Opuscula Math. 38, no. 5 (2018), 681–698
https://doi.org/10.7494/OpMath.2018.38.5.681 Opuscula Mathematica

KREIN–VON NEUMANN EXTENSION OF AN EVEN ORDER DIFFERENTIAL OPERATOR ON A FINITE INTERVAL

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Communicated by A.A. Shkalikov

Abstract. We describe the Krein–von Neumann extension of minimal operator associated with the expression $\mathcal{A} := (-1)^n \frac{d^{2n}}{dx^{2n}}$ on a finite interval (a, b) in terms of boundary conditions. All non-negative extensions of the operator A as well as extensions with a finite number of negative squares are described.

Keywords: non-negative extension, Friedrichs' extension, Krein–von Neumann extension, boundary triple, Weyl function.

Mathematics Subject Classification: 47A05.

1. INTRODUCTION

Let A be a semi-bounded symmetric operator in a separable Hilbert space $\mathfrak{H}$. It is well known that the operator A admits self-adjoint extensions preserving the lower bound (see [1, Chapter VIII] and [29, I]). According to the classical Krein's result [29, I], in the set $\mathrm{Ext}_A(0,\infty)$ of all non-negative self-adjoint extensions of the operator A, there exist two "extreme" extensions $\widehat{A}_F$ and $\widehat{A}_K$ uniquely determined by the following inequalities:

$$\left(\widehat{A}_F + x\right)^{-1} \le \left(\widetilde{A} + x\right)^{-1} \le \left(\widehat{A}_K + x\right)^{-1}, \quad x \in (0,\infty), \quad \widetilde{A} \in \mathrm{Ext}_A(0,\infty). \tag{1.1}$$

The extension $\widehat{A}_F$ is called Friedrichs' (or a hard), and the extension $\widehat{A}_K$ is called Krein–von Neumann (or a soft), see [29, I]. In the case of positively definite operator $A > \varepsilon I > 0$, M.G. Krein showed (see [29, I]) that

$$\widehat{A}_K = A^* \upharpoonright (\mathrm{dom}\, A \dotplus \ker A^*)\,. \tag{1.2}$$

In the case of positively definite operator A, extensions of the form (1.2) were first introduced and investigated by J. von Neumann in his seminal paper [33]. However, he has not discovered its extreme property described by (1.1).

In the case of non-negative operator $A \geq 0$, with zero lower bound, the extensions $\widehat{A}_F$ and $\widehat{A}_K$ were first described in [12] in terms of abstract boundary conditions. Namely, it was shown that

$$\begin{aligned} \operatorname{dom} \widehat{A}_K &= \{f \in \operatorname{dom} A^* : \Gamma_1 f = M(0)\Gamma_0 f\}, \\ \operatorname{dom} \widehat{A}_F &= \{f \in \operatorname{dom} A^* : \Gamma_1 f = M(-\infty)\Gamma_0 f\}, \end{aligned} \tag{1.3}$$

where $M(0) = M(0-)$ is a limit value of the Weyl function at zero, and $M(-\infty)$ is a limit value of the Weyl function at infinity (see Definition 2.3).

Description of the Friedrichs extension independent of (1.3) is known in many cases. For instance, M.G. Krein showed that for ordinary differential operators on a finite interval extension $\widehat{A}_F$ is generated by the Dirichlet problem ([29, II]).

H. Kalf in [27] investigated the general three-term Sturm-Liouville differential expression

$$\tau u = \frac{1}{k}\left[-(pu')' + qu\right] \tag{1.4}$$

on an interval $(0, \infty)$ under the following assumptions on coefficients:

(i) $k, p > 0$ a.e. on $(0, \infty)$; $k, 1/p \in L^1_{\mathrm{loc}}(0, \infty)$; $q \in L^1_{\mathrm{loc}}(0, \infty)$ is real-valued;
(ii) There exists a number $\mu \in \mathbb{R}$ and functions $g_0, g_\infty \in AC_{\mathrm{loc}}(0, \infty)$ with $pg_0', pg_\infty' \in AC_{\mathrm{loc}}(0, \infty)$ and $g_0 > 0$ near 0, $g_\infty > 0$ near ∞ such that

$$\int_0 \frac{1}{pg_0^2} = \int^\infty \frac{1}{pg_\infty^2} = \infty \tag{1.5}$$

and

$$q \geq \frac{(pg_0')'}{g_0} - \mu k \quad \text{near } 0, \quad q \geq \frac{(pg_\infty')'}{g_\infty} - \mu k \quad \text{near } \infty. \tag{1.6}$$

The main result of the paper [27] is the following description of Friedrichs' extension $\widehat{T}_F$ of the minimal operator $T_{\min}$ associated with (1.4):

$$\operatorname{dom} \widehat{T}_F = \left\{ u \in \operatorname{dom} T_{\max} : \int_0 pg_0^2 \left| \left(\frac{u}{g_0}\right)' \right|^2 < \infty, \int^\infty pg_\infty^2 \left| \left(\frac{u}{g_\infty}\right)' \right|^2 < \infty \right\}. \tag{1.7}$$

For more information see [27, Theorem 1] and related remarks.

This result has been extended in [16] to the case of singular differential operators on arbitrary intervals $(a, b) \subseteq \mathbb{R}$ associated with four-term general differential expressions of the type

$$\tau u = \frac{1}{k}\left(-(u^{[1]})' + su^{[1]} + qu\right),$$

where

$$u^{[1]} := p[u' + su],$$

and the coefficients p, q, k, s, are real-valued and Lebesgue measurable on (a,b), with $p \neq 0$, $k > 0$ a.e. on (a,b), and p^{-1}, q, k, $s \in L^1_{\mathrm{loc}}((a,b);dx)$, and u is supposed to satisfy

$$u \in AC_{\mathrm{loc}}(a,b), \quad u^{[1]} \in AC_{\mathrm{loc}}(a,b).$$

In particular, this setup implies that τ permits a distributional potential coefficient, including potentials in $H^{-1}_{\mathrm{loc}}(a,b)$.

Imposing additional to (1.5)–(1.6) assumptions on coefficients, the authors characterize the Friedrichs extension of $T_{\min}$ by the same conditions (1.7). For more details see [16, Theorems 11.17 and 11.19].

In [16] it is also described the Krein–von Neumann extension of $T_{\min}$ on a finite interval (a,b) in the special case where τ is regular (i.e. p^{-1}, q, k and s are integrable near a and b). A description is given as follows:

$$\operatorname{dom} \widehat{T}_K = \left\{ g \in \operatorname{dom} T_{\max} : \quad \begin{pmatrix} g(b) \\ g^{[1]}(b) \end{pmatrix} = R_K \begin{pmatrix} g(a) \\ g^{[1]}(a) \end{pmatrix} \right\},$$

where

$$R_K = \frac{1}{u_1^{[1]}(a)} \begin{pmatrix} -u_2^{[1]}(a) & 1 \\ u_1^{[1]}(a)u_2^{[1]}(b) - u_1^{[1]}(b)u_2^{[1]}(a) & u_1^{[1]}(b) \end{pmatrix},$$

and $u_j(\cdot), j \in \{1,2\}$, are positive solutions of $\tau u = 0$ determined by the conditions

$$\begin{aligned} u_1(a) &= 0, \quad u_1(b) = 1, \\ u_2(a) &= 1, \quad u_2(b) = 0. \end{aligned}$$

For more details see [16, Theorem 12.3].

Several papers (see [16, 17, 19, 20, 27, 28] and the references therein) are devoted to the spectral analysis of boundary value problems for the one-parametric Bessel's differential expression

$$\tau_\nu = -\frac{d^2}{dx^2} + \frac{\nu^2 - \frac{1}{4}}{x^2}, \quad \nu \in [0,1) \setminus \{1/2\}. \tag{1.8}$$

We especially note the papers of H. Kalf and W. Everitt [17, 27], where the explicit form of the Weyl-Titchmarsh $m-$coefficient of the expression τ_ν in $L^2(\mathbb{R}_+)$ was found.

In [2, 11, 17, 27], there were described domains of the Friedrichs extension for the minimal operator $A_{\nu,\infty}$ associated with expression (1.8) in $L^2(\mathbb{R}_+)$. In [17] the same was done for all self-adjoint extensions of the operator $A_{\nu,\infty}$. The most complete result was obtained in [2]. Namely, $\widehat{A}_{\nu,\infty,F}$ and $\widehat{A}_{\nu,\infty,K}$ are the restrictions of the maximal operator $A^*_{\nu,\infty} = A_{\nu,\infty,\max}$ to the domains

$$\operatorname{dom} \widehat{A}_{\nu,\infty,F} = \left\{ f \in \operatorname{dom} A^*_{\nu,\infty} : [f, x^{\frac{1}{2}+\nu}]_0 = 0 \right\}$$

and

$$\operatorname{dom} \widehat{A}_{\nu,\infty,K} = \begin{cases} \{f \in \operatorname{dom} A^*_{\nu,\infty} : [f, x^{\frac{1}{2}-\nu}]_0 = 0\}, & \nu \in (0,1), \\ \{f \in \operatorname{dom} A^*_{0,\infty} : [f, x^{\frac{1}{2}}]_0 = 0\}, & \nu = 0, \end{cases}$$

respectively, where

$$\operatorname{dom} A^*_{\nu,\infty} = \begin{cases} H^2_0(\mathbb{R}_+)\dot{+}\operatorname{span}\{x^{1/2+\nu}\xi(x), x^{1/2-\nu}\xi(x)\}, & \nu \in (0,1), \\ H^2_0(\mathbb{R}_+) \dot{+} \operatorname{span}\{x^{1/2}\xi(x), x^{1/2}\log(x)\xi(x)\}, & \nu = 0. \end{cases}$$

Here $[f,g]_x := f(x)\overline{g'(x)} - f'(x)\overline{g(x)}$ for all $x \in \mathbb{R}_+$, and $\xi \in C^2_0(\mathbb{R}_+)$ is a function such that $\xi(x) = 1$ whenever $x \in [0,1]$. For more details see [2, Proposition 5.7 and Remark 5.8].

Friedrichs' and Krein–von Neumann extensions $\widehat{A}_{\nu,b,F}$ and $\widehat{A}_{\nu,b,K}$ of the minimal operator corresponding to (1.8) on a finite interval $(0,b)$ were also described there (see [2, Proposition 4.5]).

M.G. Krein ([29, II]) investigated certain extensions of the minimal operator $T_{\min}$ associated in $L^2(a,b)$ with the following quasi-derivative expression

$$Tf := f^{[2n]}. \tag{1.9}$$

Here

$$f^{[k]}(x) = f^{(k)}(x), \quad k \in \{0,\ldots,n-1\}, \quad f^{[n]}(x) = p_0(x)f^{(n)}(x),$$
$$f^{[n+k]}(x) = p_k(x)f^{(n-k)}(x) - \frac{d}{dx}f^{[n+k-1]}(x), \quad k \in \{1,\ldots,n\}.$$

In the case of sufficiently smooth coefficients $p_k, k \in \{0,1,\ldots,n\}$, expression (1.9) can be written in the Jacobi–Bertrand form:

$$f^{[2n]} = \sum_{k=0}^{n}(-1)^k \frac{d^k}{dx^k}\left(p_{n-k}\frac{d^k f}{dx^k}\right).$$

In [29, II] Friedrichs' extension of the minimal operator $T_{\min}$ corresponds to Dirichlet realization:

$$\operatorname{dom} \widehat{T}_F = \{f \in \operatorname{dom} T_{\max} : f^{[k]}(a) = f^{[k]}(b) = 0,\ k \in \{0,1,\ldots,n-1\}\}.$$

In the paper by A.A. Lunyov [30] the spectral properties of the operator A generated in $L^2(\mathbb{R}_+)$ by the differential expression

$$l := (-1)^n \frac{d^{2n}}{dx^{2n}}$$

are investigated, and the Krein–von Neumann extension of the corresponding minimal operator $A_{\min}$ in terms of boundary conditions is described in the following way:

$$y^{(n)}(0) = y^{(n+1)}(0) = \ldots = y^{(2n-1)}(0) = 0.$$

Using the technique of boundary triples and the corresponding Weyl functions the author found explicit form of the characteristic matrix and the corresponding spectral function for the Friedrichs and Krein–von Neumann extensions of the minimal operator $A_{\min}$ (see [30, Theorems 1 and 2]).

Also it is shown in [32] that if $\{A_j\}_{j=1}^{\infty}$ is a sequence of densely defined closed symmetric and non-negative operators in $\mathfrak{H}_j$, and $\widehat{A}_{j,F}$ and $\widehat{A}_{j,K}$ are the Friedrichs and Krein–von Neumann extensions of A_j, respectively, and $A := \bigoplus_{j=1}^{\infty} A_j$, then

$$\widehat{A}_F = \bigoplus_{j=1}^{\infty} \widehat{A}_{j,F} \quad \text{and} \quad \widehat{A}_K = \bigoplus_{j=1}^{\infty} \widehat{A}_{j,K}.$$

For more details see [32, Corollary 3.10].

In [5] the unitary equivalence of the inverse of the Krein–von Neumann extension (on the orthogonal complement of its kernel) of a densely defined, closed, strictly positive operator, $S \geq \varepsilon I_{\mathcal{H}}$ for some $\varepsilon > 0$ in a Hilbert space $\mathcal{H}$ to an abstract buckling problem operator is proved.

Friedrichs' and Krein–von Neumann extensions for elliptic operators on bounded and unbounded domains are discussed among other problems in several papers. For instance, these questions are treated by M.Sh. Birman [9], G. Grubb [24], and M.M. Malamud [31] (elliptic operators on bounded and unbounded domains with smooth compact boundary), J. Behrndt *et al.* [7, 8] (elliptic operators on Lipschitz domains), F. Gesztesy and M. Mitrea [21] (Laplacian on domains with non-smooth boundary).

In [4] the authors study spectral properties for $\widehat{H}_{K,\Omega}$, the Krein–von Neumann extension of the perturbed Laplacian $-\Delta+V$ defined on $C_0^{\infty}(\Omega)$, where V is measurable, bounded and nonnegative, in a bounded open set $\Omega \subset \mathbb{R}^n$ belonging to a class of nonsmooth domains which contains all convex domains, along with all domains of class $C^{1,r}$, $r > 1/2$.

See also [3, 6, 10, 22, 25, 26] and the references therein, as well as the recent monograph [15].

However, the problem of finding $M(0)$ is nontrivial even in the case of positively definite operator. Its solution is known in some cases – see papers [11, Theorem 1.1], [2, Proposition 4.5 (ii), Proposition 5.7 (ii)], [10, Theorem 1], [30, Theorem 2] mentioned above.

Here we consider the minimal operator $A := A_{\min}$ associated with the differential expression $\mathcal{A} := (-1)^n \frac{d^{2n}}{dx^{2n}}$ on a finite interval (a, b), i.e.

$$\begin{gathered} A = \mathcal{A} \upharpoonright \operatorname{dom} A_{\min}, \\ \operatorname{dom} A_{\min} = \{y \in W^{2n,2}(a,b) : y^{(k)}(a) = y^{(k)}(b) = 0,\ k \in \{0, \dots, 2n-1\}\}. \end{gathered} \tag{1.10}$$

We describe its Krein–von Neumann extension in terms of boundary conditions. In this way we find $M(0)$ for special (natural) boundary triple for A^*. Note that the corresponding boundary operator is expressed by means of blocks of certain auxiliary Toeplitz matrix (see (3.4)). Using the technique of boundary triples and the corresponding Weyl functions developed in [12] (see also [15, Chapter VIII]) we describe all non-negative extensions of $A_{\min}$ as well as extensions with the finite negative spectrum.

2. PRELIMINARIES

Let A be a densely defined closed symmetric operator in a separable Hilbert space $\mathfrak{H}$ with equal deficiency indices $\mathrm{n}_\pm(A) = \dim(\mathfrak{N}_{\pm \mathrm{i}}) \leq \infty$, where $\mathfrak{N}_z := \ker(A^* - z)$ is the defect subspace.

Definition 2.1. A closed extension A' of A is called a *proper one* if $A \subset A' \subset A^*$. The set of all proper extensions of A completed by the (non-proper) extensions A and A^* is denoted by Ext_A.

Assume that operator $A \in \mathcal{C}(\mathfrak{H})$ is non-negative. Then the set $\mathrm{Ext}_A(0, \infty)$ of its non-negative self-adjoint extensions is non-empty (see [1, 18, 28]). Moreover, there is a maximal non-negative extension $\widehat{A}_F$ (also called Friedrichs' or hard extension), and there is a minimal non-negative extension $\widehat{A}_K$ (Krein–von Neumann or soft extension) satisfying (1.1). For details we refer the reader to [1, 23].

Definition 2.2 ([23]). A triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is called a boundary triple for the adjoint operator A^* if $\mathcal{H}$ is an auxiliary Hilbert space and $\Gamma_0, \Gamma_1 : \operatorname{dom} A^* \to \mathcal{H}$ are linear mappings such that the abstract Green identity

$$(A^* f, g)_\mathfrak{H} - (f, A^* g)_\mathfrak{H} = (\Gamma_1 f, \Gamma_0 g)_\mathcal{H} - (\Gamma_0 f, \Gamma_1 g)_\mathcal{H}, \quad f, g \in \operatorname{dom} A^*,$$

holds and the mapping $\Gamma := \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} : \operatorname{dom} A^* \to \mathcal{H} \oplus \mathcal{H}$ is surjective.

First, note that a boundary triple for A^* exists whenever the deficiency indices of A are equal, $\mathrm{n}_+(A) = \mathrm{n}_-(A)$. Moreover, $\mathrm{n}_\pm(A) = \dim \mathcal{H}$ and $\ker \Gamma = \ker \Gamma_0 \cap \ker \Gamma_1 = \operatorname{dom} A$. Note also that Γ is a bounded mapping from $\mathfrak{H}_+ = \operatorname{dom} A^*$ equipped with the graph norm to $\mathcal{H} \oplus \mathcal{H}$.

A boundary triple for A^* is not unique. Moreover, for any self-adjoint extension $\widetilde{A} := \widetilde{A}^*$ of A there exists a boundary triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ such that $\ker \Gamma_0 = \operatorname{dom} \widetilde{A}$.

Definition 2.3 ([12]). Let A be a densely defined closed symmetric operator in $\mathfrak{H}$ with equal deficiency indices, and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for A^*. The operator valued functions $\gamma(\cdot) : \rho(A_0) \to \mathcal{B}(\mathcal{H}, \mathfrak{H})$ and $M(\cdot) : \rho(A_0) \to \mathcal{B}(\mathcal{H})$, $A_0 := A^* \upharpoonright \ker \Gamma_0$ defined by

$$\gamma(z) := \left(\Gamma_0 \upharpoonright \mathfrak{N}_z\right)^{-1} \qquad \text{and} \qquad M(z) := \Gamma_1 \gamma(z), \qquad z \in \rho(A_0),$$

are called the *γ-field* and the *Weyl function*, respectively, corresponding to the boundary triple Π.

Remark 2.4 ([34], [1, Chapter VIII]). In the case of $\mathrm{n}_\pm(A) = m < \infty$, the set of all self-adjoint extensions of the operator A is parametrized as follows:

$$\begin{gathered} \mathrm{Ext}_A \ni \widetilde{A} = \widetilde{A}^* = A_{C,D} = A^* \upharpoonright \ker(D\Gamma_1 - C\Gamma_0), \\ \text{where} \quad CD^* = DC^*, \quad \det(CC^* + DD^*) \neq 0, \quad C, D \in \mathbb{C}^{m \times m}. \end{gathered} \tag{2.1}$$

Definition 2.5. Let T be a self-adjoint operator in $\mathfrak{H}$, and let $E_T(\cdot)$ be its spectral measure. It is said that the operator T has κ negative eigenvalues if

$$\kappa_-(T) := \dim E_T(-\infty, 0) = \kappa.$$

In the following proposition all self-adjoint extensions of an operator $A \geq 0$ with a finite negative spectrum are described.

Proposition 2.6 ([12,14])**.** *Let A be a densely defined non-negative symmetric operator in $\mathfrak{H}$, $\mathrm{n}_\pm(A) = m < \infty$, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for A^* such that $A_0 \geq 0$, and let $A_{C,D}$ be an arbitrary self-adjoint extension of the form* (2.1)*. Let also $M(\cdot)$ be the corresponding Weyl function. Then the following assertions hold.*

(i) *There exist strong resolvent limits*

$$M(0) := s - R - \lim_{x \uparrow 0} M(x), \qquad M(-\infty) := s - R - \lim_{x \downarrow -\infty} M(x).$$

(ii) $\operatorname{dom} A_0 \cap \operatorname{dom} \widehat{A}_K = \operatorname{dom} A$ $(\operatorname{dom} A_0 \cap \operatorname{dom} \widehat{A}_F = \operatorname{dom} A)$ *if and only if* $M(0) \in \mathbb{C}^{m \times m}$ $(M(-\infty) \in \mathbb{C}^{m \times m})$*. Moreover, in this case*

$$\widehat{A}_K = A^* \restriction \ker (\Gamma_1 - M(0)\Gamma_0), \quad \left(\widehat{A}_F = A^* \restriction \ker (\Gamma_1 - M(-\infty)\Gamma_0)\right).$$

(iii) $A_0 = \widehat{A}_F$ $(A_0 = \widehat{A}_K)$ *if and only if*

$$\lim_{x \downarrow -\infty} (M(x)f, f) = -\infty \quad \left(\lim_{x \uparrow 0} (M(x)f, f) = +\infty\right), \quad f \in \mathcal{H} \setminus \{0\}.$$

(iv) *If $A_0 = \widehat{A}_F$, then the following identity holds:*

$$\kappa_-(A_{C,D}) = \kappa_-(CD^* - DM(0)D^*).$$

In particular, $A_{C,D} \geq 0$ if and only if $CD^* - DM(0)D^* \geq 0$.

(v) *The extension $A_B = A^* \restriction \ker(\Gamma_1 - B\Gamma_0)$ is symmetric* (*self-adjoint*) *if and only if B is symmetric* (*self-adjoint*).

Theorem 2.7 ([29, I, Theorem 14])**.** *Let A be a symmetric positively definite operator. Then* $\operatorname{dom} \widehat{A}_K = \operatorname{dom} A \dotplus \mathfrak{N}_0$*, and*

$$\widehat{A}_K(f + f_0) = Af \quad \text{for any} \quad f \in \operatorname{dom} A, \ f_0 \in \mathfrak{N}_0.$$

3. MAIN RESULT

Let $A := A_{\min}$ be the minimal operator generated in $\mathfrak{H} = L^2(a,b), -\infty < a < b < \infty$ by the differential expression (1.10). In view of [13], the boundary triple for $A^* := A_{\max}$ can be taken as

$$\mathcal{H} = \mathbb{C}^{2n}, \quad \Gamma_0 f = \begin{pmatrix} f(a) \\ \vdots \\ f^{(n-1)}(a) \\ f(b) \\ \vdots \\ f^{(n-1)}(b) \end{pmatrix}, \quad \Gamma_1 f = \begin{pmatrix} (-1)^{n-1} f^{(2n-1)}(a) \\ \vdots \\ f^{(n)}(a) \\ (-1)^n f^{(2n-1)}(b) \\ \vdots \\ -f^{(n)}(b) \end{pmatrix}. \tag{3.1}$$

The main result of this paper is presented by the following theorem.

Theorem 3.1. *Let A be the minimal operator defined by* (1.10). *Let also $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be the boundary triple for A^* defined by relations* (3.1). *Then the following assertions hold.*

(i) *The domain of Krein–von Neumann extension $\widehat{A}_K$ is of the form*

$$\operatorname{dom} \widehat{A}_K = \left\{ f \in W^{2n,2}(a,b) : \begin{pmatrix} f^{(2n-1)}(b) \\ \vdots \\ f(b) \end{pmatrix} = T \begin{pmatrix} f^{(2n-1)}(a) \\ \vdots \\ f(a) \end{pmatrix} \right\}, \tag{3.2}$$

where T is the Toeplitz lower-triangular $2n \times 2n$ matrix of the form

$$T = \begin{pmatrix} 1 & \dots & & & \mathbf{0} \\ b-a & 1 & \dots & & \\ \dots & \dots & \dots & \dots & \dots \\ \frac{(b-a)^{2n-1}}{(2n-1)!} & \frac{(b-a)^{2n-2}}{(2n-2)!} & \dots & b-a & 1 \end{pmatrix}.$$

(ii) *Krein–von Neumann extension $\widehat{A}_K$ is given by*

$$\operatorname{dom} \widehat{A}_K = \left\{ f \in W^{2n,2}(a,b) : \Gamma_1 f = B_K \Gamma_0 f \right\}, \tag{3.3}$$

where

$$B_K = \begin{pmatrix} QT_2^{-1}T_1 S & -QT_2^{-1}S \\ -QT_1T_2^{-1}T_1 S & QT_1T_2^{-1}S \end{pmatrix}, \tag{3.4}$$

and T_1, T_2, Q, S are the following $n \times n$ matrices:

$$T_1 = \begin{pmatrix} 1 & \dots & & \mathbf{0} \\ b-a & 1 & \dots & \\ \dots & \dots & \dots & \dots \\ \frac{(b-a)^{n-1}}{(n-1)!} & \frac{(b-a)^{n-2}}{(n-2)!} & \dots \; b-a & 1 \end{pmatrix},$$

$$T_2 = \begin{pmatrix} \frac{(b-a)^n}{n!} & \frac{(b-a)^{n-1}}{(n-1)!} & \dots & b-a \\ \frac{(b-a)^{n+1}}{(n+1)!} & \frac{(b-a)^n}{n!} & \dots & \frac{(b-a)^2}{2!} \\ \dots & \dots & \dots & \dots \\ \frac{(b-a)^{2n-1}}{(2n-1)!} & \frac{(b-a)^{2n-2}}{(2n-2)!} & \dots & \frac{(b-a)^n}{n!} \end{pmatrix}, \tag{3.5}$$

$$Q = \begin{pmatrix} (-1)^n & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & -1 \end{pmatrix}, \quad S = \begin{pmatrix} \mathbf{0} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \mathbf{0} \end{pmatrix}.$$

Proof. (i) Let us consider the k-th row in (3.2):

$$f^{(2n-k)}(b) = \sum_{m=1}^{k} \frac{f^{(2n-m)}(a)}{(2n-m)!}(b-a)^{2n-m}, \quad k \in \{1, 2, \dots, n\}. \tag{3.6}$$

Due to the Theorem 2.7, it suffices to prove (3.6) for $\ker A^* = \operatorname{span}\left\{1, x, \dots, x^{2n-1}\right\}$. Since $\ker A^*$ consists of polynomials of degree not greater than $2n-1$, the formula (3.6) follows from Tailor's one for polynomials.

(ii) Let

$$U_1 = \begin{pmatrix} f^{(n-1)}(b) \\ \vdots \\ f(b) \end{pmatrix}, \quad U_2 = \begin{pmatrix} f^{(n-1)}(a) \\ \vdots \\ f(a) \end{pmatrix},$$

$$U_3 = \begin{pmatrix} f^{(2n-1)}(b) \\ \vdots \\ f^{(n)}(b) \end{pmatrix}, \quad U_4 = \begin{pmatrix} f^{(2n-1)}(a) \\ \vdots \\ f^{(n)}(a) \end{pmatrix},$$

$$U_{1,t} = \begin{pmatrix} f(b) \\ \vdots \\ f^{(n-1)}(b) \end{pmatrix}, \quad U_{2,t} = \begin{pmatrix} f(a) \\ \vdots \\ f^{(n-1)}(a) \end{pmatrix}.$$

Then

$$\Gamma_0 f = \begin{pmatrix} SU_2 \\ SU_1 \end{pmatrix} = \begin{pmatrix} U_{2,t} \\ U_{1,t} \end{pmatrix}, \quad \Gamma_1 f = \begin{pmatrix} -QU_4 \\ QU_3 \end{pmatrix},$$

and hence the equality in (3.2) takes the form

$$\begin{pmatrix} U_3 \\ U_1 \end{pmatrix} = \begin{pmatrix} T_1 & \mathbb{O} \\ T_2 & T_1 \end{pmatrix} \begin{pmatrix} U_4 \\ U_2 \end{pmatrix} \quad \text{or} \quad \begin{cases} U_3 = T_1 U_4 + \mathbb{O} U_2, \\ U_1 = T_2 U_4 + T_1 U_2. \end{cases}$$

Expressing U_4 and U_3 from the latter we get

$$\begin{cases} U_4 = T_2^{-1}U_1 - T_2^{-1}T_1U_2, \\ U_3 = T_1T_2^{-1}U_1 - T_1T_2^{-1}T_1U_2. \end{cases}$$

Multiplying from the left the first equality by $-Q$ and the second one by Q we obtain

$$\begin{cases} -QU_4 = -QT_2^{-1}U_1 + QT_2^{-1}T_1U_2, \\ \quad QU_3 = QT_1T_2^{-1}U_1 - QT_1T_2^{-1}T_1U_2. \end{cases} \tag{3.7}$$

Since $U_1 = SU_{1,t}$, $U_2 = SU_{2,t}$ then (3.7) yields

$$\begin{cases} -QU_4 = QT_2^{-1}T_1SU_{2,t} - QT_2^{-1}SU_{1,t}, \\ \quad QU_3 = -QT_1T_2^{-1}T_1SU_{2,t} + QT_1T_2^{-1}SU_{1,t}, \end{cases}$$

or

$$\Gamma_1 f = \begin{pmatrix} QT_2^{-1}T_1S & -QT_2^{-1}S \\ -QT_1T_2^{-1}T_1S & QT_1T_2^{-1}S \end{pmatrix} \Gamma_0 f.$$

Thus, we arrive at the representation $\Gamma_1 f = B_K\Gamma_0 f$, and the equality (3.4) is proved. □

Theorem 3.2. *The matrix B_K is self-adjoint, i.e., $B_K = B_K^*$.*

Proof. Obviously, B_K is self-adjoint in accordance with Proposition 2.6 (v). Let us prove this fact directly. It is necessary to show that the following equalities hold:

$$QT_2^{-1}T_1S = \left(QT_2^{-1}T_1S\right)^*, \tag{3.8}$$

$$QT_1T_2^{-1}S = \left(QT_1T_2^{-1}S\right)^*, \tag{3.9}$$

$$QT_2^{-1}S = \left(QT_1T_2^{-1}T_1S\right)^*. \tag{3.10}$$

Denote $V = ST_2$. Let us prove the equality (3.8). We start with the following obvious relation:

$$QT_2^{-1}T_1S = QT_2^{-1}SST_1S = QV^{-1}T_1^*.$$

Let us check that inverse matrix $T_1^{-1*}VQ$ is self-adjoint.

We will numerate matrix entries of V starting from its right low corner (j is the number of a column and k is the number of a row): $v_{j,k} = \frac{(b-a)^{j+k-1}}{(j+k-1)!}$.

The entry $T_1^{-1*}V$ (denoted by $\varphi_{j,k}$) has the following form:

$$\varphi_{j,k} = \sum_{l=0}^{k-1} \frac{(a-b)^l}{l!} v_{j,k-l} = (b-a)^{j+k-1} \sum_{l=0}^{k-1} (-1)^l \frac{1}{l!(j+k-l-1)!}. \tag{3.11}$$

The symmetric one is

$$\varphi_{k,j} = (b-a)^{j+k-1} \sum_{m=0}^{j-1} (-1)^m \frac{1}{m!(j+k-m-1)!}.$$

Substituting $l = j + k - m - 1$ we get

$$\varphi_{k,j} = (b-a)^{j+k-1} \sum_{l=k}^{j+k-1} (-1)^{j+k-l-1} \frac{1}{l!(j+k-l-1)!}.$$

Now we multiply the matrix $T_1^{-1*}V$ from the right by Q. This means that odd columns are multiplied by -1. To finish the proof of the self-adjointness of $T_1^{-1*}VQ$, one must show that $\varphi_{j,k} - (-1)^{j+k}\varphi_{k,j} = 0$. We have

$$\begin{aligned} \frac{\varphi_{j,k} - (-1)^{j+k}\varphi_{k,j}}{(b-a)^{j+k-1}} &= \sum_{l=0}^{k-1} (-1)^l \frac{1}{l!(j+k-l-1)!} \\ &\quad - \sum_{l=k}^{j+k-1} (-1)^{j+k-l-1} \frac{1}{l!(j+k-l-1)!} \\ &= \sum_{l=0}^{j+k-1} \frac{(-1)^l}{l!(j+k-l-1)!} \\ &= \frac{1}{(j+k-1)!} \sum_{l=0}^{j+k-1} (-1)^l \binom{j+k-1}{l} = 0. \end{aligned} \tag{3.12}$$

The equality (3.8) is proved.

The equality (3.9) is implied by both (3.8) and the following relations:

$$QT_1T_2^{-1}S = QT_1V^{-1}, \quad VT_1^{-1}Q = Q\left(T_1^{-1*}VQ\right)^*Q.$$

Now let us prove the equality (3.10). Passing to inverse matrices in (3.10) and taking into account the relations $V = ST_2$, $V = V^*$, $ST_1^{-1} = T_1^{-1*}S$ we obtain

$$VQ = \left(T_1^{-1*}VT_1^{-1}Q\right)^* = QT_1^{-1*}VT_1^{-1}. \tag{3.13}$$

Multiplying the second equality in (3.13) from the right by T_1^{-1} we get

$$VQT_1^{-1} = QT_1^{-1*}V. \tag{3.14}$$

Now let us prove (3.14). The entry VQ has the form $\frac{(-1)^k}{(j+k-1)!}(b-a)^{j+k-1}$. Therefore, the entry VQT_1^{-1} equals

$$\begin{aligned} \psi_{j,k} &= (b-a)^{j+k-1} \sum_{m=0}^{j-1} \frac{(-1)^{j+m}}{m!(j+k-m-1)!} \\ &= (b-a)^{j+k-1}(-1)^k \sum_{l=k}^{j+k-1} \frac{(-1)^{l+1}}{l!(j+k-l-1)!}. \end{aligned}$$

To calculate entries of $QT_1^{-1*}V$, we must multiply the matrix $T_1^{-1*}V$ from the left by Q. This means that odd rows are multiplied by -1. Then, in accordance with (3.11), the entry of the matrix $QT_1^{-1*}V$ is

$$\mu_{j,k} = (b-a)^{j+k-1}(-1)^k \sum_{l=0}^{k-1} \frac{(-1)^l}{l!(j+k-l-1)!}.$$

It is easily seen that $\psi_{j,k} = \mu_{j,k}$ (similarly to (3.12)). Equalities (3.14) and (3.10) are established, and the theorem is completely proved directly. $\square$

Proposition 3.3. *Let* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *be the boundary triple for* A^* *defined by* (3.1), *and let* $M(\cdot)$ *be the corresponding Weyl function. Then* $B_K = M(0) = B_K^*$.

Proof. Combining Proposition 2.6 (ii) with Theorem 3.1 (ii) we arrive at the desired result. $\square$

In the following theorem we describe all non-negative extensions of the operator A as well as extensions having exactly κ negative squares.

Theorem 3.4. *Let* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *be the boundary triple for operator* A^* *defined by* (3.1), *and let* B_K *be the matrix defined by* (3.4). *Let also matrices* $C, D \in \mathbb{C}^{2n\times 2n}$ *satisfy the conditions* $CD^* = DC^*, \det(CC^* + DD^*) \neq 0$, *and*

$$A_{C,D} = A^* \restriction \ker(D\Gamma_1 - C\Gamma_0) = A^*_{C,D}.$$

The the following assertions hold.

(i) *The following equivalence holds:*

$$\kappa_-(A_{C,D}) = \kappa \quad \Longleftrightarrow \quad \kappa_-(CD^* - DB_KD^*) = \kappa.$$

In particular, $A_{C,D} \geq 0 \Longleftrightarrow CD^* - DB_KD^* \geq 0$.

(ii) *The operator* $A_{C,D}$ *is positively definite if and only if the matrix* $CD^* - DB_KD^*$ *is positive definite too.*

Proof. Due to Proposition 3.3, one has $B_K = M(0)$. To complete the proof, it suffices to use Proposition 2.6 (iv). $\square$

Corollary 3.5. *Let* $\{A_j\}_{j=1}^{\infty} := \{A_{j,\min}\}_{j=1}^{\infty}$ *be the sequence of minimal operators generated in* $\mathfrak{H}_j = L^2(a_j, b_j)$, *respectively, by the differential expression* (1.10). *Let also* $A := \bigoplus_{j=1}^{\infty} A_j$. *Then*

$$\widehat{A}_K = \bigoplus_{j=1}^{\infty} \widehat{A}_{j,K},$$

where $\widehat{A}_{j,K}$ *is given by* (3.2) *and* (3.3).

Proof. Combining Theorem 3.1 with Corollary 3.10 from [32] we arrived at the desired result. $\square$

4. EXAMPLES

To facilitate the reading, let us provide four examples for $a = 0$, $b = 1$ and $n \in \{1, 2, 3, 4\}$.

Example 4.1. Let $n = 1$, i.e., $Ay = -y''$. Then

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

and the boundary conditions from (3.2) take the form:

$$\begin{cases} f'(1) = f'(0), \\ f(1) = f'(0) + f(0). \end{cases}$$

It follows from (3.5) and (3.4) that

$$T_1 = (1), \quad T_2 = (1), \quad Q = (-1), \quad S = (1),$$

and

$$B_K = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

is symmetric as required.

Example 4.2. Let $n = 2$, i.e., $Ay = y^{(iv)}$. Then

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \frac{1}{2} & 1 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & 1 & 1 \end{pmatrix},$$

and the boundary conditions from (3.2) take the form:

$$\begin{cases} f'''(1) = f'''(0), \\ f''(1) = f'''(0) + f''(0), \\ f'(1) = \frac{1}{2}f'''(0) + f''(0) + f'(0), \\ f(1) = \frac{1}{6}f'''(0) + \frac{1}{2}f''(0) + f'(0) + f(0). \end{cases}$$

It follows from (3.5) and (3.4) that

$$T_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} \frac{1}{2} & 1 \\ \frac{1}{6} & \frac{1}{2} \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and

$$B_K = \begin{pmatrix} -12 & -6 & 12 & -6 \\ -6 & -4 & 6 & -2 \\ 12 & 6 & -12 & 6 \\ -6 & -2 & 6 & -4 \end{pmatrix}.$$

Example 4.3. Let $n = 3$, i.e., $Ay = -y^{(vi)}$. Then

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 1 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{2} & 1 & 1 & 0 & 0 \\ \frac{1}{24} & \frac{1}{6} & \frac{1}{2} & 1 & 1 & 0 \\ \frac{1}{120} & \frac{1}{24} & \frac{1}{6} & \frac{1}{2} & 1 & 1 \end{pmatrix},$$

and the boundary conditions are the following:

$$\begin{cases} f^{(v)}(1) = f^{(v)}(0), \\ f^{(iv)}(1) = f^{(v)}(0) + f^{(iv)}(0), \\ f'''(1) = \frac{1}{2}f^{(v)}(0) + f^{(iv)}(0) + f'''(0), \\ f''(1) = \frac{1}{6}f^{(v)}(0) + \frac{1}{2}f^{(iv)}(0) + f'''(0) + f''(0), \\ f'(1) = \frac{1}{24}f^{(v)}(0) + \frac{1}{6}f^{(iv)}(0) + \frac{1}{2}f'''(0) + f''(0) + f'(0), \\ f(1) = \frac{1}{120}f^{(v)}(0) + \frac{1}{24}f^{(iv)}(0) + \frac{1}{6}f'''(0) + \frac{1}{2}f''(0) + f'(0) + f(0). \end{cases}$$

Both (3.5) and (3.4) imply that

$$T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{1}{2} & 1 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} \frac{1}{6} & \frac{1}{2} & 1 \\ \frac{1}{24} & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{120} & \frac{1}{24} & \frac{1}{6} \end{pmatrix},$$

$$Q = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and

$$B_K = \begin{pmatrix} -720 & -360 & -60 & 720 & -360 & 60 \\ -360 & -192 & -36 & 360 & -168 & 24 \\ -60 & -36 & -9 & 60 & -24 & 3 \\ 720 & 360 & 60 & -720 & 360 & -60 \\ -360 & -168 & -24 & 360 & -192 & 36 \\ 60 & 24 & 3 & -60 & 36 & -9 \end{pmatrix}.$$

Example 4.4. Let $n = 4$, i.e., $Ay = y^{(viii)}$. Then

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{2} & 1 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{24} & \frac{1}{6} & \frac{1}{2} & 1 & 1 & 0 & 0 & 0 \\ \frac{1}{120} & \frac{1}{24} & \frac{1}{6} & \frac{1}{2} & 1 & 1 & 0 & 0 \\ \frac{1}{720} & \frac{1}{120} & \frac{1}{24} & \frac{1}{6} & \frac{1}{2} & 1 & 1 & 0 \\ \frac{1}{5040} & \frac{1}{720} & \frac{1}{120} & \frac{1}{24} & \frac{1}{6} & \frac{1}{2} & 1 & 1 \end{pmatrix},$$

and the boundary conditions are the following:

$$\begin{cases} f^{(vii)}(1) = f^{(vii)}(0), \\ f^{(vi)}(1) = f^{(vii)}(0) + f^{(vi)}(0), \\ f^{(v)}(1) = \frac{1}{2}f^{(vii)}(0) + f^{(vi)}(0) + f^{(v)}(0), \\ f^{(iv)}(1) = \frac{1}{6}f^{(vii)}(0) + \frac{1}{2}f^{(vi)}(0) + f^{(v)}(0) + f^{(iv)}(0), \\ f'''(1) = \frac{1}{24}f^{(vii)}(0) + \frac{1}{6}f^{(vi)}(0) + \frac{1}{2}f^{(v)}(0) + f^{(iv)}(0) + f'''(0) \\ f''(1) = \frac{1}{120}f^{(vii)}(0) + \frac{1}{24}f^{(vi)}(0) + \frac{1}{6}f^{(v)}(0) + \frac{1}{2}f^{(iv)}(0) + f'''(0) + f''(0), \\ f'(1) = \frac{1}{720}f^{(vii)}(0) + \frac{1}{120}f^{(vi)}(0) + \frac{1}{24}f^{(v)}(0) + \frac{1}{6}f^{(iv)}(0) + \frac{1}{2}f'''(0) + f''(0) \\ \qquad + f'(0), \\ f(1) = \frac{1}{5040}f^{(vii)}(0) + \frac{1}{720}f^{(vi)}(0) + \frac{1}{120}f^{(v)}(0) + \frac{1}{24}f^{(iv)}(0) + \frac{1}{6}f'''(0) + \frac{1}{2}f''(0) \\ \qquad + f'(0) + f(0). \end{cases}$$

Both (3.5) and (3.4) imply that

$$T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \frac{1}{2} & 1 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & 1 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} \frac{1}{24} & \frac{1}{6} & \frac{1}{2} & 1 \\ \frac{1}{120} & \frac{1}{24} & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{720} & \frac{1}{120} & \frac{1}{24} & \frac{1}{6} \\ \frac{1}{5040} & \frac{1}{720} & \frac{1}{120} & \frac{1}{24} \end{pmatrix},$$
$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and

$$B_K = \begin{pmatrix} -100800 & -50400 & -10080 & -840 & 100800 & -50400 & 10080 & -840 \\ -50400 & -25920 & -5400 & -480 & 50400 & -24480 & 4680 & -360 \\ -10080 & -5400 & -1200 & -120 & 10080 & -4680 & 840 & -60 \\ -840 & -480 & -120 & -16 & 840 & -360 & 60 & -4 \\ 100800 & 50400 & 10080 & 840 & -100800 & 50400 & -10080 & 840 \\ -50400 & -24480 & -4680 & -360 & 50400 & -25920 & 5400 & -480 \\ 10080 & 4680 & 840 & 60 & -10080 & 5400 & -1200 & 120 \\ -840 & -360 & -60 & -4 & 840 & -480 & 120 & -16 \end{pmatrix}.$$

Acknowledgements
The authors express their gratitude to Prof. M. Malamud for posing the problem and permanent attention to the work, and to A. Ananieva for useful comments. The authors also express their gratitude to Prof. Gesztesy for his guide in the literature.

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Received: August 21, 2017.
Revised: December 15, 2017.
Accepted: December 20, 2017.