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PSEUDO ALMOST PERIODIC SOLUTIONS OF INFINITE CLASS UNDER THE LIGHT OF MEASURE THEORY

Abstract. The aim of this work is to present new approach to study weighted pseudo almost periodic functions with infinite delay using the measure theory. We present a new concept of weighted ergodic functions which is more general than the classical one. Then we establish many interesting results on the functional space of such functions. We study the existence and uniqueness of (μ, ν)-pseudo almost periodic solutions of infinite class for some neutral partial functional differential equations in a Banach space when the delay is distributed on $]-\infty, 0]$ using the spectral decomposition of the phase space developed in Adimy and co-authors.

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1. Introduction

In this work, we present a new approach dealing with weighted pseudo almost periodic functions with infinite delay and their applications in evolution equations and partial functional differential equations. Here we use the measure theory to define an ergodic function and we investigate many interesting properties of such functions. Weighted pseudo almost periodic functions started recently and becomes an interesting field in dynamical systems. The study of existence of almost periodic, asymptotically almost periodic, almost automorphic, asymptotically almost automorphic and pseudo almost periodic solutions is one of the most attractive topics in the qualitative theory of differential equations due both to its mathematical interest and applications in physics, mathematical biology, and control theory, among other areas. Most of these problems need to be studied in abstract spaces and the operators are defined over non-dense domains. In this context the literature is very scarce (see [1, 2, 4] and the bibliography therein).

In this work, we study the existence and uniqueness of (μ,ν)-pseudo almost periodic and automorphic solutions of infinite class for the following neutral partial functional differential equation

$$u'(t) = A\,u(t) + L(u_t) + f(t) \qquad \text{for } t \in \mathbb{R}, \tag{1}$$

where A is a linear operator on a Banach space X satisfying the Hille-Yosida condition, that is, there exist $M_0 \geqslant 1$ and $\omega \in \mathbb{R}$ such that $]\omega,+\infty[\subset \rho(A)$ and

$$|R(\lambda,A)^n| \leqslant \frac{M_0}{(\lambda-\omega)^n} \qquad \text{for } n \in \mathbb{N} \text{ and } \lambda > \omega,$$

where $\rho(A)$ is the resolvent set of A and $R(\lambda,A) = (\lambda I - A)^{-1}$ for $\lambda \in \rho(A)$. In sequel, without lost of generality, we suppose that $M_0 = 1$. The phase space $\mathcal{B}$ is a normed linear space of functions mapping $]-\infty,0]$ into X satisfying axioms which will be described in the sequel, for every $t \geqslant 0$, the history $u_t \in \mathcal{B}$ is defined by

$$u_t(\theta) = u(t+\theta) \qquad \text{for } \theta \in]-\infty,0],$$

$f : \mathcal{B} \to X$ is a continuous function and L is a bounded linear operator from $\mathcal{B}$ into X. In the literature devoted to equations with finite delay, the state space is the space of all continuous functions on $[-r,0]$, $r \geqslant 0$, endowed with the uniform norm topology.

When the delay is finite some recent contributions concerning pseudo almost periodic solutions for abstract differential equations similar to equation (1) have been made. For example in [2] the authors have shown that if the inhomogeneous term f depends only on variable t and it is a pseudo almost periodic function, then equation (1) has a unique pseudo almost periodic solution. In [4] the authors have proven that if $f : \mathbb{R} \times X_0 \to X$ is a suitable continuous function, where $X_0 = \overline{D(A)}$, the problem

$$x'(t) = A\,x(t) + f(t, x(t)), \qquad t \in \mathbb{R}$$

has a unique pseudo almost periodic solution, while in [1] the authors have treated the existence of almost periodic solutions for a class of partial neutral functional differential equations defined by a linear operator of Hille-Yosida type with non-dense domain. In [3], the authors studied the existence and uniqueness of pseudo almost periodic solutions for a first-order abstract functional differential equation with a linear part dominated by a Hille-Yosida type operator with a non-dense domain.

In [9], the authors introduce some new classes of functions called weighted pseudo-almost periodic functions, which implement in a natural fashion the classical pseudo-almost periodic functions due to Zhang [15–17]. Properties of these weighted pseudo-almost periodic functions are discussed, including a composition result for weighted pseudo-almost periodic functions. The results obtained are subsequently utilized to study the existence and uniqueness of a weighted pseudo-almost periodic solution to the heat equation with Dirichlet conditions.

In [6], the authors present new approach to study weighted pseudo almost periodic functions using the measure theory. They present a new concept of weighted ergodic functions which is more general than the classical one. Then they establish many interesting results on the functional space of such functions like completeness and composition theorems. The theory of their work generalizes the classical results on weighted pseudo almost periodic functions. More details can be found in book [10] where the authors give basic definitions and facts, concerning the subject discussed in the current paper.

The aim of this work is to prove the existence of (μ, ν)-pseudo almost periodic and automorphic solutions of equation (1) when the delay is distributed on $]-\infty, 0]$. Our approach is based on the spectral decomposition of the phase space developed in [3] and a new approach developped in [6].

This work is organised as follow, in Section 2 we recall some prelimary results on spectral decomposition. In Section 3, we recall some prelimary results on

(μ,ν)-pseudo almost periodic functions and neutral partial functional differential equations that will be used in this work. In Section 4, we give some properties of (μ,ν)-pseudo almost periodic functions of infinite class. In Section 5, we discuss the main result of this paper. Using the strict contraction principle we show the existence and uniqueness of (μ,ν)-pseudo almost periodic solution of infinite class for equation (1). Section 6 is devoted to some applications arising in population dynamics.

2. Variation of constants formula and spectral decomposition

In this work, we assume that the state space $(\mathcal{B}, |\cdot|_{\mathcal{B}})$ is a normed linear space of functions mapping $]-\infty, 0]$ into X. In what follows, we give some examples of normed linear space $(\mathcal{B}, |\cdot|_{\mathcal{B}})$.

Example 1. $\mathcal{B} = BC$, where BC is the space of bounded continuous functions defined from $]-\infty, 0]$ to X, with the the following norm

$$|\varphi|_{\mathcal{B}} = \sup_{\theta \leqslant 0} |\varphi(\theta)| \qquad \text{for all } \varphi \in \mathcal{B}.$$

Example 2. $\mathcal{B} = C_\gamma$, $\gamma > 0$, where

$$C_\gamma = \{\varphi \in C(]-\infty, 0]; X) : \lim_{\theta \to -\infty} e^{\gamma\theta}\,\varphi(\theta) \ \text{ exist in } \ X\}$$

with the the following norm

$$|\varphi|_\gamma = \sup_{\theta \leqslant 0} \left| e^{\gamma\theta}\,\varphi(\theta)\right|.$$

We assume that $\mathcal{B}$ satisfies the following fundamental axioms:

($\mathbf{A}_1$) There exist a positive constant H and functions $K(\cdot), M(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$, with K continuous and M locally bounded, such that for any $\sigma < 0$ and $a > 0$, if $u :]-\infty, a] \to X$, $u_\sigma \in \mathcal{B}$, and $u(\cdot)$ is continuous on $[\sigma, \sigma + a]$, then for every $t \in [\sigma, \sigma + a]$ the following conditions hold

- (**i**) $u_t \in \mathcal{B}$,
- (**ii**) $|u(t)| \leqslant H|u_t|_{\mathcal{B}}$, which is equivalent to $|\varphi(0)| \leqslant H|\varphi|_{\mathcal{B}}$ for every $\varphi \in \mathcal{B}$,

(iii) $|u_t|_{\mathcal{B}} \leqslant K(t-\sigma) \sup_{\sigma \leqslant s \leqslant t} |u(s)| + M(t-\sigma)|u_\sigma|_{\mathcal{B}}$.

($\mathbf{A}_2$) For the function $u(\cdot)$ in ($\mathbf{A}_1$), $t \mapsto u_t$ is a $\mathcal{B}$-valued continuous function for $t \in [\sigma, \sigma + a]$.

($\mathbf{B}$) The space $\mathcal{B}$ is a Banach space.

In the whole of this work, we suppose that $\mathcal{B}$ satisfies axioms ($\mathbf{A}_1$), ($\mathbf{A}_2$) and ($\mathbf{B}$). We also assume that:

($\mathbf{C_1}$) If $(\varphi_n)_{n\geqslant 0}$ is a sequence in $\mathcal{B}$ such that $\varphi_n \to 0$ in $\mathcal{B}$ as $n \to +\infty$, then for all $\theta \leqslant 0$, $(\varphi_n(\theta))_{n\geqslant 0}$ converges to 0 in X.

Let $C(]-\infty, 0], X)$ be the space of continuous functions from $]-\infty, 0]$ into X. We suppose the following assumptions:

($\mathbf{C_2}$) $\mathcal{B} \subset C(]-\infty, 0], X)$.

($\mathbf{C_3}$) There exists $\lambda_0 \in \mathbb{R}$ such that, for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \lambda_0$ and $x \in X$ we have $e^{\lambda \cdot} x \in \mathcal{B}$, where $(e^{\lambda \cdot} x)(\theta) = e^{\lambda\theta} x$ for $\theta \in]-\infty, 0]$ and $x \in X$ and

$$K_0 = \sup_{\substack{\operatorname{Re}\lambda > \lambda_0,\, x\in X \\ x \neq 0}} \frac{|e^{\lambda \cdot} x|_{\mathcal{B}}}{|x|} < \infty.$$

To equation (1), we associate the following initial value problem

$$\begin{cases} \dfrac{d}{dt} u_t = A\, u_t + L(u_t) + f(t) & \text{for } t \geqslant 0, \\ u_0 = \varphi \in \mathcal{B}, \end{cases} \tag{2}$$

where $f : \mathbb{R}^+ \to X$ is a continuous function.

Let us introduce the part A_0 of the operator A in $\overline{D(A)}$ which defined by

$$\begin{cases} D(A_0) = \left\{ x \in D(A) \colon A\, x \in \overline{D(A)} \right\} \\ A_0\, x = A\, x \qquad \text{for } x \in D(A_0). \end{cases}$$

We make the following assumption:

($\mathbf{H_0}$) A satisfies the Hille-Yosida condition.

Lemma 3. *[1] A_0 generates a strongly continuous semigroup $(T_0(t))_{t\geqslant 0}$ on $\overline{D(A)}$.*

The phase space $\mathcal{B}_A$ of equation (2) is defined by

$$\mathcal{B}_A = \{\varphi \in \mathcal{B}\colon \varphi(0) \in \overline{D(A)}\}.$$

For each $t \geqslant 0$, we define the linear operator $\mathcal{U}(t)$ on $\mathcal{B}_A$ by

$$\mathcal{U}(t) = v_t(\cdot, \varphi)$$

where $v_t(\cdot, \varphi)$ is the solution of the following homogeneous equation

$$\begin{cases} \dfrac{d}{dt} v_t = A\, v_t + L(v_t) & \text{for } t \geqslant 0, \\ v_0 = \varphi \in \mathcal{B}. \end{cases}$$

Proposition 4. *[3] $(\mathcal{U}(t))_{t\geqslant 0}$ is a strongly continuous semigroup of linear operators on $\mathcal{B}_A$. Moreover, $(\mathcal{U}(t))_{t\geqslant 0}$ satisfies, for $t \geqslant 0$ and $\theta \in]-\infty, 0]$, the following translation property*

$$\big(\mathcal{U}(t)\,\varphi\big)(\theta) = \begin{cases} \big(\mathcal{U}(t+\theta)\,\varphi\big)(0) & \text{for } t+\theta \geqslant 0, \\ \varphi(t+\theta) & \text{for } t+\theta \leqslant 0. \end{cases}$$

Theorem 5. *[3] Assume that $\mathcal{B}$ satisfies $(\boldsymbol{A_1})$, $(\boldsymbol{A_2})$, $(\boldsymbol{B})$, $(\boldsymbol{C_1})$ and $(\boldsymbol{C_2})$. Then $\mathcal{A}_{\mathcal{U}}$ defined on $\mathcal{B}_A$ by*

$$\begin{cases} D(\mathcal{A}_{\mathcal{U}}) = \Big\{\varphi \in C^1(]-\infty, 0]; X) \cap \mathcal{B}_A\colon \varphi' \in \mathcal{B}_A,\ \varphi(0) \in \overline{D(A)} \\ \qquad\qquad \text{and } \varphi'(0) = A\varphi(0) + L(\varphi)\Big\} \\ \mathcal{A}_{\mathcal{U}}\varphi = \varphi' \quad \text{for } \varphi \in D(\mathcal{A}_{\mathcal{U}}). \end{cases}$$

is the infinitesimal generator of the semigroup $(\mathcal{U}(t))_{t\geqslant 0}$ on $\mathcal{B}_A$.

Let $\langle X_0 \rangle$ be the space defined by

$$\langle X_0 \rangle = \{X_0\, x\colon x \in X\}$$

where the function $X_0\, x$ is defined by

$$(X_0\, x)(\theta) = \begin{cases} 0 & \text{if } \theta \in]-\infty, 0[\,, \\ x & \text{if } \theta = 0. \end{cases}$$

The space $\mathcal{B}_A \oplus \langle X_0 \rangle$ equipped with the norm $|\phi + X_0\, c|_{\mathcal{B}} = |\phi|_{\mathcal{B}} + |c|$ for $(\phi, c) \in \mathcal{B}_A \times X$ is a Banach space and consider the extension $\mathcal{A}_{\mathcal{U}}$ defined on $\mathcal{B}_A \oplus \langle X_0 \rangle$ by

$$\begin{cases} D(\widetilde{\mathcal{A}_{\mathcal{U}}}) = \left\{\varphi \in C^1(]-\infty, 0]; X) \colon \varphi \in D(A) \text{ and } \varphi' \in \overline{D(A)}\right\} \\ \widetilde{\mathcal{A}_{\mathcal{U}}}\varphi = \varphi' + X_0\big(A\,\varphi + L(\varphi) - \varphi'\big). \end{cases}$$

Lemma 6. *[3] Assume that $\mathcal{B}$ satisfies ($\boldsymbol{A_1}$), ($\boldsymbol{A_2}$), ($\boldsymbol{B}$), ($\boldsymbol{C_1}$), ($\boldsymbol{C_2}$) and ($\boldsymbol{C_3}$). Then, $\widetilde{\mathcal{A}_{\mathcal{U}}}$ satisfies the Hille-Yosida condition on $\mathcal{B}_A \oplus \langle X_0 \rangle$.*

Now, we can state the variation of constants formula associated to equation (2).

Let C_{00} be the space of X-valued continuous function on $]-\infty, 0]$ with compact support. We assume that:

(**D**) If $(\varphi_n)_{n\geqslant 0}$ is a Cauchy sequence in $\mathcal{B}$ and converges compactly to φ on $]-\infty, 0]$, then $\varphi \in \mathcal{B}$ and $|\varphi_n - \varphi| \to 0$.

Theorem 7. *[3] Assume that ($\boldsymbol{C_1}$), ($\boldsymbol{C_2}$) and ($\boldsymbol{C_3}$) hold. Then the integral solution u of equation (2) is given by the following variation of constants formula*

$$u_t = \mathcal{U}(t)\,\varphi + \lim_{\lambda\to+\infty} \int_0^t \mathcal{U}(t-s)\,\widetilde{B}_\lambda\big(X_0\, f(s)\big)\, ds \qquad \textit{for } t \geqslant 0,$$

where $\widetilde{B}_\lambda = \lambda\,(\lambda\, I - \widetilde{\mathcal{A}_{\mathcal{U}}})^{-1}$.

Let $(S_0(t))_{t\geqslant 0}$ be the strongly continuous semigroup defined on the subspace

$$\mathcal{B}_0 = \big\{\varphi \in \mathcal{B} \colon \varphi(0) = 0\big\}$$

by

$$\big(S_0(t)\,\phi\big)(\theta) = \begin{cases} \phi(t+\theta) & \text{if } t+\theta \leqslant 0, \\ 0 & \text{if } t+\theta \geqslant 0. \end{cases}$$

Definition 8. *Assume that the space $\mathcal{B}$ satisfies axioms **(B)** and **(D)**, $\mathcal{B}$ is said to be a fading memory space, if for all $\varphi \in \mathcal{B}_0$,*

$$\big|S_0(t)\big| \to 0 \qquad \textit{as } t \to +\infty \textit{ in } \mathcal{B}_0.$$

Moreover, $\mathcal{B}$ is said to be a uniform fading memory space, if

$$\big|S_0(t)\big| \to 0 \qquad \textit{as } t \to +\infty.$$

Lemma 9. *If $\mathcal{B}$ is a uniform fading memory space, then we can choose the function K constant and the function M such that $M(t) \to 0$ as $t \to +\infty$.*

Proposition 10. *If the phase space $\mathcal{B}$ is a fading memory space, then the space $BC(]-\infty, 0], X)$ of bounded continuous X-valued functions on $]-\infty, 0]$ endowed with the uniform norm topology, is continuous embedding in $\mathcal{B}$. In particular $\mathcal{B}$ satisfies ($\boldsymbol{C_3}$), for $\lambda_0 > 0$.*

For the sequel, we make the following assumption:

($\mathbf{H_1}$) $T_0(t)$ is compact on $\overline{D(A)}$ for every $t > 0$.

($\mathbf{H_2}$) $\mathcal{B}$ is a uniform fading memory space.

Theorem 11. *[3] Assume that $\mathcal{B}$ satisfies ($\boldsymbol{A_1}$), ($\boldsymbol{A_2}$), ($\boldsymbol{B}$), ($\boldsymbol{C_1}$) and ($\boldsymbol{H_0}$), ($\boldsymbol{H_1}$), ($\boldsymbol{H_2}$) hold. Then the semigroup $(\mathcal{U}(t))_{t\geqslant 0}$ is decomposed on $\mathcal{B}_A$ as follows*

$$\mathcal{U}(t) = \mathcal{U}_1(t) + \mathcal{U}_2(t) \qquad \text{for } t \geqslant 0,$$

where $(\mathcal{U}_1(t))_{t\geqslant 0}$ is an exponentially stable semigroup on $\mathcal{B}_A$, which means that there are positive constants α_0 and N_0 such that

$$\left|\mathcal{U}_1(t)\right| \leqslant N_0\, e^{-\alpha_0 t}\, |\varphi| \qquad \text{for } t \geqslant 0 \quad \text{and} \quad \varphi \in \mathcal{B}_A$$

and $(\mathcal{U}_2(t))_{t\geqslant 0}$ is compact for for every $t > 0$.

We have the following result on the spectral decomposition of the phase space $\mathcal{B}_A$.

Theorem 12. *[3] Assume that $\mathcal{B}$ satisfies ($\boldsymbol{A_1}$), ($\boldsymbol{A_2}$), ($\boldsymbol{B}$), ($\boldsymbol{C_1}$), and ($\boldsymbol{H_0}$), ($\boldsymbol{H_1}$), ($\boldsymbol{H_2}$) hold. Then the space $\mathcal{B}_A$ is decomposed as a direct sum*

$$\mathcal{B}_A = S \oplus U$$

of two $\mathcal{U}(t)$ invariant closed subspaces S and U such that the restricted semigroup on $\mathcal{U}$ is a group and there exist positive constants $\overline{M}$ and ω such that

$$\begin{aligned} \left|\mathcal{U}(t)\,\varphi\right| &\leqslant \overline{M}\, e^{-\omega t}\, |\varphi| && \text{for } t \geqslant 0 \quad \text{and } \varphi \in S, \\ \left|\mathcal{U}(t)\,\varphi\right| &\leqslant \overline{M}\, e^{\omega t}\, |\varphi| && \text{for } t \leqslant 0 \quad \text{and } \varphi \in U, \end{aligned}$$

where S and U are called the stable and unstable space respectively.

3. (μ, ν)-Pseudo almost periodic functions

In this section, we recall some properties about μ-pseudo almost periodic functions. The notion of μ-pseudo almost periodicity is a generalization of the pseudo almost periodicity introduced by Zhang [15–17]; it is also a generalization of weighted pseudo almost periodicity given by Diagana [9]. Let $BC(\mathbb{R}; X)$ be the space of all bounded and continuous function from $\mathbb{R}$ to X equipped with the uniform topology norm.

We denote by $\mathcal{N}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measures μ on $\mathcal{N}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a, b]) < \infty$, for all $a, b \in \mathbb{R}$, $a \leqslant b$.

Definition 13. *A bounded continuous function $\phi : \mathbb{R} \to X$ is called almost periodic if for each $\varepsilon > 0$, there exists a relatively dense subset of $\mathbb{R}$ denote by $\mathcal{K}(\varepsilon, \phi, X)$ such that $|\phi(t + \tau) - \phi(t)| < \varepsilon$ for all $(t, \tau) \in \mathbb{R} \times \mathcal{K}(\varepsilon, \phi, X)$.*

We denote by $AP(\mathbb{R}; X)$, the space of all such functions.

Definition 14. *Let X_1 and X_2 be two Banach spaces. A bounded continuous function $\phi : \mathbb{R} \times X_1 \to X_2$ is called almost periodic in $t \in \mathbb{R}$ uniformly in $x \in X_1$ if for each $\varepsilon > 0$ and all compact $K \subset X_1$, there exists a relatively dense subset of $\mathbb{R}$ denote by $\mathcal{K}(\varepsilon, \phi, K)$ such that $|\phi(t + \tau, x) - \phi(t, x)| < \varepsilon$ for all $t \in \mathbb{R}$, $x \in K$, $\tau \in \mathcal{K}(\varepsilon, \phi, K)$.*

We denote by $AP(\mathbb{R} \times X_1; X_2)$, the space of all such functions.

The next lemma is also a characterization of almost periodic functions.

Lemma 15. *A function $\phi \in C(\mathbb{R}, X)$ is almost periodic if and only if the space of functions $\{\phi_\tau \colon \tau \in \mathbb{R}\}$, where $(\phi_\tau)(t) = \phi(t+\tau)$, is relatively compact in $BC(\mathbb{R}; X)$.*

In the sequel, we recall some preliminary results concerning the (μ, ν)-pseudo almost periodic functions with infinite delay.

$\mathcal{E}(\mathbb{R}; X, \mu, \nu)$ stands for the space of functions

$$\mathcal{E}(\mathbb{R}; X, \mu, \nu) = \Big\{u \in BC(\mathbb{R}; X) \colon \lim_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{+\tau} |u(t)| \, d\mu(t) = 0\Big\}.$$

To study delayed differential equations for which the history belong to $\mathcal{B}$, we need to introduce the space

$$\mathcal{E}(\mathbb{R}; X, \mu, \nu, \infty) = \\ = \Big\{u \in BC(\mathbb{R}; X)\colon \lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \Big(\sup_{\theta\in\,]-\infty,t]} |u(\theta)|\Big)\, d\mu(t) = 0\Big\}.$$

In addition to above-mentioned space, we consider the following spaces

$$\mathcal{E}(\mathbb{R} \times X_1, X_2, \mu, \nu) = \\ = \Big\{u \in BC(\mathbb{R} \times X_1; X_2)\colon \lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} |u(t,x)|_{X_2}\, d\mu(t) = 0\Big\},$$

$$\mathcal{E}(\mathbb{R} \times X_1; X_2, \mu, \nu, \infty) = \Big\{u \in BC(\mathbb{R} \times X_1; X_2)\colon \\ \lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \Big(\sup_{\theta\in\,]-\infty,t]} |u(\theta,x)|_{X_2}\Big)\, d\mu(t) = 0\Big\},$$

where in both cases the limit (as $\tau \to +\infty$) is uniform in compact subset of X_1.

In view of previous definitions, it is clear that the spaces $\mathcal{E}(\mathbb{R}; X, \mu, \nu, \infty)$ and $\mathcal{E}(\mathbb{R} \times X_1; X_2, \mu, \nu, \infty)$ are continuously embedded in $\mathcal{E}(\mathbb{R}; X, \mu, \nu)$ and $\mathcal{E}(\mathbb{R} \times X_1, X_2, \mu, \nu)$, respectively. On the other hand, one can observe that a ρ-weighted pseudo almost periodic functions is μ-pseudo almost periodic, where the measure μ is absolutely continuous with respect to the Lebesgue measure and its Radon-Nikodym derivative is ρ:

$$d\mu(t) = \rho(t)\, dt$$

and ν is the usual Lebesgue measure on $\mathbb{R}$, i.e. $\nu([-\tau,\tau]) = 2\tau$ for all $\tau \geqslant 0$.

Example 16. [6] Let ρ be a nonnegative $\mathcal{N}$-measurable function. Denote by μ the positive measure defined by

$$\mu(A) = \int_A \rho(t)\, dt \qquad \text{for } A \in \mathcal{N}, \tag{3}$$

where dt denotes the Lebesgue measure on $\mathbb{R}$. The function ρ which occurs in equation (3) is called the Radon-Nikodym derivative of μ with respect to the Lebesgue measure on $\mathbb{R}$.

Definition 17. *Let $\mu, \nu \in \mathcal{M}$. A bounded continuous function $\phi : \mathbb{R} \to X$ is called (μ, ν)-pseudo almost periodic if $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AP(\mathbb{R}, X)$ and $\phi_2 \in \mathcal{E}(\mathbb{R}; X, \mu, \nu)$.*

We denote by $PAP(\mathbb{R}; X, \mu, \nu)$ the space of all such functions.

Definition 18. *Let $\mu, \nu \in \mathcal{M}$ and X_1 and X_2 be two Banach spaces. A bounded continuous function $\phi : \mathbb{R} \times X_1 \to X_2$ is called uniformly (μ, ν)-pseudo almost periodic if $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AP(\mathbb{R} \times X_1; X_2)$ and $\phi_2 \in \mathcal{E}(\mathbb{R} \times X_1, X_2, \mu, \nu)$.*

We denote by $PAP(\mathbb{R} \times X_1; X_2, \mu, \nu)$, the space of all such functions.

Definition 19. *$\mu, \nu \in \mathcal{M}$. A bounded continuous function $\phi : \mathbb{R} \to X$ is called (μ, ν)-pseudo almost periodic of infinite class if $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AP(\mathbb{R}; X)$ and $\phi_2 \in \mathcal{E}(\mathbb{R}; X, \mu, \nu, \infty)$. We denote by $PAP(\mathbb{R}; X, \mu, \nu, \infty)$, the space of all such functions.*

Definition 20. *$\mu, \nu \in \mathcal{M}$. Let X_1 and X_2 be two Banach spaces. A bounded continuous function $\phi : \mathbb{R} \times X_1 \to X_2$ is called uniformly (μ, ν)-pseudo almost periodic of infinite class if $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AP(\mathbb{R} \times X_1; X_2)$ and $\phi_2 \in \mathcal{E}(\mathbb{R} \times X_1; X_2, \mu, \nu, \infty)$.*

We denote by $PAP(\mathbb{R} \times X_1; X_2, \mu, \nu, \infty)$, the space of all such functions.

4. Properties of (μ, ν)-pseudo almost periodic functions of infinite class

From $\mu, \nu \in \mathcal{M}$, we formulate the following hypothese.

$(\mathbf{H_3})$ Let $\mu, \nu \in \mathcal{M}$ be such that $\limsup_{\tau \to +\infty} \dfrac{\mu([-\tau, \tau])}{\nu([-\tau, \tau])} = \alpha < \infty$.

We have the following result.

Lemma 21. *Assume that $(\boldsymbol{H_3})$ holds. The space $\mathcal{E}(\mathbb{R}; X, \mu, \nu, \infty)$ endowed with the uniform topology norm is a Banach space.*

Proof. We can see that $\mathcal{E}(\mathbb{R}; X, \mu, \nu, \infty)$ is a vector subspace of $BC(\mathbb{R}; X)$. To complete the proof, it is enough to prove that $\mathcal{E}(\mathbb{R}; X, \mu, \nu, \infty)$ is closed in $BC(\mathbb{R}; X)$.

Let $(z_n)_n$ be a sequence in $\mathcal{E}(\mathbb{R}; X, \mu, \nu, \infty)$ such that $\lim_{n\to+\infty} z_n = z$ uniformly in $\mathbb{R}$. From $\nu(\mathbb{R}) = +\infty$, it follows $\nu([-\tau, \tau]) > 0$ for τ sufficiently large. Let $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, $\|z_n - z\|_\infty < \varepsilon$. Let $n \geqslant n_0$, then we have

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in\,]-\infty,t]}|z(\theta)|\Big)\,d\mu(t) \leqslant \\
&\qquad \leqslant \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in\,]-\infty,t]}|z_n(\theta)-z(\theta)|\Big)\,d\mu(t) + \\
&\qquad\quad + \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in\,]-\infty,t]}|z_n(\theta)|\Big)\,d\mu(t) \leqslant \\
&\qquad\quad \leqslant \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{t\in\mathbb{R}}|z_n(t)-z(t)|\Big)\,d\mu(t) + \\
&\qquad\quad + \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in\,]-\infty,t]}|z_n(\theta)|\Big)\,d\mu(t) \leqslant \\
&\qquad \leqslant \|z_n - z\|_\infty \times \frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} + \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in\,]-\infty,t]}|z_n(\theta)|\Big)\,d\mu(t).
\end{aligned}$$

We deduce that

$$\limsup_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in\,]-\infty,t]}|z(\theta)|\Big)\,d\mu(t) \leqslant \alpha\,\varepsilon \qquad \text{for any } \ \varepsilon > 0.$$

□

From the definition of $PAP(\mathbb{R}; X, \mu, \nu, \infty)$, we deduce the following result.

Proposition 22. $\mu, \nu \in \mathcal{M}$. *The space* $PAP(\mathbb{R}; X, \mu, \nu, \infty)$ *endowed with the uniform topology norm is a Banach space.*

Next result is a characterization of (μ, ν)-ergodic functions of infinite class.

Theorem 23. *Assume that* (***H₃***) *holds and let* $\mu, \nu \in \mathcal{M}$ *and* I *be a bounded interval (eventually* $I = \emptyset$*). Assume that* $f \in BC(\mathbb{R}, X)$*. Then the following assertions are equivalent:*

i) $f \in \mathcal{E}(\mathbb{R}, X, \mu, \nu, \infty)$.

ii) $\displaystyle\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in\,]-\infty,t]}|f(\theta)|\Big)\,d\mu(t) = 0.$

$$iii)\ \textit{For any } \varepsilon>0,\ \lim_{\tau\to+\infty}\frac{\mu\Big(\big\{t\in[-\tau,\tau]\setminus I\colon \sup\limits_{\theta\in\,]-\infty,t]}|f(\theta)|>\varepsilon\big\}\Big)}{\nu([-\tau,\tau]\setminus I)}=0.$$

Proof. The proof is made like the proof of Theorem 2.13 in [6].

$i) \Leftrightarrow ii)$ Denote by $A=\nu(I)$, $B=\displaystyle\int_I\big(\sup_{\theta\in\,]-\infty,t]}|f(\theta)|\big)\,d\mu(t)$. We have A and $B\in\mathbb{R}$, since the interval I is bounded and the function f is bounded and continuous. For $\tau>0$ such that $I\subset[-\tau,\tau]$ and $\nu([-\tau,\tau]\setminus I)>0$, we have

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in\,]-\infty,t]}|f(\theta)|\Big)\,d\mu(t)=\\
&\qquad=\frac{1}{\nu([-\tau,\tau])-A}\Big[\int_{[-\tau,\tau]}\Big(\sup_{\theta\in\,]-\infty,t]}|f(\theta)|\Big)\,d\mu(t)-B\Big]=\\
&=\frac{\nu([-\tau,\tau])}{\nu([-\tau,\tau])-A}\Big[\frac{1}{\nu([-r,r])}\int_{[-\tau,\tau]}\Big(\sup_{\theta\in\,]-\infty,t]}|f(\theta)|\Big)\,d\mu(t)-\frac{B}{\nu([-\tau,\tau])}\Big].
\end{aligned}$$

From above equalities and the fact that $\nu(\mathbb{R})=+\infty$, we deduce that $ii)$ is equivalent to

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in\,]-\infty,t]}|f(\theta)|\Big)d\mu(t)=0,$$

that is $i)$.

$iii) \Rightarrow ii)$ Denote by A_τ^ε and B_τ^ε the following sets

$$A_\tau^\varepsilon=\Big\{t\in[-\tau,\tau]\setminus I\colon \sup_{\theta\in\,]-\infty,t]}|f(\theta)|>\varepsilon\Big\}$$

and

$$B_\tau^\varepsilon=\Big\{t\in[-\tau,\tau]\setminus I\colon \sup_{\theta\in\,]-\infty,t]}|f(\theta)|\leqslant\varepsilon\Big\}.$$

Assume that $iii)$ holds, that is

$$\lim_{\tau\to+\infty}\frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)}=0. \tag{4}$$

From the equality

$$\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in\,]-\infty,t]}|f(\theta)|\Big)d\mu(t)=$$
$$=\int_{A^{\varepsilon}_{\tau}}\Big(\sup_{\theta\in\,]-\infty,t]}|f(\theta)|\Big)\,d\mu(t)+\int_{B^{\varepsilon}_{\tau}}\Big(\sup_{\theta\in\,]-\infty,t]}|f(\theta)|\Big)\,d\mu(t),$$

we deduce that for τ sufficiently large

$$\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in\,]-\infty,t]}|f(\theta)|\Big)d\mu(t)\leqslant$$
$$\leqslant\|f\|_{\infty}\,\frac{\mu(A^{\varepsilon}_{\tau})}{\nu([-\tau,\tau]\setminus I)}+\varepsilon\,\frac{\mu(B^{\varepsilon}_{\tau})}{\nu([-\tau,\tau]\setminus I)}.$$

By using $(\mathbf{H_3})$, it follows that

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in\,]-\infty,t]}|f(\theta)|\Big)\,d\mu(t)\leqslant\alpha\,\varepsilon\qquad\text{for any }\varepsilon>0,$$

consequently $ii)$ holds.

$ii)\Rightarrow iii)$ Assume that $ii)$ holds. From the following inequality

$$\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in\,]-\infty,t]}|f(\theta)|\Big)d\mu(t)\geqslant\int_{A^{\varepsilon}_{\tau}}\Big(\sup_{\theta\in\,]-\infty,t]}|f(\theta)|\Big)d\mu(t)$$
$$\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in\,]-\infty,t]}|f(\theta)|\Big)d\mu(t)\geqslant\varepsilon\,\frac{\mu(A^{\varepsilon}_{\tau})}{\nu([-\tau,\tau]\setminus I)}$$
$$\frac{1}{\varepsilon\,\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in\,]-\infty,t]}|f(\theta)|\Big)\,d\mu(t)\geqslant\frac{\mu(A^{\varepsilon}_{\tau})}{\nu([-\tau,\tau]\setminus I)},$$

for τ sufficiently large, we obtain equation (4), that is $iii)$. □

For $\mu\in\mathcal{M}$, we formulate the following hypotheses.

$(\mathbf{H_4})$ For all a, b and $c\in\mathbb{R}$, such that $0\leqslant a<b\leqslant c$, there exist δ_0 and $\alpha_0>0$ such that

$$|\delta|\geqslant\delta_0\quad\Longrightarrow\quad\mu(a+\delta,b+\delta)\geqslant\alpha_0\,\mu(\delta,c+\delta).$$

$(\mathbf{H_5})$ For all $\tau\in\mathbb{R}$, there exist $\beta>0$ and a bounded interval I such that

$$\mu(\{a+\tau\colon a\in A\})\leqslant\beta\,\mu(A)\qquad\text{when }A\in\mathcal{N}\text{ satisfies }A\cap I=\emptyset.$$

We have the following results due to [6].

Lemma 24. *[6] Hypothesis ($\boldsymbol{H_5}$) implies ($\boldsymbol{H_4}$).*

Proposition 25. *[6] $\mu, \nu \in \mathcal{M}$ satisfy ($\boldsymbol{H_4}$) and $f \in PAP(\mathbb{R}; X, \mu, \nu)$ be such that*

$$f = g + h$$

where $g \in AP(\mathbb{R}, X)$ and $h \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$. Then

$$\{g(t),\ t \in \mathbb{R}\} \subset \overline{\{f(t),\ t \in \mathbb{R}\}} \qquad (\textit{the closure of the range of } f).$$

Corollary 26. *[6] Assume that ($\boldsymbol{H_4}$) holds. Then the decomposition of a (μ, ν)-pseudo almost periodic function in the form $f = g + \phi$ where $g \in AP(\mathbb{R}; X)$ and $\phi \in \mathcal{E}(\mathbb{R}; X, \mu, \nu)$, is unique.*

The following proposition is a consequence of Proposition 25.

Proposition 27. *Let $\mu, \nu \in \mathcal{M}$. Assume ($\boldsymbol{H_4}$) holds. Then the decomposition of a (μ, ν)-pseudo-almost periodic function $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AP(\mathbb{R}; X)$ and $\phi_2 \in \mathcal{E}(\mathbb{R}; X, \mu, \nu, \infty)$, is unique.*

Proof. In fact, since as a consequence of Corollary 26, the decomposition of a (μ, ν)-pseudo-almost periodic function $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AP(\mathbb{R}; X)$ and $\phi_2 \in \mathcal{E}(\mathbb{R}; X, \mu, \nu)$, is unique. Since $PAP(\mathbb{R}; X, \mu, \nu, \infty) \subset PAP(\mathbb{R}; X, \mu, \nu)$, we get the desired result. $\square$

Definition 28. *Let $\mu_1, \mu_2 \in \mathcal{M}$. We say that μ_1 is equivalent to μ_2, denoting this as $\mu_1 \sim \mu_2$ if there exist constants α and $\beta > 0$ and a bounded interval I (eventually $I = \emptyset$) such that*

$$\alpha\, \mu_1(A) \leqslant \mu_2(A) \leqslant \beta\, \mu_1(A), \qquad \textit{when } A \in \mathcal{N} \textit{ satisfies } A \cap I = \emptyset.$$

From [6] $\sim$ is a binary equivalence relation on $\mathcal{M}$. The equivalence class of a given measure $\mu \in \mathcal{M}$ will then be denoted by

$$cl(\mu) = \big\{\varpi \in \mathcal{M}\colon \mu \sim \varpi\big\}.$$

Theorem 29. *Let $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathcal{M}$. If $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$, then*

$$PAP(\mathbb{R}; X, \mu_1, \nu_1, \infty) = PAP(\mathbb{R}; X, \mu_2, \nu_2, \infty).$$

Proof. Since $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$ there exist some constants $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ and a bounded interval I (eventually $I = \emptyset$) such that $\alpha_1\, \mu_1(A) \leqslant \mu_2(A) \leqslant \beta_1\, \mu_1(A)$ and $\alpha_2\, \nu_1(A) \leqslant \nu_2(A) \leqslant \beta_2\, \nu_1(A)$ for each $A \in \mathcal{N}$ satisfies $A \cap I = \emptyset$, i.e.

$$\frac{1}{\beta_2\, \nu_1(A)} \leqslant \frac{1}{\nu_2(A)} \leqslant \frac{1}{\alpha_2\, \nu_1(A)}.$$

Since $\mu_1 \sim \mu_2$ and $\mathcal{N}$ is the Lebesgue σ-field, for τ sufficiently large, we obtain

$$\begin{aligned}
&\frac{\alpha_1\, \mu_1\Big(\big\{t \in [-\tau, \tau] \setminus I\colon \sup\limits_{\theta \in\,]-\infty, t]} |f(\theta)| > \varepsilon\big\}\Big)}{\beta_2\, \nu_1([-\tau, \tau] \setminus I)} \leqslant \\
&\qquad \leqslant \frac{\mu_2\left(\big\{t \in [-\tau, \tau] \setminus I\colon \sup\limits_{\theta \in\,]-\infty, t]} |f(\theta)| > \varepsilon\big\}\right)}{\nu_2([-\tau, \tau] \setminus I)} \leqslant \\
&\qquad\qquad \leqslant \frac{\beta_1\, \mu_1\Big(\big\{t \in [-\tau, \tau] \setminus I\colon \sup\limits_{\theta \in\,]-\infty, t]} |f(\theta)| > \varepsilon\big\}\Big)}{\alpha_2\, \nu_1([-\tau, \tau] \setminus I)}.
\end{aligned}$$

By using Theorem 23 we deduce that $\mathcal{E}(\mathbb{R}, X, \mu_1, \nu_1, \infty) = \mathcal{E}(\mathbb{R}, X, \mu_2, \nu_2, \infty)$. From the definition of a (μ, ν)-pseudo almost periodic function, we deduce that $PAP(\mathbb{R}; X, \mu_1, \nu_1, \infty) = PAP(\mathbb{R}; X, \mu_2, \nu_2, \infty)$. □

For $\mu, \nu \in \mathcal{M}$ we denote

$$cl(\mu, \nu) = \{\varpi_1, \varpi_2 \in \mathcal{M}\colon \mu \sim \varpi_2 \text{ and } \nu \sim \varpi_2\}.$$

Proposition 30. *[8] Let $\mu, \nu \in \mathcal{M}$ satisfy $(\boldsymbol{H_5})$. Then $PAP(\mathbb{R}, X, \mu, \nu)$ is invariant by translation, that is $f \in PAP(\mathbb{R}, X, \mu, \nu)$ implies $f_\alpha \in PAP(\mathbb{R}, X, \mu, \nu)$ for all $\alpha \in \mathbb{R}$.*

In what follows, we prove some preliminary results concerning the composition of (μ, ν)-pseudo almost periodic functions of infinite class.

Theorem 31. *Let* $\mu, \nu \in \mathcal{M}$, $\phi \in PAP(\mathbb{R} \times X_1; X_2, \mu, \nu, \infty)$ *and* $h \in PAP(\mathbb{R}; X_1, \mu, \nu, \infty)$. *Assume that there exists a function* $L_\phi : \mathbb{R} \to [0, +\infty[$ *satisfies*

$$\big|\phi(t, x_1) - \phi(t, x_2)\big| \leqslant L_\phi(t)|x_1 - x_2| \qquad \text{for } \ t \in \mathbb{R} \ \text{ and for } \ x_1, x_2 \in X_1. \tag{5}$$

If

$$\frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \Big(\sup_{\theta \in \,]-\infty, t]} L_\phi(\theta) \Big)\, d\mu(t) < \infty \tag{6}$$

and

$$\lim_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{+\tau} \Big(\sup_{\theta \in \,]-\infty, t]} L_\phi(\theta) \Big)\, \xi(t)\, d\mu(t) = 0 \tag{7}$$

for each $\xi \in \mathcal{E}(\mathbb{R}, \mu, \nu)$ *and for almost* $\tau > 0$, *then the function* $t \to \phi(t, h(t))$ *belongs to* $PAP(\mathbb{R}; X_2, \mu, \nu, \infty)$.

Proof. Assume that $\phi = \phi_1 + \phi_2$, $h = h_1 + h_2$ where $\phi_1 \in AP(\mathbb{R} \times X_1; X_2)$, $\phi_2 \in \mathcal{E}(\mathbb{R} \times X_1; X_2, \mu, \nu, \infty)$ and $h_1 \in AP(\mathbb{R}; X_1)$, $h_2 \in \mathcal{E}(\mathbb{R}; X_1, \mu, \nu, \infty)$. Consider the following decomposition

$$\phi(t, h(t)) = \phi_1(t, h_1(t)) + \big[\phi(t, h(t)) - \phi(t, h_1(t))\big] + \phi_2(t, h_1(t)).$$

From [7, 14], $\phi_1(\cdot, h_1(\cdot)) \in AP(\mathbb{R}; X_2)$. It remains to prove that both $\phi(\cdot, h(\cdot)) - \phi(\cdot, h_1(\cdot))$ and $\phi_2(\cdot, h_1(\cdot))$ belong to $\mathcal{E}(\mathbb{R}; X_2, \mu, \nu, \infty)$.

Using equation (5), it follows that

$$\begin{aligned}
&\frac{\mu\Big(\big\{t \in [-\tau, \tau] \colon \sup_{\theta \in \,]-\infty, t]} |\phi(\theta, h(\theta)) - \phi(\theta, h_1(\theta))| > \varepsilon\big\}\Big)}{\nu([-\tau, \tau])} \leqslant \\
&\leqslant \frac{\mu\Big(\big\{t \in [-\tau, \tau] \colon \sup_{\theta \in \,]-\infty, t]} (L_\phi(\theta)|h_2(\theta)|) > \varepsilon\big\}\Big)}{\nu([-\tau, \tau])} \leqslant \\
&\leqslant \frac{\mu\Big(\big\{t \in [-\tau, \tau] \colon \big(\sup_{\theta \in \,]-\infty, t]} L_\phi(\theta)\big)\big(\sup_{\theta \in \,]-\infty, t]} |h_2(\theta)|\big) > \varepsilon\big\}\Big)}{\nu([-\tau, \tau])}.
\end{aligned}$$

Since h_2 is (μ, ν)-ergodic of infinite class, Theorem 23 and equations (6)–(7) yield that for the above-mentioned ε, we have

$$\lim_{\tau \to +\infty} \frac{\mu\Big(\big\{t \in [-\tau, \tau] \colon \big(\sup_{\theta \in \,]-\infty, t]} L_\phi(\theta)\big)\big(\sup_{\theta \in \,]-\infty, t]} |h_2(\theta)|\big) > \varepsilon\big\}\Big)}{\nu([-\tau, \tau])} = 0,$$

and then we obtain

$$\lim_{\tau\to+\infty} \frac{\mu\Big(\big\{t\in[-\tau,\tau]\colon \sup\limits_{\theta\in\,]-\infty,t]} |\phi(\theta,h(\theta))-\phi(\theta,h_1(\theta))|>\varepsilon\big\}\Big)}{\nu([-\tau,\tau])}=0, \tag{8}$$

By Theorem 23, equation (8) shows that $t\mapsto \phi(t,h(t))-\phi(t,h_1(t))$ is (μ,ν)-ergodic of infinite class.

Now to complete the proof, it is enough to prove that $t\mapsto \phi_2(t,h(t))$ is (μ,ν)-ergodic of infinite class. Since ϕ_2 is uniformly continuous on the compact set $K=\overline{\{h_1(t)\colon t\in\mathbb{R}\}}$ with respect to the second variable x, we deduce that for given $\varepsilon>0$, there exists $\delta>0$ such that, for all $t\in\mathbb{R}$, ξ_1 and $\xi_2\in K$, one has

$$\|\xi_1-\xi_2\|\leqslant\delta \implies \|\phi_2(t,\xi_1(t))-\phi_2(t,\xi_2(t))\|\leqslant\varepsilon.$$

Therefore, there exist $n(\varepsilon)\in\mathbb{N}$ and $\{z_i\}_{i=1}^{n(\varepsilon)}\subset K$, such that

$$K\subset\bigcup_{i=1}^{n(\varepsilon)} B_\delta(z_i,\delta)$$

and then

$$\|\phi_2(t,h_1(t))\|\leqslant\varepsilon+\sum_{i=1}^{n(\varepsilon)}\|\phi_2(t,z_i)\|.$$

Since

$$\forall\, i\in\{1,\dots,n(\varepsilon)\}\quad \lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in\,]-\infty,t]}|\phi_2(\theta,z_i)|\Big)\,d\mu(t)=0,$$

we deduce that

$$\forall\,\varepsilon>0\quad \limsup_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in\,]-\infty,t]}|\phi_2(\theta,h_1(t))|\Big)\,d\mu(t)\leqslant\varepsilon,$$

which implies

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in\,]-\infty,t]}|\phi_2(\theta,h_1(t))|\Big)\,d\mu(t)=0.$$

Consequently $t\mapsto\phi_2(t,h(t))$ is (μ,ν)-ergodic of infinite class. $\square$

We have the following result.

Theorem 32. *Assume that* ($\boldsymbol{H_4}$) *holds. Let* $\mu, \nu \in \mathcal{M}$ *and* $\phi \in PAP(\mathbb{R}; X, \mu, \nu, \infty)$*, then the function* $t \to \phi_t$ *belongs to* $PAP(\mathcal{B}; X, \mu, \nu, \infty)$.

Proof. Assume that $\phi = g + h$ where $g \in AP(\mathbb{R}; X)$ and $h \in \mathcal{E}(\mathbb{R}; X, \mu, \nu, \infty)$. Then we can see that, $\phi_t = g_t + h_t$ and g_t is almost periodic. On the other hand, we have

$$\frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \Big(\sup_{\theta \in]-\infty,t]} \Big[\sup_{\xi \in]-\infty,0]} |h(\theta+\xi)| \Big] \Big) \, d\mu(t) \leqslant \\ \leqslant \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \Big(\sup_{\theta \in]-\infty,t]} |h(\theta)| \Big) \, d\mu(t),$$

which shows that ϕ_t belongs to $PAP(\mathcal{B}, \mu, \nu, \infty)$. Thus, we obtain the desired result. □

5. (μ, ν)-Pseudo almost periodic solutions of infinite class

In what follows, we will be looking at the existence of bounded integral solutions of infinite class of equation (1).

Theorem 33. *[3] Assume that* $\mathcal{B}$ *satisfies* ($\boldsymbol{A_1}$), ($\boldsymbol{A_2}$), ($\boldsymbol{B}$), ($\boldsymbol{C_1}$), ($\boldsymbol{C_2}$) *and* ($\boldsymbol{H_1}$)*, holds. If* $f \in BC(\mathbb{R}; X)$*, then there exists a unique bounded solution* u *of equation* (1) *on* $\mathbb{R}$*, given by*

$$u_t = \lim_{\lambda \to +\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\, \Pi^s \big(\widetilde{B}_\lambda\, X_0\, f(s)\big)\, ds + \\ + \lim_{\lambda \to +\infty} \int_{+\infty}^{t} \mathcal{U}^u(t-s)\, \Pi^u \big(\widetilde{B}_\lambda\, X_0\, f(s)\big)\, ds \qquad \textit{for} \;\; t \in \mathbb{R},$$

where Π^s *and* Π^u *are the projections of* $\mathcal{B}_A$ *onto the stable and unstable subspaces, respectively.*

Proposition 34. *[12] Let $h \in AP(\mathbb{R};X)$ and Γ be the mapping defined for $t \in \mathbb{R}$ by*

$$\Gamma h(t) = \Big[\lim_{\lambda\to+\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\,\Pi^s\big(\widetilde{B}_\lambda\, X_0\, h(s)\big)\, ds + \\ + \lim_{\lambda\to+\infty} \int_{+\infty}^{t} \mathcal{U}^u(t-s)\,\Pi^u\big(\widetilde{B}_\lambda\, X_0\, h(s)\big)\, ds\Big](0).$$

Then $\Gamma h \in AP(\mathbb{R}, X)$.

Theorem 35. *Let $\mu,\nu \in \mathcal{M}$ satisfy ($\boldsymbol{H_4}$) and $g \in \mathcal{E}(\mathbb{R};X,\mu,\nu,\infty)$. Then $\Gamma g \in \mathcal{E}(\mathbb{R};X,\mu,\nu,\infty)$.*

Proof. In fact, for $\tau > 0$ we get

$$\begin{aligned}
&\int_{-\tau}^{\tau}\Big(\sup_{\theta\in\,]-\infty,t]} |\Gamma h(\theta)|\, ds\Big) d\mu(t) \leqslant \\
&\qquad \leqslant \overline{M}\,\widetilde{M} \int_{-\tau}^{\tau}\Big(\sup_{\theta\in\,]-\infty,t]} \int_{-\infty}^{\theta} e^{-\omega(\theta-s)}\,|\Pi^s|\,|g(s)|\, ds\Big) d\mu(t) + \\
&\qquad + \overline{M}\,\widetilde{M} \int_{-\tau}^{\tau}\Big(\sup_{\theta\in\,]-\infty,t]} \int_{\theta}^{+\infty} e^{\omega(\theta-s)}\,|\Pi^u|\,|g(s)|\, ds\Big) d\mu(t) \leqslant \\
&\quad \leqslant \overline{M}\,\widetilde{M}\,|\Pi^s| \int_{-\tau}^{\tau} \sup_{s\in\,]-\infty,t]} |g(s)| \Big(\sup_{\theta\in\,]-\infty,t]} \int_{-\infty}^{\theta} e^{-\omega(\theta-s)}\, ds\Big) d\mu(t) + \\
&\quad + \overline{M}\,\widetilde{M}\,|\Pi^u| \int_{-\tau}^{\tau} \sup_{s\in\,]-\infty,-\theta]} |g(s)| \Big(\sup_{\theta\in\,]-\infty,t]} \int_{\theta}^{+\infty} e^{\omega(\theta-s)}\, ds\Big) d\mu(t) \leqslant \\
&\qquad\qquad \leqslant \frac{\overline{M}\,\widetilde{M}\,|\Pi^s| + \overline{M}\,\widetilde{M}\,|\Pi^u|}{\omega} \int_{-\tau}^{\tau}\Big(\sup_{s\in\,]-\infty,t]} |g(s)|\Big) d\mu(t).
\end{aligned}$$

Consequently

$$\begin{aligned}
&\frac{1}{\nu[-\tau,\tau]} \int_{-\tau}^{\tau}\Big(\sup_{\theta\in\,]-\infty,t]} (\Gamma g)(\theta)\Big) d\mu(t) \leqslant \\
&\qquad \leqslant \frac{\overline{M}\,\widetilde{M}\,|\Pi^s| + \overline{M}\,\widetilde{M}\,|\Pi^u|}{\omega} \Big(\frac{1}{\nu[-\tau,\tau]} \int_{-\tau}^{\tau}\Big(\sup_{s\in\,]-\infty,t]} |g(s)|\Big) d\mu(t)\Big),
\end{aligned}$$

which converges to zero as $\tau \to +\infty$. Thus, we obtain the desired result. $\square$

For the existence of (μ,ν)-pseudo almost periodic solution of infinite class, we make the following assumption.

$(\mathbf{H_6})$ $f:\mathbb{R}\to X$ is in $cl(\mu,\nu)$-pseudo almost periodic of infinite class.

Proposition 36. *Assume that $\mathcal{B}$ satisfies $(\boldsymbol{A_1})$, $(\boldsymbol{A_2})$, $(\boldsymbol{B})$, $(\boldsymbol{C_1})$, $(\boldsymbol{C_2})$ and $(\boldsymbol{H_0})$, $(\boldsymbol{H_1})$, $(\boldsymbol{H_4})$ and $(\boldsymbol{H_6})$ hold. Then equation* (1) *has a unique $cl(\mu,\nu)$-pseudo almost periodic solution of infinite class.*

Proof. Since f is a (μ,ν)-pseudo almost periodic function, f has a decomposition $f=f_1+f_2$, where $f_1\in AP(\mathbb{R};X)$ and $f_2\in\mathcal{E}(\mathbb{R};X,\mu,\nu,\infty)$. Using Proposition 33, Proposition 34 and Theorem 35, we get the desired result. $\square$

Our next objective is to show the existence of (μ,ν)-pseudo almost periodic solutions of infinite class for the following problem

$$u'(t)=A\,u(t)+L(u_t)+f(t,u_t)\qquad\text{for}\quad t\in\mathbb{R},\tag{9}$$

where $f:\mathbb{R}\times\mathcal{B}\to X$ is continuous.

For the sequel, we make the following assumption.

$(\mathbf{H_7})$ Let $\mu,\nu\in\mathcal{M}$ and $f:\mathbb{R}\times\mathcal{B}\to X$ $cl(\mu,\nu)$-pseudo almost periodic of infinite class such that there exists a continuous function $L_f:\mathbb{R}\to[0,+\infty[$ such that

$$\big|f(t,\varphi_1)-f(t,\varphi_2)\big|\leqslant L_f(t)|\varphi_1-\varphi_2|_{\mathcal{B}}\qquad\text{for all}\quad t\in\mathbb{R}\quad\text{and}\quad\varphi_1,\varphi_2\in\mathcal{B}$$

and L_f satisfies inequality (6).

Theorem 37. *Assume that $\mathcal{B}$ is a uniform fading memory space and $(\boldsymbol{A_1})$, $(\boldsymbol{A_2})$, $(\boldsymbol{C_1})$, $(\boldsymbol{C_2})$, $(\boldsymbol{H_0})$, $(\boldsymbol{H_1})$, $(\boldsymbol{H_2})$, $(\boldsymbol{H_3})$, $(\boldsymbol{H_5})$ and $(\boldsymbol{H_7})$ hold. If*

$$\overline{M}\,\widetilde{M}\,C\sup_{t\in\mathbb{R}}\Big(|\Pi^s|\int_{-\infty}^{t}e^{-\omega(t-s)}\,L_f(s)\,ds+|\Pi^u|\int_{t}^{+\infty}e^{\omega(t-s)}\,L_f(s)\,ds\Big)<\frac{1}{2},$$

where $C=\max\big\{\sup_{t\in\mathbb{R}}|M(t)|,\ \sup_{t\in\mathbb{R}}|K(t)|\big\}$, then equation (9) *has a unique $cl(\mu,\nu)$-pseudo almost periodic solution of infinite class.*

Proof. Let x be a function in $PAP(\mathbb{R};X,\mu,\nu,\infty)$, from Theorem 32 the function $t\to x_t$ belongs to $PAP(\mathcal{B},\mu,\infty)$. Hence Theorem 31 implies that the function

$g(\cdot) := f(\cdot, x_\cdot)$ is in $PAP(\mathbb{R}; X, \mu, \infty)$. Consider the mapping

$$\mathcal{H} : PAP(\mathbb{R}; X, \mu, \nu, \infty) \to PAP(\mathbb{R}; X, \mu, \nu, \infty)$$

defined for $t \in \mathbb{R}$ by

$$(\mathcal{H}x)(t) = \Big[\lim_{\lambda \to +\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\, \Pi^s \big(\widetilde{B}_\lambda\, X_0\, f(s, x_s)\big)\, ds + \\ + \lim_{\lambda \to +\infty} \int_{+\infty}^{t} \mathcal{U}^u(t-s)\, \Pi^u \big(\widetilde{B}_\lambda\, X_0\, f(s, x_s)\big)\, ds \Big](0).$$

From Proposition 33, Proposition 34 and taking into account Theorem 35, it suffices now to show that the operator $\mathcal{H}$ has a unique fixed point in $PAP(\mathbb{R}; X, \mu, \nu, \infty)$. Since $\mathcal{B}$ is a uniform fading memory space, by the Lemma 9, we can choose the function K constant and the function M such that $M(t) \to 0$ as $t \to +\infty$. Let $C = \max\big\{ \sup_{t\in\mathbb{R}} |M(t)|,\ \sup_{t\in\mathbb{R}} |K(t)| \big\}$ and $x_1, x_2 \in PAP(\mathbb{R}; X, \mu, \nu, \infty)$, then we have

$$\begin{aligned}
&\big|\mathcal{H}x_1(t) - \mathcal{H}x_2(t)\big| \leqslant \\
&\leqslant \Big| \lim_{\lambda \to +\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\, \Pi^s \big(\widetilde{B}_\lambda\, X_0 \big[f(s, x_{1s}) - f(s, x_{1s})\big]\big)\, ds \Big| + \\
&+ \Big| \lim_{\lambda \to +\infty} \int_{+\infty}^{t} \mathcal{U}^s(t-s)\, \Pi^u \big(\widetilde{B}_\lambda\, X_0\, \big[f(s, x_{2s}) - f(s, x_{2s})\big]\big)\, ds \Big| \leqslant \\
&\leqslant \overline{M}\, \widetilde{M} \Big(|\Pi^s| \int_{-\infty}^{t} e^{-\omega(t-s)}\, L_f(s)\, |x_{1s} - x_{2s}|_{\mathcal{B}}\, ds + \\
&+ |\Pi^u| \int_{t}^{+\infty} e^{\omega(t-s)}\, L_f(s)\, |x_{1s} - x_{2s}|_{\mathcal{B}}\, ds \Big) \leqslant \\
&\leqslant \overline{M}\, \widetilde{M} \Big[|\Pi^s| \int_{-\infty}^{t} e^{-\omega(t-s)}\, L_f(s) \Big(K(s) \sup_{0 \leqslant \xi \leqslant s} |x_1(\xi) - x_2(\xi)| + \\
&+ M(s)\, |x_{1_0} - x_{2_0}|_{\mathcal{B}} \Big) ds + \\
&+ |\Pi^u| \int_{t}^{+\infty} e^{\omega(t-s)}\, L_f(s) \Big(K(s) \sup_{0 \leqslant \xi \leqslant s} |x_1(\xi) - x_2(\xi)| + M(s)\, |x_{1_0} - x_{2_0}|_{\mathcal{B}} \Big) ds \Big],
\end{aligned}$$

which implies

$$\left|\mathcal{H}x_1(t) - \mathcal{H}x_2(t)\right| \leqslant 2\,\overline{M}\,\widetilde{M}\,C \sup_{t\in\mathbb{R}} \Big(|\Pi^s| \int_{-\infty}^{t} e^{-\omega(t-s)}\, L_f(s)\, ds + \\ + |\Pi^u| \int_{t}^{+\infty} e^{\omega(t-s)}\, L_f(s)\, ds\Big)\, |x_1 - x_2|.$$

This means that $\mathcal{H}$ is a strict contraction. Thus by Banach's fixed point theorem, $\mathcal{H}$ has a unique fixed point u in $PAP(\mathbb{R}; X, \mu, \nu, \infty)$. We conclude that equation (9), has one and only one $cl(\mu,\nu)$-pseudo almost periodic solution of infinite class. □

Proposition 38. *Assume that $\mathcal{B}$ is a uniform fading memory space and ($\boldsymbol{A_1}$), ($\boldsymbol{A_2}$), ($\boldsymbol{C_1}$), ($\boldsymbol{C_2}$), ($\boldsymbol{H_0}$), ($\boldsymbol{H_1}$), ($\boldsymbol{H_2}$), ($\boldsymbol{H_3}$) and, ($\boldsymbol{H_5}$) and f is lipschitz continuous with respect the second argument. If*

$$Lip(f) < \frac{\omega}{2\,\overline{M}\,\widetilde{M}\,C\,(|\Pi^s| + |\Pi^u|)}$$

then equation (9) *has a unique $cl(\mu,\nu)$-pseudo almost periodic solution of infinite class, where $Lip(f)$ is the lipschitz constant of f.*

Proof. Let us pose $k = Lip(f)$, we have

$$\left|\mathcal{H}x_1(t) - \mathcal{H}x_2(t)\right| \leqslant \\ \leqslant 2\,\overline{M}\,\widetilde{M}\,C \sup_{t\in\mathbb{R}} \Big(|\Pi^s| \int_{-\infty}^{t} e^{-\omega(t-s)}\, k\, ds + |\Pi^u| \int_{t}^{+\infty} e^{\omega(t-s)}\, k\, ds\Big)\, |x_1 - x_2| \leqslant \\ \leqslant \frac{2\,k\,\overline{M}\,\widetilde{M}\,C\,(|\Pi^s| + |\Pi^u|)}{\omega}\, |x_1 - x_2|.$$

Consequently $\mathcal{H}$ is a strict contraction if $k < \dfrac{\omega}{2\,\overline{M}\,\widetilde{M}\,C\,(|\Pi^s| + |\Pi^u|)}$. □

6. Application

For illustration, we propose to study the existence of solutions for the following model

$$\frac{\partial}{\partial t}z(t,x) = \frac{\partial^2}{\partial x^2}z(t,x) + \int_{-\infty}^{0} G(\theta)\, z(t+\theta,x)\, d\theta + \big(\sin t + \sin(\sqrt{2}t)\big) + \\ + \arctan(t) + \int_{-\infty}^{0} h\big(\theta, z(t+\theta,x)\big)\, d\theta \qquad \text{for } \ t\in\mathbb{R} \ \text{ and } \ x\in[0,\pi] \quad (10)$$

with conditions

$$z(t,0) = z(t,\pi) = 0 \qquad \text{for } \ t\in\mathbb{R}, \quad (11)$$

where $G\ :]-\infty,0] \to \mathbb{R}$ is a continuous function and $h\ :]-\infty,0]\times\mathbb{R}\to\mathbb{R}$ is continuous and lipschitzian with respect to the second argument. To rewrite equation (10) in the abstract form, we introduce the space $X = C_0([0,\pi];\mathbb{R})$ of continuous function from $[0,\pi]$ to $\mathbb{R}^+$ equipped with the uniform norm topology. Let $A : D(A) \to X$ be defined by

$$\begin{cases} D(A) = \big\{y \in X \cap C^2([0,\pi],\mathbb{R}) \colon y'' \in X\big\} \\ A\,y = y''. \end{cases}$$

Then A satisfied the Hille-Yosida condition in X. Moreover the part A_0 of A in $\overline{D(A)}$ is the generator of strongly continuous compact semigroup $(T_0(t))_{t\geqslant 0}$ on $\overline{D(A)}$. It follows that $(\mathbf{H_0})$ and $(\mathbf{H_1})$ are satisfied.

The phase space $\mathcal{B} = C_\gamma$, $\gamma > 0$ where

$$C_\gamma = \big\{\varphi \in C(]-\infty,0];X) \colon \lim_{\theta\to-\infty} e^{\gamma\theta}\,\varphi(\theta) \ \text{ exist in } \ X\big\}$$

with the the following norm

$$|\varphi|_\gamma = \sup_{\theta\leqslant 0} |e^{\gamma\theta}\,\varphi(\theta)|.$$

This space is a uniform fading memory space, that is $(\mathbf{H_2})$, and it satisfies $(\mathbf{C_1})$, $(\mathbf{C_2})$.

We define $f : \mathbb{R} \times C \to X$ and $L : C \to X$ as follows

$$f(t,\varphi)(x) = \sin t + \sin(\sqrt{2}t) + \arctan(t) + \\ + \int_{-\infty}^{0} h\big(\theta, \varphi(\theta)(x)\big)\, d\theta \qquad \text{for } \ x \in [0,\pi] \ \text{ and } \ t \in \mathbb{R},$$

$$L(\varphi)(x) = \int_{-\infty}^{0} G(\theta)\, \varphi(\theta)(x)\, d\theta \qquad \text{for } \ \theta \in]-\infty, 0] \ \text{ and } \ x \in [0,\pi].$$

Let us pose $v(t) = z(t,x)$. Then equation (10) takes the following abstract form

$$v'(t) = A\, v(t) + L(v_t) + f(t, v_t) \qquad \text{for } \ t \in \mathbb{R}. \tag{12}$$

Consider the measures μ and ν where its Radon-Nikodym derivative are respectively $\rho_1, \rho_2 : \mathbb{R} \to \mathbb{R}$ defined by

$$\rho_1(t) = \begin{cases} 1 & \text{for } \ t > 0, \\ e^t & \text{for } \ t \leqslant 0, \end{cases}$$

and

$$\rho_2(t) = |t| \qquad \text{for } \ t \in \mathbb{R},$$

i.e. $d\mu(t) = \rho_1(t)\, dt$ and $d\nu(t) = \rho_2(t)\, dt$ where dt denotes the Lebesgue measure on $\mathbb{R}$ and

$$\mu(A) = \int_A \rho_1(t)\, dt \qquad \text{for } \ \nu(A) = \int_A \rho_2(t)\, dt \qquad \text{for } \ A \in \mathcal{B}.$$

From [6] $\mu, \nu \in \mathcal{M}$, μ, ν satisfy Hypothesis $(\mathbf{H_5})$ and $\sin t + \sin(\sqrt{2}\, t) + \dfrac{\pi}{2}$ is almost periodic.

We have

$$\limsup_{\tau \to +\infty} \frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} = \limsup_{\tau \to +\infty} \frac{\displaystyle\int_{-\tau}^{0} e^t\, dt + \int_0^{\tau} dt}{\displaystyle 2\int_0^{\tau} t\, dt} = \limsup_{\tau \to +\infty} \frac{1 - e^{-\tau} + \tau}{\tau^2} = 0 < \infty,$$

which implies that $(\mathbf{H_3})$ is satisfied.

Since for all $\theta \in \mathbb{R}$, $\dfrac{-\pi}{2} < \arctan\theta < \dfrac{\pi}{2}$, then we have

$$\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in\,]-\infty,0]}|\arctan(\theta)|\,d\mu(t) \leqslant \frac{\pi}{2}\times\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}d\mu(t) \leqslant$$
$$\leqslant \frac{\pi}{2}\times\frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} \to 0 \qquad \text{as } \ \tau\to+\infty.$$

It follows that $t \mapsto \arctan t$ is (μ,ν)-ergodic of infinite class consequently, f is uniformly (μ,ν)-pseudo almost periodic of infinite class. Moreover, L is a bounded linear operator from $\mathcal{B}$ to X.

In fact for $\varphi \in C_\gamma$, we have $\varphi \in C(]-\infty,0];X)$ and $\lim\limits_{\theta\to-\infty} e^{\gamma\theta}\,\varphi(\theta) = x_0$ exist in X, then there exists $M \geqslant 0$ such that $|e^{\gamma\theta}\varphi(\theta)| \leqslant M$ for all $\theta \in]-\infty,0]$. We have for $x \in X$

$$|L(\varphi)(x)| \leqslant \int_{-\infty}^{0}\big|G(\theta)\,\varphi(\theta)(x)\big|\,d\theta \leqslant$$
$$\leqslant \int_{-\infty}^{0}\big|e^{(\gamma+1)\theta}\,e^{-\gamma\theta}\,e^{\gamma\theta}\,\varphi(\theta)(x)\big|\,d\theta \leqslant M\int_{-\infty}^{0}e^{\theta}\,d\theta < \infty$$

and

$$|L(\varphi)(x)| \leqslant \int_{-\infty}^{0}\big|G(\theta)\,\varphi(\theta)(x)\big|\,d\theta \leqslant$$
$$\leqslant \int_{-\infty}^{0}\big|e^{(\gamma+1)\theta}\,e^{-\gamma\theta}\,e^{\gamma\theta}\,\varphi(\theta)(x)\big|\,d\theta \leqslant \Big(\int_{-\infty}^{0}e^{\theta}\,d\theta\Big)\,|\varphi|_\gamma,$$

which implies that L is well defined and L is a bounded linear operator from $\mathcal{B}$ to X. We suppose that there exists a function $k_1(\cdot) \in L^1(]-\infty,0];\mathbb{R}^+)$ such that

$$\big|h(\theta,x_1) - h(\theta,x_2)\big| \leqslant k(\theta)\,|x_1 - x_2| \qquad \text{for } \ \theta \leqslant 0 \ \text{ and } \ x_1,x_2 \in \mathbb{R}, \tag{13}$$
$$h(\theta,0) = 0. \tag{14}$$

For example, we can take $h(\theta,x) = e^{-\theta^2}\sin\left(\frac{x}{2}\right)$ for $(\theta,x) \in]-\infty,0]\times\mathbb{R}$ and $k_1(\theta) = e^{-\theta^2}$. We can see that $h(\theta,0) = 0$ and $|h(\theta,x_1) - h(\theta,x_2)| \leqslant \frac{1}{2}\,|x_1 - x_2|$. Assumptions (13) and (14) imply that $f(\varphi) \in X$. In fact, $\varphi \in \mathcal{B}$, then

$$f(\varphi)(x) = \int_{-\infty}^{0} h\big(\theta,\varphi(\theta)(x)\big)\,d\theta \qquad \text{for } \ x \in [0,\pi]$$

and

$$|f(\varphi)(x)| \leqslant \int_{-\infty}^{0} k_1(\theta) \left|\psi(\theta)(x)\right| d\theta.$$

Consequently

$$|f(\psi)| \leqslant \Big(\int_{-\infty}^{0} k_1(\theta)\, d\theta\Big)\, |\psi|_{\mathcal{B}}.$$

Moreover assumption (14) implies that

$$f(\psi)(0) = f(\psi)(\pi) = 0.$$

Using the dominated convergence theorem, one can show that $f(\varphi)$ is a continuous function on $[0, \pi]$. Moreover, for every $\varphi_1, \varphi_2 \in \mathcal{B}$, we have

$$\begin{aligned}
\big|f(t,\varphi_1) - f(t,\varphi_2)\big| &= \sup_{0\leqslant x\leqslant\pi} \big|f(\varphi_1)(x) - f(\varphi_2)(x)\big| \leqslant \\
&\leqslant \sup_{0\leqslant x\leqslant\pi} \int_{-\infty}^{0} \big|h\big(\theta, \varphi_1(\theta)(x)\big) - h\big(\theta, \varphi_2(\theta)(x)\big)\big|\, d\theta \leqslant \\
&\leqslant \sup_{0\leqslant x\leqslant\pi} \int_{-\infty}^{0} k_1(\theta)\big|\varphi_1(\theta)(x) - \varphi_2(\theta)(x)\big|\, d\theta \leqslant \\
&\leqslant \Big(\int_{-\infty}^{0} k_1(\theta)\, d\theta\Big) \sup_{\substack{-\infty<\theta\leqslant 0\\ 0\leqslant x\leqslant\pi}} \big|\varphi_1(\theta)(x) - \varphi_2(\theta)(x)\big|.
\end{aligned}$$

Consequently, we conclude that f is Lipschitz continuous and $cl(\mu,\nu)$-pseudo almost periodic of infinite class. Then by Proposition 38 we deduce the following result.

Theorem 39. *Under the above assumptions, equation* (12) *has a unique* $cl(\mu,\nu)$*-pseudo almost periodic solution* v *of infinite class.*

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