin{aligned}
\|P'\mathbf{0} - \mathbf{0}\|_h &= \sup_{t \in \mathbb{N}_0} \frac{\|P'\mathbf{0} - \mathbf{0}\|}{h(t)} = \sup_{t \in \mathbb{N}_0} \frac{1}{h(t)} \Big\| \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t - \rho(s))^{\overline{\alpha - 1}} \mathbf{f}(s, \mathbf{0}) \Big\| \\
&\leq \sup_{t \in \mathbb{N}_0} \frac{1}{h(t)} \Big[\frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t - \rho(s))^{\overline{\alpha - 1}} \|\mathbf{f}(s, \mathbf{0})\| \Big] \\
&= \sup_{t \in \mathbb{N}_0} \frac{1}{h(t)} \Big[\frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t - \rho(s))^{\overline{\alpha - 1}} h(s) \frac{\|\mathbf{f}(s, \mathbf{0})\|}{h(s)} \Big] \\
&\leq \|\mathbf{f}\|_h \sup_{t \in \mathbb{N}_0} \frac{1}{h(t)} \Big[\frac{1}{\Gamma(\alpha)} \sum_{s=1}^{t} (t - \rho(s))^{\overline{\alpha - 1}} h(s) \Big] \\
&= \|\mathbf{f}\|_h \sup_{t \in \mathbb{N}_0} \frac{1}{h(t)} \Big[\nabla_0^{-\alpha} h(t) \Big] < \frac{M}{\lambda - K} \sup_{t \in \mathbb{N}_0} \frac{1}{\lambda h(t)} \Big[h(t) - 1 \Big] \\
&= \frac{M}{\lambda(\lambda - K)} \sup_{t \in \mathbb{N}_0} \Big[1 - \frac{1}{h(t)} \Big] = \Big(1 - \frac{K}{\lambda} \Big) q.
\end{aligned}$$

Hence, by Theorem 3.20, P' has a unique fixed point $\mathbf{u} \in B_q^h(\mathbf{0})$. Similarly, we can prove that P has a unique fixed point $\mathbf{u} \in B_q^h(\mathbf{0})$. Hence the proof is complete. $\square$

Corollary 3.26 (The autonomous case)**.** *If* $\mathbf{f}$ *is constant with respect to its first argument and let* (L2') *and* (B1') *hold, then there exists a unique bounded solution of the initial value problems* (1.4)–(1.5) *and* (1.6)–(1.7) *in* $B_q^h(\mathbf{0})$.

The following example demonstrates the applicability of Corollary 3.26.

Example 3.27. Consider the scalar initial value problem

$$\nabla_{0*}^{0.5}u = (0.25)u^2 + 1, \quad t \in \mathbb{N}_1, \tag{3.24}$$

$$u(0) = c. \tag{3.25}$$

We have $f(u) = (0.25)u^2 + 1$ is continuous. Let $M > 0$ be an arbitrary constant. For (t, u), $(t, v) \in D$, consider

$$\begin{aligned} |f(u) - f(v)| &= (0.25)|u^2 - v^2| = (0.25)|u - v||u + v| \\ &\leq (0.25)|u - v|(|u| + |v|) \leq (0.5)M|u - v|. \end{aligned}$$

Here $K = (0.5)M$. Let $\lambda = 0.75$. We choose $M = 1 < \frac{\lambda}{(0.5)}$ so that $K < \lambda$ implies (3.22) holds. Now consider

$$\|f\|_h = \sup_{t\in\mathbb{N}_0} \frac{\|f(0)\|}{h(t)} = \sup_{t\in\mathbb{N}_0} \frac{1}{h(t)} = 1 < \frac{M}{\lambda - K},$$

and thus (3.23) holds. Thus, all the assumptions of Corollary 3.26 are satisfied and hence the initial value problem (3.24)–(3.25) has a unique bounded solution in an open ball having center 0 and radius $q = 16$.

Remark 3.28. Note that the problem in the previous example does not satisfy (L2) in the global sense and thus Corollary 3.10 does not apply.

4. APPLICATIONS

In this section we investigate existence and uniqueness of solutions of initial value problems associated with logistic, prey - predator and SIR epidemic models using the conditions established in Section 3.

Discrete Fractional Order Logistic Model: Let $0 < \alpha < 1$, $r > 0$ and $c > 0$. Consider the initial value problem of the discrete fractional order logistic equation

$$\nabla_{0*}^{\alpha}u = ru[1 - u], \quad t \in \mathbb{N}_1, \tag{4.1}$$

$$u(0) = c. \tag{4.2}$$

Here $f(u) = ru[1 - u]$ is continuous. Let $M > 0$ be an arbitrary constant. For $|u|$, $|v| \leq M$, consider

$$\begin{aligned} |f(u) - f(v)| &= r|[u - v] - [u^2 - v^2]| = r|u - v||1 - [u + v]| \\ &\leq r|u - v|[1 + |u| + |v|] \leq r(1 + 2M)|u - v|. \end{aligned}$$

Here $K = r(1+2M) > 0$ and choose $M < \frac{\lambda - r}{2r}$ so that $K < \lambda$. Clearly $0 < r < \lambda < 1$. Also,

$$\sup_{t\in\mathbb{N}_0} \frac{\|f(0)\|}{h(t)} = 0 < \frac{M}{\lambda - K}.$$

Thus the initial value problem (4.1)–(4.2) has a unique bounded solution in $B_q^h(0)$. For example, if we take $r = 0.5$, $\lambda = 0.75$ and choose $M = 0.2 < \frac{\lambda - r}{2r}$, then the initial value problem (4.1)–(4.2) has a unique bounded solution in an open ball having center 0 and radius $q = 80$.

Discrete Fractional Order Lotka-Volterra Predator – Prey Model: Let $P(t)$ and $Q(t)$ denote prey and predator populations at time t, respectively; α, a, b, c and d are positive constants such that $0 < \alpha < 1$. Under the standard assumptions, consider the initial value problem of the discrete fractional order Lotka-Volterra predator - prey model

$$\nabla_*^{\alpha} P = P(a - bQ), \quad t \in \mathbb{N}_1, \tag{4.3}$$
$$\nabla_*^{\alpha} Q = Q(cP - d), \quad t \in \mathbb{N}_1, \tag{4.4}$$
$$P(0) = P_0 \text{ and } Q(0) = Q_0. \tag{4.5}$$

The initial value problem (4.3)–(4.5) can be written in the form

$$\nabla_*^{\alpha} \mathbf{u} = \mathbf{f}(t, \mathbf{u}), \quad t \in \mathbb{N}_1,$$
$$\mathbf{u}(0) = \mathbf{u}_0,$$

where

$$\mathbf{u} = (P,\ Q), \quad \mathbf{f} = (aP - bPQ,\ cPQ - dQ) \quad \text{and} \quad \mathbf{u}_0 = (P(0),\ Q(0)).$$

Let $M > 0$ be an arbitrary constant. Clearly, $\mathbf{f}$ is continuously differentiable on D. We have $\|\mathbf{u}\| = \sqrt{P^2 + Q^2} \le M$ implies $|P|$, $|Q| \le M$. Consider,

$$\left\| \frac{\partial \mathbf{f}}{\partial P} \right\| = \|(a - bQ, cQ)\| = \sqrt{(a - bQ)^2 + (cQ)^2} \le a + (b + c)M,$$

and

$$\left\| \frac{\partial \mathbf{f}}{\partial Q} \right\| = \|(-bP, cP - d)\| = \sqrt{(-bP)^2 + (cP - d)^2} \le d + (b + c)M.$$

Let $K = \max\{a + (b + c)M, d + (b + c)M\}$. Then,

$$\left\| \frac{\partial \mathbf{f}}{\partial P} \right\| \le K \quad \text{and} \quad \left\| \frac{\partial \mathbf{f}}{\partial Q} \right\| \le K$$

and hence from Theorem 3.4, we get

$$\|\mathbf{f}(t, \mathbf{u}) - \mathbf{f}(t, \mathbf{v})\| \le K\|\mathbf{u} - \mathbf{v}\|.$$

Choose

$$M < \min\left\{\frac{\lambda - a}{b+c}, \frac{\lambda - d}{b+c}\right\} \quad \text{so that} \quad K < \lambda.$$

Clearly $0 < a,\ d < \lambda < 1$. Also,

$$\sup_{t\in\mathbb{N}_0} \frac{\|\mathbf{f}(\mathbf{0})\|}{h(t)} = 0 < \frac{M}{\lambda - K}.$$

Thus the initial value problem (4.3)–(4.5) has unique bounded solution in $B_q^h(0)$. For example, if we take $a = 0.1$, $b = 2$, $c = 3$, $d = 0.4$, $\lambda = 0.75$ and choose $M = 0.06 < \min\{0.13, 0.07\}$ so that $K = \max\{0.4, 0.7\} = 0.7 < \lambda$. Then the initial value problem (4.3)–(4.5) has a unique bounded solution in an open ball having center 0 and radius $q = 24$.

Discrete Fractional Order SIR Epidemic Model: Let $S(t)$, $I(t)$ and $R(t)$ denote susceptible, infective and recovered populations at time t, respectively; N is the total population present in the system; α, β and γ are positive constants such that $0 < \alpha < 1$. Under the standard assumptions, consider the initial value problem of the discrete fractional order SIR epidemic model without vital dynamics

$$\nabla_*^\alpha S = -\beta IS, \quad t \in \mathbb{N}_1, \tag{4.6}$$

$$\nabla_*^\alpha I = \beta IS - \gamma I, \quad t \in \mathbb{N}_1, \tag{4.7}$$

$$\nabla_*^\alpha R = \gamma I, \quad t \in \mathbb{N}_1, \tag{4.8}$$

$$S(0) = S_0,\ I(0) = I_0 \text{ and } R(0) = R_0, \tag{4.9}$$

such that

$$N = S(t) + I(t) + R(t).$$

The initial value problem (4.6)–(4.9) can be written in the form

$$\nabla_*^\alpha \mathbf{u} = \mathbf{f}(t, \mathbf{u}), \quad t \in \mathbb{N}_1,$$
$$\mathbf{u}(0) = \mathbf{u}_0,$$

where

$$\mathbf{u} = (S,\ I,\ R), \quad \mathbf{f} = (-\beta IS,\ \beta IS - \gamma I,\ \gamma I) \quad \text{and} \quad \mathbf{u}_0 = (S(0),\ I(0),\ R(0)).$$

Let $M > 0$ be an arbitrary constant. Clearly, $\mathbf{f}$ is continuously differentiable on D. We have $\|\mathbf{u}\| = \sqrt{S^2 + I^2 + R^2} \le M$ implies $|P|$, $|Q|$, $|R| \le M$. Consider,

$$\left\|\frac{\partial \mathbf{f}}{\partial S}\right\| = \|(-\beta I, \beta I, 0)\| = \sqrt{(-\beta I)^2 + (\beta I)^2} \le \sqrt{2}\beta M,$$

$$\left\|\frac{\partial \mathbf{f}}{\partial I}\right\| = \|(-\beta S, \beta S - \gamma, \gamma)\| = \sqrt{(-\beta S)^2 + (\beta S - \gamma)^2 + \gamma^2} \le \sqrt{2}(\beta M + \gamma),$$

and

$$\left\|\frac{\partial \mathbf{f}}{\partial R}\right\| = \|(0, 0, 0)\| = 0.$$

Let $K = \max\{\sqrt{2}\beta M, \sqrt{2}(\beta M + \gamma), 0\} = \sqrt{2}(\beta M + \gamma)$. Then,

$$\left\|\frac{\partial \mathbf{f}}{\partial S}\right\| \leq K, \quad \left\|\frac{\partial \mathbf{f}}{\partial I}\right\| \leq K \quad \text{and} \quad \left\|\frac{\partial \mathbf{f}}{\partial R}\right\| \leq K$$

and hence from Theorem 3.4, we get

$$\|\mathbf{f}(t, \mathbf{u}) - \mathbf{f}(t, \mathbf{v})\| \leq K\|\mathbf{u} - \mathbf{v}\|.$$

Choose $M < \frac{\lambda - \sqrt{2}\gamma}{\sqrt{2}\beta}$ so that $K < \lambda$. Clearly $0 < \sqrt{2}\gamma < \lambda < 1$. Also,

$$\sup_{t \in \mathbb{N}_0} \frac{\|\mathbf{f}(\mathbf{0})\|}{h(t)} = 0 < \frac{M}{\lambda - K}.$$

Thus the initial value problem (4.6)–(4.9) has a unique bounded solution in $B_q^h(0)$. For example, if we take $\beta = 2$, $\gamma = 0.4$, $\lambda = 0.75$ and choose $M = 0.06 < \frac{\lambda - \sqrt{2}\gamma}{\sqrt{2}\beta}$ so that $K = 0.7354 < \lambda$. Then the initial value problem (4.6)–(4.9) has a unique bounded solution in an open ball having center 0 and radius $q = 281$.

Acknowledgments
The authors would like to thank the referees for their suggestions and comments which considerably helped to improve the content of this paper.
The research was supported by Dr. Jagan Mohan Jonnalagadda's Research Initiation Grant BITS / GAU / RIG / 45 "Applications of Discrete Fractional Calculus in Science and Engineering" sponsored by Birla Institute of Technology and Science Pilani, Hyderabad Campus.

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Received: July 5, 2015.
Revised: September 14, 2015.
Accepted: September 22, 2015.