Scientific Journals
of the Maritime University of Szczecin

Zeszyty Naukowe
Akademii Morskiej w Szczecinie

2017, 52 (124), 97–102
ISSN 1733-8670 (Printed)
ISSN 2392-0378 (Online)
DOI: 10.17402/250

Received: 23.10.2017
Accepted: 30.11.2017
Published: 15.12.2017

An application of the selected graph theory domination concepts to transportation networks modelling

Sambor Guze

Gdynia Maritime University, Faculty of Navigation, Department of Mathematics
81/83 Morska St., 81-225 Gdynia, Poland, e-mail: s.guze@wn.am.gdynia.pl

Key words: domination, edge-subdivision, connected bondage number, bondage-connected number, transportation network, modelling

Abstract

One of the possibilities when modelling a transport network is to use a graph with vertices and edges. They represent the nodes and arcs of such a network respectively. There are dozens of parameters or characteristics that we can describe in graphs, including the different types of domination number and the problems related to it. The main aim of this paper has been to show the possibilities of the application of the selected domination-oriented concepts to modelling and improving the transportation and/or logistics networks. Firstly, the basic description of domination in graph theory has been introduced. The edge-subdivision and bondage number notations and their implementations to the transportation network description and modelling were then proposed. Furthermore, the possible usage of distinguishing concepts in an exemplary academic transportation network has been shown. Finally, the conclusions and future directions of the work have been presented.

Introduction

Transportation systems are the basis of today's national and world economies. It is a major component of each country's Gross Domestic Product. Therefore, it is recognized in many countries as one of the critical infrastructures. In the European Union, it was first introduced in 2006 by the European Programme for Critical Infrastructure Protection (Commission of the European Communities, 2006) and legally fixed by the Directive 2008/114/EC in 2008 (Council Directive, 2008). According to (Cascetta, 2001), a transportation system can be defined as the combination of elements and their interactions, which produce the demand for travel within a given area, and the supply of transportation services to satisfy this demand. These items are means of transport, infrastructure, and people. The definition is general, but also flexible. It gives the possibility of specifying the structure by the problem that needs to be solved. The networks usually describe the transportation systems. The best and most natural way to represent them is with an analogy for their structure and flows (Newell, 1980; Rodrigue, Comtois & Slack, 2017).

Research on transportation systems relates to various aspects of their functionality. One of the main aspects is their safety and reliability problems. When the transport systems are considered as complex technical systems, then the methods included in (Guze, 2009; Kołowrocki, 2004; Kołowrocki & Soszyńska-Budny, 2011; Blokus-Roszkowska, 2016) can be used to analyze and model their safety and reliability. In case of two modes of transport, i.e., air and maritime transport, these aspects are of particular importance. There have been studies in stochastic congestion models for waterways (Gucma et al., 2016), traffic flows in transportation networks (Newell, 1980), and the development of new methods in ship movement prediction (Borkowski, 2017).

Nowadays, the decision support systems' importance is growing. Some concepts and algorithms that are helpful for decision support in collision situations at sea have been presented in (Gucma et al., 2016; Guze, Smolarek & Weintrit, 2016; Borkowski, 2017; Pietrzykowski, Wołejsza & Borkowski, 2017).

The second aspect that is important for transportation systems is improvement and optimization. There have been many studies on methods for optimization of transport systems due to different criteria (Kołowrocki & Soszyńska-Budny, 2011). It is increasingly important to include not only a one-criterion optimization, but two or three. The multi-criteria optimization methods have been discussed in (Venter, 2010; Ming-Hua, Jung-Fa & Chian-Son, 2012), and examples of their applications were included in (Guze, Neumann & Wilczyński, 2017).

On the other hand, there are more advanced mathematical methods of nonlinear analysis, referring to biology, and they have been used to describe the safe operation of stratospheric balloons (Guze & Janczewska, 2015).

It can be difficult to use these methods depending on the difficulty of a considered problem. Thus, in such situations, it is possible to apply discrete methods like graph theory. An example of the use of the graph theory approach to analyse and model transport systems is the well-known problem of finding the shortest path, which has been presented in (Newell, 1980; Venter, 2010; Neumann, 2016; Guze, Neumann & Wilczyński, 2017; Rodrigue, Comtois & Slack, 2017).

The main aim of this article has been the application of the theory of domination, in graphs and related concepts, to transportation networks analysis and modelling.

Graph theory topics review

This section contains the basic notations of graph theory, some definitions and parameters of domination and edge-subdivision terms based on results given in (Harrary, 1969; Hartnell & Rall, 1994; Haynes, Hedetniemi & Slater, 1998; Bhattacharya & Vijayakumar, 2002; Ruan et al., 2004).

Basic notations

In the whole article, we have considered the connected, simple, undirected graph $G = (V, E)$, where V is the set of vertices (nodes) and E is the set of edges (arcs). In other words, for a graph G, $V(G)$ and $E(G)$ respectively denote its vertex-set and the edge-set. These assumptions are very important, because the connectivity of transport or logistics networks is fundamental to the functioning of these networks.

The set of all adjacent vertices to vertex $v \in V$ in G is called the neighborhood and denoted by $N_G(v)$ or $N(v)$. The close neighbourhood of this vertex is defined as $v \in V \cup \{v\}$ and denoted by $N_G[v]$. The other basic parameter for the graphs is the degree of vertex $v \in V$, which is defined as the number of vertices in $N_G(v)$ and denoted by $\deg(v)$. The minimum and maximum degrees are defined as $\delta(G) = \min\{x \in V{:}\deg(x)\}$ and $\Delta(G) = \max\{x \in V{:}\deg(x)\}$, respectively. Moreover, the set of all edges incident to the vertex $v \in V$ is denoted by $I_G \in (v)$.

For any set $A \subseteq V$, the neighbourhood is given by $N(A) = \bigcup_{v \in A} N(v)$. The induced subgraph defined on A is denoted by $G[A]$.

Domination and bondage numbers in graphs

According to (Harrary, 1969; Haynes, Hedetniemi & Slater, 1998), the definitions of two dominating sets and domination numbers have been introduced in this subsection.

Generally, a set $D \subseteq V(G)$ is a *dominating set* of graph G, if for any $v \in V$ either $v \in D$ or $N_G(v) \cap D = \varnothing$. While the minimum cardinality of a dominating set of graph G is called the *domination number* of G and denoted as $\gamma(G)$. The example of a dominating set is presented in Figure 1.

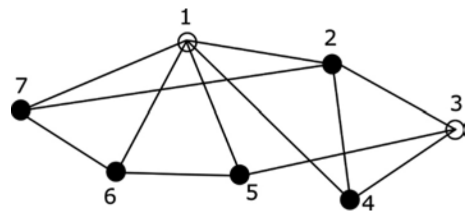

Figure 1. Dominating set $D = \{1, 3\}$ and domination number $\gamma(G) = 2$

Furthermore, a set $D_C \subseteq V(G)$ is called a *connected dominating set* of graph G, if every vertex of $V \setminus D_C$ is adjacent to a vertex in D_C and the subgraph induced by D_C is connected (see Figure 2). The minimum cardinality of a connected dominating set of graph G is called a connected domination number of G and denoted as $\gamma_C(G)$.

The domination theory has various applications, and the analysis of communication networks is the one that has been most discussed in literature. Fink et al. (Fink et al., 1990) examined a question concerning the vulnerability of the communications network under link failure. They proposed the hypothetical situation, where someone does not know which nodes in the network act as transmitters, but

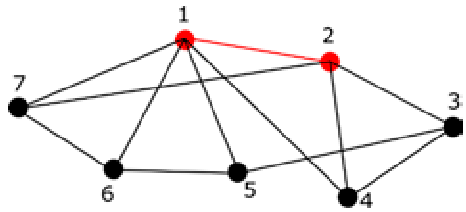

Figure 2. Connected dominating set $D_C = \{1,2\}$ and connected domination number $\gamma_C(G) = 2$

does know that the set of such nodes can built a minimum dominating set in the related graph. Thus, they investigated the fewest number of communication links that must be severed so that at least one additional transmitter would be required so that communication with all sites would be possible. In this way, they introduced a new parameter called the bondage number of a graph. It is defined in the following way; The bondage number $b(G)$ of nonempty graph G is the minimum cardinality among all sets of edges E for which $\gamma(G - E) > \gamma(G)$ (Fink et al., 1990; Hartnell & Rall, 1994). Thus, the bondage number of graph G describes the smallest number of edges whose removal from G results in a graph with a domination number larger than that of G (see Figure 3).

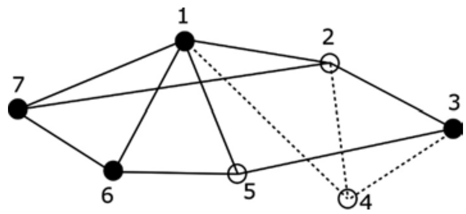

Figure 3. Graph with bondage number $b(G) = 3$ after removing dotted edges (1,4), (2,4), (3,4)

The exemplary graph considered in this section has a bondage number equal to 3, because the removal of the three edges (dotted in Figure 3) is enough to increase the domination number.

Domination and edge-subdivision in graphs

Another, interesting concept in the theory of domination is edge-subdivision. It was introduced by S. Arumugam and S. Velammal (Velammal, 1997). This approach is based on the operation of subdividing graph G, which was defined in (Bhattacharya & Vijayakumar, 2002) in the following way.

Definition 1 (Bhattacharya & Vijayakumar, 2002). Let G be a graph and uv be an edge of G. By *subdividing* the edge uv we mean forming a graph H from G by adding a new vertex w and replacing the edge uv by uw and wv. (Formally, $V(H) = V(G) \cup \{w\}$ and $E(H) = (E(G) - \{uv\}) \cup \{uw, wv\}$.) The graph obtained from G by subdividing each edge exactly once is denoted by $S(G)$.

Moreover, this concept is used under the assumption that the domination number of graph $S(G)$, obtained from G by subdividing every edge exactly once is more than that of G, i.e. $\gamma(S(G)) > \gamma(G)$ (Bhattacharya & Vijayakumar, 2002). The parameter related to subdividing is the subdivision number, defined as follows (Bhattacharya & Vijayakumar, 2002).

Definition 2 (Bhattacharya & Vijayakumar, 2002). Let G be a graph with $\Delta(G) > 1$. The smallest number that can be the cardinality of a set of edges such that subdividing each of them exactly once results in a graph with a domination number of more than that of G, is called the *subdivision number* of G and is denoted by $\xi(G)$.

The application of the subdivision number for the exemplary graph G has been presented in Figure 4.

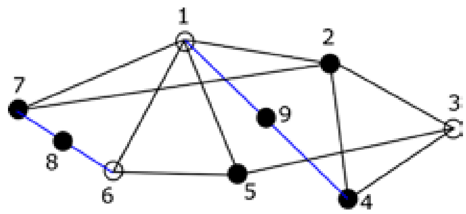

Figure 4. Graph G with subdivision number $\xi(G) = 2$ and subdividing blue edges (1,4), (6,7)

The authors of (Bhattacharya & Vijayakumar, 2002) showed that for any graph H obtained from G by subdividing some edges of G, the domination number of graphs G and H satisfy the inequality $\gamma(H) \geq \gamma(G)$. Furthermore, there are two proven theorems about the upper estimation of the subdivision number of graph G (Bhattacharya & Vijayakumar, 2002). The first theorem gives the estimation related to the domination number, where for a connected graph with at least 3 vertices, the subdivision number is estimated by (Bhattacharya & Vijayakumar, 2002):

$$\xi(G) \leq \gamma(G) + 1 \tag{1}$$

In the second theorem, the upper estimation is dependent on the number of vertices, and for a connected graph of large order n, the subdivision number's estimation is given by (Bhattacharya & Vijayakumar, 2002):

$$\xi(G) \leq 4\sqrt{n}\ln(n) + 5 \tag{2}$$

Both approximations are helpful in finding the subdivision number.

The problem of the minimum dominating set in the theory of complexity is NP-hard, i.e. there is no algorithm to find this set in polynomial time. In this way, finding the bondage number is NP-hard and the subdivision number is an NP-complete class of problems.

Implementations of domination-related concepts in transportation networks

The results presented in Section *Introduction* can be applied to the analysis and modelling of the transportation networks. This section presents the most efficient way to do this. Moreover, the connected bondage number and the bondage-connected number are defined as the author's new concepts in domination-related problems. Furthermore, the translation of the edge-subdividing concept to the transportation network analysis and modelling problem has been presented.

The theory of the bondage number introduced by Fink et al. (Fink et al., 1990) can be implemented to model transport networks, only after redefinition. All the algorithms used to find the bondage number (Fink et al., 1990; Hartnell & Rall, 1994) do not require connectivity of the graph after edge removal. It is an undesirable phenomenon in a transportation network. Thus, the author has proposed two new bondage numbers, i.e. a connected bondage number for the connected dominating set, and a bondage-connected number for the dominating set.

Firstly, the definition for the connected domination number has been proposed.

Definition 3. The connected bondage number $b_C(G)$ of nonempty graph G is the minimum cardinality among all sets of edges E for which $\gamma_C(G-E) > \gamma_C(G)$.

The next definition has been introduced as the best way to apply the analysis and modelling to the transportation network.

Definition 4. The bondage-connected number $b_C(G)$ of nonempty graph G is the minimum cardinality among all sets of edges E for which $\gamma(G-E) > \gamma(G)$ and graph $G-E$ is connected.

Both numbers can be used to approximate the vulnerability of a transportation network because of the temporary or permanent exclusion of a road or railway connection. The number defined in Definition 3 can be useful to answer the question about how many broken connections are needed to increase the number of nodes formed from the connected root of a transportation network by one. In the case of the number proposed in Definition 4, it gives the possibility of getting the solution to how many broken connections increase the number of production and storage centres by one. In this situation, it is not necessary that each production centre has a direct connection to each other.

Whereas, the results presented in Subsection *Domination and edge-subdivision in graphs* can be useful for redesigning the functioning transport or logistics network. This operation can be provoked by the necessity of serving more customers. The way of modelling by using the edge-subdividing method can help to select the proper connections to upgrade by adding new nodes. As we know according to the results, for graphs, i.e. networks, two approximations for the edge-subdivision number are given by (1) and (2). Both can be applied depending on the problem. According to this approach it is possible to state the question of how many new customers we need to grow the number of operation centres. From a mathematical point of view, how many subdividing operations need to be done, so that the domination number grows?

Applications

Let us consider the exemplary transportation network with 12 nodes and arcs presented in Figure 5. We have taken into account two cases:

Case 1. The transportation network has fixed main nodes, which correspond to the dominating set in the represented graph.

Case 2. The transportation network can be designed by finding the main nodes, i.e. the minimal dominating set.

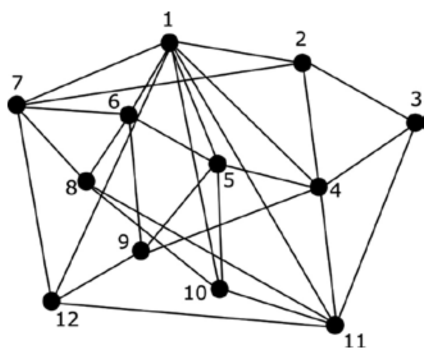

Figure 5. Exemplary transportation network

Case 1. The main nodes are selected and they form the dominating set $D = \{1,5,11\}$. At the same time, it is also the minimum cardinality of this set. Thus, the domination number is equal to 3 (see Figure 6).

If there is no possibility to change the main nodes, then both the bondage and bondage-connected

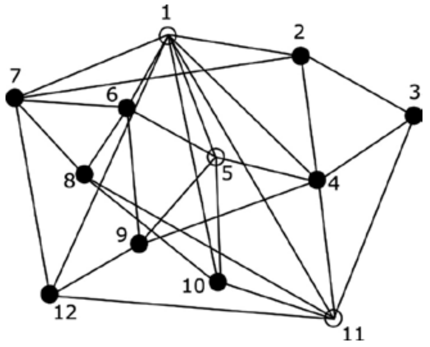

Figure 6. Dominating set for transportation network (empty circles)

numbers are equal to 2, i.e. $\gamma(G) = \gamma_C(G) = 2$, and extension of the transportation network main nodes are as given in Figure 7. These numbers give the information about the vulnerability of the considered network to temporarily or permanently broken infrastructure.

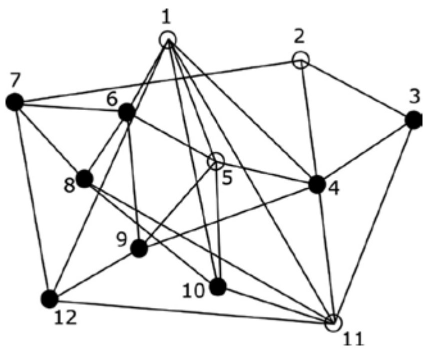

Figure 7. The transportation network without two arcs (1,2) and (1,7)

Now we want to know how many new customers are needed to grow the transportation network. To answer the question, edge-subdividing can be applied. The exemplary results for the considered network are presented in Figure 8.

It has been shown, that only one additional customer is needed to increase the domination number, i.e. the minimal dominating set. It satisfies the estimation given in (1) and (2).

Case 2. The transportation network can be designed by finding the main nodes, i.e. the minimal dominating set.

The dominating set can be the same as the one presented in Figure 6, but it is only one of many possibilities, e.g. other $D = \{1,9,11\}$. These sets form the potential main nodes in the transportation network without the assumption about the connectivity of the induced subgraph.

The most important question is about the vulnerability of the network represented by the graph in

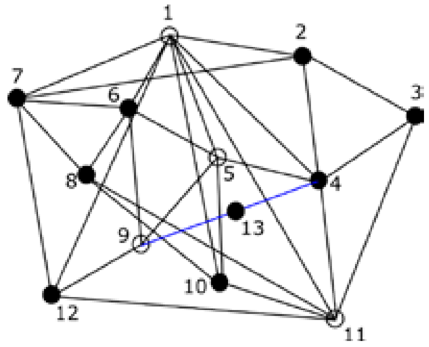

Figure 8. The transportation network edge-subdividing one arc (4,9) to extend the dominating set to {1,5,9,11}

Figure 5 according to the bondage-connected number. For this network, this number is equal to 9 (see results in Figure 9).

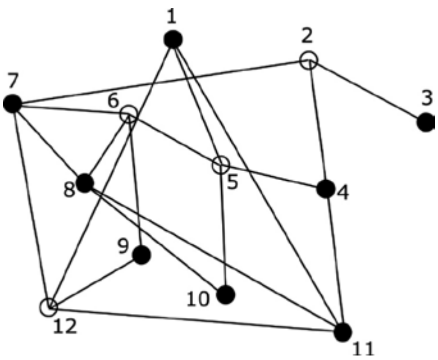

Figure 9. Exemplary graph after reduction of 9 edges, $b_C(G) = 9$, dominating set $D = \{2,5,6,12\}$

Next, similarly to case 1, the question is about the number of additional customers necessary to increase the number of main nodes to serve the service. One of the possible solutions is given in Figure 10 on the blue edges.

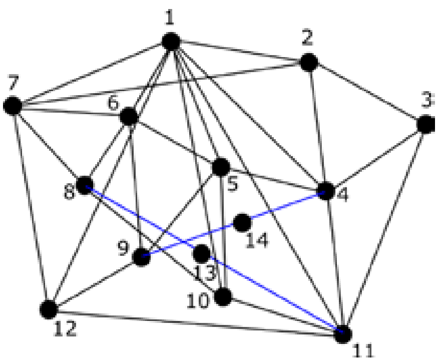

Figure 10. The transportation network edge-subdividing two arcs (4,9) and (8,11) and the dominating set {1,8, 9,11}

The graph in Figure 10 shows that one of the possible uses of the edge-subdividing operation can be done for edges (4,9) and (8,11).

Conclusions

The paper has presented the possibilities for modelling a transport network with the graph theory approach using the domination parameters and the edge-subdivision concept.

First, a review of the literature on the methods of modelling and the optimization of technical and transport systems, with particular emphasis on maritime transport, was conducted.

Next, the basics of domination in graph theory were introduced. The domination number, bondage number, and the author's new concepts of the connected bondage number and the bondage-connected number have been proposed. The edge-subdivision methods for vertex-domination in graphs have been described and implemented for the transportation and logistics networks.

Finally, the application of the previously mentioned and defined methods has been presented as used on the exemplary transportation network in two cases.

The presented methods are universal and helpful in modelling every type of network, not only transportation networks. Future research will be concerned with the possibility of using the discussed methods to optimize transport networks.

References

1. Bhattacharya, A. & Vijayakumar, G. R. (2002) Effect of edge-subdivision on vertex-domination in a graph. *Discussiones Mathematicae Graph Theory* 22, 2, pp. 335–347.
2. Blokus-Roszkowska, A. (2016) *Reliability Analysis of the Bulk Cargo Loading System IncludingDependent Components*. Proc. The International Conference of Numerical Analysis and Applied Mathematics – ICNAAM 2015, Symposium/Workshop on Safety of Critical Infrastructures, Rhodes, Greece, 2015, AIP Conf. Proc. 1738, T. Simos, Ch. Tsitouras (eds.), 1738A, 440006-5-440006-9.
3. Borkowski, P. (2017) The ship movement trajectory prediction algorithm using navigational data fusion. *Sensors* 17, 6 (1432), pp. 1–12.
4. Cascetta, E. (2001) *Transportation Systems in Transportation Systems Engineering: Theory and Methods*. E. Cascetta, Ed. Boston, MA: Springer US, pp. 1–22.
5. Commission of the European Communities (2006) Communication from the Commission on a European Programme for Critical Infrastructure Protection, Brussels.
6. Council Directive (2008) 2008/114/EC of 8 December 2008 on the identification and designation of European critical infrastructures and the assessment of the need to improve their protection. Official Journal of the European Union L 345/75 (23.12.2008).
7. Fink, J.F., Jacobson, M.S., Kinch, L.F. & Roberts, J. (1990) The bondage number of graph, *Discrete Mathematics* 86, 1–3, pp. 47–57.
8. Gucma, L., Bąk, A., Sokołowska, S. & Hajduk, J. (2016) Stochastic model of ship traffic congestion in waterways applied to determine the influence of Liquefied Petroleum Gas tanker introduction on ship traffic on the Świnoujście–Szczecin waterway. *Scientific Journals of the Maritime University of Szczecin, Zeszyty Naukowe Akademii Morskiej w Szczecinie* 45 (117), pp. 69–74.
9. Guze, S. (2009) *Reliability analysis of multi-state ageing series-consecutive "m out of n: F" systems*. Proc. European Safety and Reliability Conference – ESREL 2009, Prague, Vol. 3, pp. 1629–1635.
10. Guze, H. & Janczewska, J. (2015) Subcritical bifurcation of free elastic shell of biological cluster. *Nonlinear Analysis: Real World Applications* 24, pp. 61–72.
11. Guze, S., Neumann, T. & Wilczyński, P. (2017) Multi-Criteria Optimisation of Liquid Cargo Transport According to Lingustic Approach to the Route Selection Task. *Polish Maritime Research* 24(s1), pp. 89–96.
12. Guze, S., Smolarek, L. & Weintrit, A. (2016) The area--dynamic approach to the assessment of the risks of ship collision in the restricted water. *Scientific Journals of the Maritime University of Szczecin, Zeszyty Naukowe Akademii Morskiej w Szczecinie* 45 (117), pp. 88–93.
13. Harrary, F. (1969) *Graph Theory*. Reading, MA: Addison-Wesley.
14. Hartnell, B.L. & Rall, D.F. (1994) Bounds on the bondage number of a graph. *Discrete Mathematics* 128, 1, pp. 173–177.
15. Haynes, T.W., Hedetniemi, S. & Slater, P. (1998) *Fundamentals of Domination in Graphs*. CRC Press.
16. Kołowrocki, K. (2004) *Reliability of Large and Complex Systems*. Amsterdam, Boston, Heidelberd, London, New York, Oxford, Paris, San Diego, San Francisco, Singapore, Sidney, Tokyo: Elsevier.
17. Kołowrocki, K. & Soszyńska-Budny, J. (2011) *Reliability and Safety of Complex Technical Systems and Processes: Modeling – Identification – Prediction – Optimization*. London, Dordrecht, Heildeberg, New York: Springer.
18. Ming-Hua, L., Jung-Fa, T. & Chian-Son, Y. (2012) A Review of Deterministic Optimization Methods in Engineering and Management. *Mathematical Problems in Engineering* 2012.
19. Neumann, T. (2016) The Shortest Path Problem with Uncertain Information in Transport Networks. In: J. Mikulski, Ed. *Challenge of Transport Telematics*, Springer International Publishing.
20. Newell, G.F. (1980) *Traffic flow on transportation networks*. MIT Press Series in transportation studies, Monograph 5.
21. Pietrzykowski, Z., Wołejsza, P. & Borkowski, P. (2017) Decision Support in Collision Situations at Sea. *Journal of Navigation* 70, 3, pp. 447–464.
22. Rodrigue, J.P., Comtois, C. & Slack, B. (2017) *The geography of transport systems* (4th Edition), Routledge: Taylor & Francis Group.
23. Ruan, L., Du, H., Jia X., Wu, W., Li, Y. & Ko K.-I. (2004) A greedy approximation for minimum connected dominating sets. *Theoretical Computer Science* 329, 1–3, pp 325–330.
24. Velammal, S. (1997) Studies in Graph Theory: Covering, Independence, Domination and Related Topics, Ph.D. Thesis, Manonmaniam Sundaranar University, Tirunelveli.
25. Venter, G. (2010) *Review of Optimization Techniques. Encyclopedia of Aerospace Engineering*. John Willey and Sons, Ltd.