

On the Exponential Stability of a Neutral Differential Equation of First Order

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ABSTRACT: In this work, we establish some assumptions that guaranteeing the global exponential stability (GES) of the zero solution of a neutral differential equation (NDE). We aim to extend and improves some results found in the literature.

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1. Introduction

In [1], sufficient conditions for solutions of the (NDEs) form

$$\frac{d}{dt}(x(t) + c(t)x(t - \tau)) + p(t)x(t) + q(t)x(t - \sigma) = 0 \quad (1)$$

to tend zero as $t \rightarrow \infty$ are established.

In [4, 8, 10, 15, 17], it was considered a (NDE),

$$\frac{d}{dt}(x(t) + px(t - \tau)) = -\alpha x(t) + b \tanh(x(t - \sigma)) = 0 \quad (2)$$

and the asymptotic stability (AS) of solutions are investigated.

In addition, some qualitative behaviors of solutions of equation (2) or some different models of that (NDE) were investigated in the relevant literature; for example, (S), (AS), (ES) in [2, 6, 9, 11, 14, 16, 17-25], (GES) in [3], asymptotic behaviors in [13], oscillation and non-oscillation in [5, 7] and so on.

In this paper, we deal with the following (NDE) with different variable delays:

$$\frac{d}{dt} \left[x(t) + \sum_{i=1}^2 p_i(t)x(t - \tau_i(t)) \right] + a(t)h(x(t)) - \sum_{i=1}^2 b_i(t) \tanh x(t - \sigma_i(t)) = 0, \quad (3)$$

for $t \geq 0$ where $a_i, b_i : [0, \infty) \rightarrow [0, \infty)$ are continuously differentiable functions and $\sum_{i=1}^2 a_i^2(t) \leq 1$. The functions $\tau_i(\cdot) : [0, \infty) \rightarrow [0, \tau_i]$, ($\tau_i > 0$) and $\sigma_i(\cdot) : [0, \infty) \rightarrow [0, \sigma_i]$, ($\sigma_i > 0$) are bounded and continuously differentiable, and the functions h , p_1 and p_2 are continuous with $h(0) = 0$. Let $r_i = \max\{\tau_i, \sigma_i\} > 0$, ($i = 1, 2$). Let $\mu_1, \mu_2, \mu_3, \mu_4 \in (0, 1)$ be positive constants such that $\tau_1'(t) \leq \mu_1$, $\tau_2'(t) \leq \mu_2$, $\sigma_1'(t) \leq \mu_3$ and $\sigma_2'(t) \leq \mu_4$. For each solution of (NDE) (3), we suppose that

$$x_0(\theta) = \phi(\theta), \theta \in [-r_i, 0], \quad \text{where } \phi \in C([-r_i, 0]; R).$$

2. Stability Result

Our stability result is given below.

Theorem. *Let $K, \alpha_0, \alpha_1, \alpha_2, \alpha_3$ and α_4 be positive constants. The zero solution of (NDE)(3) is global exponential stable if the following matrix inequalities hold:*

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & \Omega_{15} & 0 \\ * & \Omega_{22} & \Omega_{23} & \Omega_{24} & \Omega_{25} & 0 \\ * & * & \Omega_{33} & \Omega_{34} & \Omega_{35} & 0 \\ * & * & * & \Omega_{44} & 0 & 0 \\ * & * & * & * & \Omega_{55} & 0 \\ * & * & * & * & * & \Omega_{66} \end{bmatrix} < 0,$$

$$\Delta = \begin{bmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} & \Delta_{14} & \Delta_{15} & 0 \\ * & \Delta_{22} & \Delta_{23} & \Delta_{24} & \Delta_{25} & 0 \\ * & * & \Delta_{33} & \Delta_{34} & \Delta_{35} & 0 \\ * & * & * & \Delta_{44} & 0 & 0 \\ * & * & * & * & \Delta_{55} & 0 \\ * & * & * & * & * & \Delta_{66} \end{bmatrix} < 0, \quad (4)$$

where

$$h_1(x) = \begin{cases} h(x)x^{-1}, & x \neq 0 \\ h'(0), & x = 0 \end{cases}$$

and

$$\Omega_{11} = 2K\alpha_0\lambda_1 - 2a(t)\alpha_0 \frac{h(x)}{x} \lambda_1 + \alpha_1\lambda_1 \sum_{i=1}^2 e^{2K\tau_i} + \alpha_3\lambda_1 \sum_{i=1}^2 e^{2K\sigma_i}$$

$$\begin{aligned}
 & + \frac{\alpha_2 \lambda_1}{2K} \sum_{i=1}^2 (e^{2K\tau_i} - 1) + \frac{\alpha_4 \lambda_1}{2K} \sum_{i=1}^2 (e^{2K\sigma_i} - 1), \\
 \Omega_{12} & = 2\lambda_1 K \alpha_0 p_1(t) - \lambda_1 \alpha_0 a(t) \frac{h(x)}{x} p_1(t), \\
 \Omega_{13} & = 2\lambda_1 K \alpha_0 p_2(t) - \lambda_1 \alpha_0 a(t) \frac{h(x)}{x} p_2(t), \\
 \Omega_{14} & = \lambda_1 \alpha_0 b_1(t), \\
 \Omega_{15} & = \lambda_1 \alpha_0 b_2(t), \\
 \Omega_{22} & = 2\lambda_1 K \alpha_0 p_1^2(t) - \alpha_1 \lambda_1 (1 - \mu_1), \\
 \Omega_{23} & = 2\lambda_1 K \alpha_0 p_1(t) p_2(t), \\
 \Omega_{24} & = \lambda_1 \alpha_0 p_1(t) b_1(t), \\
 \Omega_{25} & = \lambda_1 \alpha_0 p_1(t) b_2(t), \\
 \Omega_{33} & = 2\lambda_1 K \alpha_0 p_2^2(t) - \alpha_1 \lambda_1 (1 - \mu_2), \\
 \Omega_{34} & = \lambda_1 \alpha_0 p_2(t) b_1(t), \\
 \Omega_{35} & = \lambda_1 \alpha_0 p_2(t) b_2(t), \\
 \Omega_{44} & = -\alpha_3 \lambda_1 (1 - \mu_3), \\
 \Omega_{55} & = -\alpha_3 \lambda_1 (1 - \mu_4), \\
 \Omega_{66} & = -\alpha_2 \tau_i, \\
 \Delta_{11} & = 2K \alpha_0 \lambda_2 - 2a(t) \alpha_0 \frac{h(x)}{x} \lambda_2 + \alpha_1 \lambda_2 \sum_{i=1}^2 e^{2K\tau_i} + \alpha_3 \lambda_2 \sum_{i=1}^2 e^{2K\sigma_i}, \\
 & + \frac{\alpha_2 \lambda_2}{2K} \sum_{i=1}^2 (e^{2K\tau_i} - 1) + \frac{\alpha_4 \lambda_2}{2K} \sum_{i=1}^2 (e^{2K\sigma_i} - 1), \\
 \Delta_{12} & = 2\lambda_2 K \alpha_0 p_1(t) - \lambda_2 \alpha_0 a(t) \frac{h(x)}{x} p_1(t), \\
 \Delta_{13} & = 2\lambda_2 K \alpha_0 p_2(t) - \lambda_2 \alpha_0 a(t) \frac{h(x)}{x} p_2(t), \\
 \Delta_{14} & = \lambda_2 \alpha_0 b_1(t), \\
 \Delta_{15} & = \lambda_2 \alpha_0 b_2(t), \\
 \Delta_{22} & = 2\lambda_2 K \alpha_0 p_1^2(t) - \alpha_1 \lambda_2 (1 - \mu_1), \\
 \Delta_{23} & = 2\lambda_2 K \alpha_0 p_1(t) p_2(t), \\
 \Delta_{24} & = \lambda_2 \alpha_0 p_1(t) b_1(t), \\
 \Delta_{25} & = \lambda_2 \alpha_0 p_1(t) b_2(t), \\
 \Delta_{33} & = 2\lambda_2 K \alpha_0 p_2^2(t) - \alpha_1 \lambda_2 (1 - \mu_2), \\
 \Delta_{34} & = \lambda_2 \alpha_0 p_2(t) b_1(t), \\
 \Delta_{35} & = \lambda_2 \alpha_0 p_2(t) b_2(t), \\
 \Delta_{44} & = -\alpha_3 \lambda_2 (1 - \mu_3), \\
 \Delta_{55} & = -\alpha_3 \lambda_2 (1 - \mu_4), \\
 \Delta_{66} & = -\alpha_4 \sigma_i, \\
 \lambda_1 & = \frac{1}{2} \frac{\tau_i}{\tau_i + \sigma_i}, \quad \lambda_2 = \frac{1}{2} \frac{\sigma_i}{\tau_i + \sigma_i}, \quad (i = 1, 2).
 \end{aligned}$$

Proof. Choose an auxiliary functional, that is, Lyapunov functional (LF) by

$$\begin{aligned}
V(\cdot) = V(t, x) &= e^{2Kt} \alpha_0 \left[x(t) + \sum_{i=1}^2 p_i(t) x(t - \tau_i(t)) \right]^2 \\
&+ \alpha_1 \sum_{i=1}^2 \int_{t-\tau_i(t)}^t e^{2K(s+\tau_i)} x^2(s) ds \\
&+ \alpha_2 \sum_{i=1}^2 \int_{-\tau_i}^0 \int_{t+\theta}^t e^{2K(s-\theta)} x^2(s) ds d\theta \\
&+ \alpha_3 \sum_{i=1}^2 \int_{t-\sigma_i(t)}^t e^{2K(s+\sigma_i)} \tanh^2 x(s) ds \\
&+ \alpha_4 \sum_{i=1}^2 \int_{-\sigma_i}^0 \int_{t+\theta}^t e^{2K(s-\theta)} \tanh^2 x(s) ds d\theta,
\end{aligned}$$

where $\alpha_i \in \mathfrak{R}$, ($i = 0, 1, \dots, 4$), $\alpha_i > 0$, and we choose them later.

The calculation of derivative of (LF) $V(\cdot)$ with respect to the (NDE) (3) gives that

$$\begin{aligned}
\frac{dV(\cdot)}{dt} &= 2Ke^{2Kt} \alpha_0 \left[x(t) + \sum_{i=1}^2 p_i(t) x(t - \tau_i(t)) \right]^2 \\
&+ 2e^{2Kt} \alpha_0 \left[x(t) + \sum_{i=1}^2 p_i(t) x(t - \tau_i(t)) \right] \\
&\times \left[-a(t)h(x(t)) + \sum_{i=1}^2 b_i(t) \tanh x(t - \sigma_i(t)) \right] \\
&+ \alpha_1 \sum_{i=1}^2 e^{2K(t+\tau_i)} x^2(t) - \alpha_1 \sum_{i=1}^2 (1 - \tau_i'(t)) e^{2K(t-\tau_i(t)+\tau_i)} x^2(t - \tau_i(t)) \\
&- \frac{\alpha_2}{2K} \sum_{i=1}^2 [e^{2Kt} - e^{2K(t+\tau_i)}] x^2(t) - \alpha_2 e^{2Kt} \sum_{i=1}^2 \int_{t-\tau_i}^t x^2(s) ds \\
&+ \alpha_3 \sum_{i=1}^2 e^{2K(t+\sigma_i)} \tanh^2 x(t) \\
&- \alpha_3 \sum_{i=1}^2 (1 - \sigma_i') e^{2K(t-\sigma_i(t)+\sigma_i)} \tanh^2 x(t - \sigma_i(t))
\end{aligned}$$

$$\begin{aligned}
 & -\frac{\alpha_4}{2K} \sum_{i=1}^2 [e^{2Kt} - e^{2K(t+\sigma_i)}] \tanh^2 x(t) - \alpha_4 e^{2Kt} \sum_{i=1}^2 \int_{t-\sigma_i}^t \tanh^2 x(s) ds \\
 & = 2Ke^{2Kt} \alpha_0 [x^2(t) + 2x(t)p_1(t)x(t - \tau_1(t)) + 2x(t)p_2(t)x(t - \tau_2(t))] \\
 & + p_1^2(t)x^2(t - \tau_1(t)) + p_2^2(t)x^2(t - \tau_2(t)) \\
 & + 2p_1(t)p_2(t)x(t - \tau_1(t))x(t - \tau_2(t)) \\
 & + 2e^{2Kt} \alpha_0 \left[-\alpha(t) \frac{h(x)}{x} x^2(t) + x(t)b_1(t) \tanh x(t - \sigma_1(t)) \right. \\
 & + x(t)b_2(t) \tanh x(t - \sigma_2(t)) - a(t) \frac{h(x)}{x} x(t)p_1(t)x(t - \tau_1(t)) \\
 & - a(t) \frac{h(x)}{x} x(t)p_2(t)x(t - \tau_2(t)) \\
 & + (p_1(t)x(t - \tau_1(t)) + p_2(t)x(t - \tau_2(t)))(b_1(t) \tanh x(t - \sigma_1(t)) \\
 & \left. + b_2(t) \tanh x(t - \sigma_2(t))) \right] \\
 & + \alpha_1 \sum_{i=1}^2 e^{2K(t+\tau_i)} x^2(t) - \alpha_1 (1 - \tau_1'(t)) e^{2K(t-\tau_1(t)+\tau_1)} x^2(t - \tau_1(t)) \\
 & - \alpha_1 (1 - \tau_2'(t)) e^{2K(t-\tau_2(t)+\tau_2)} x^2(t - \tau_2(t)) \\
 & + \frac{\alpha_2}{2K} e^{2Kt} \sum_{i=1}^2 [e^{2K\tau_i} - 1] x^2(t) - \alpha_2 e^{2Kt} \sum_{i=1}^2 \int_{t-\tau_i}^t x^2(s) ds \\
 & + \alpha_3 \sum_{i=1}^2 e^{2K(t+\sigma_i)} \tanh^2 x(t) \\
 & - \alpha_3 (1 - \sigma_1'(t)) e^{2K(t-\sigma_1(t)+\sigma_1)} \tanh^2 x(t - \sigma_1(t)) \\
 & - \alpha_3 (1 - \sigma_2'(t)) e^{2K(t-\sigma_2(t)+\sigma_2)} \tanh^2 x(t - \sigma_2(t)) \\
 & + \frac{\alpha_4}{2K} e^{2Kt} \sum_{i=1}^2 [e^{2K\sigma_i} - 1] \tanh^2 x(t) - \alpha_4 e^{2Kt} \sum_{i=1}^2 \int_{t-\sigma_i}^t \tanh^2 x(s) ds.
 \end{aligned}$$

The assumptions of the theorem implies

$$\begin{aligned}
 & -\alpha_1 (1 - \tau_1'(t)) e^{2K(\tau_1 - \tau_1(t))} \leq -\alpha_1 (1 - \mu_1) \\
 & -\alpha_1 (1 - \tau_2'(t)) e^{2K(\tau_2 - \tau_2(t))} \leq -\alpha_1 (1 - \mu_2) \\
 & -\alpha_3 (1 - \sigma_1'(t)) e^{2K(\sigma_1 - \sigma_1(t))} \leq -\alpha_3 (1 - \mu_3)
 \end{aligned}$$

and

$$-\alpha_3 (1 - \sigma_2'(t)) e^{2K(\sigma_2 - \sigma_2(t))} \leq -\alpha_3 (1 - \mu_4).$$

Then,

$$\begin{aligned}
\frac{dV(\cdot)}{dt} &\leq 2Ke^{2Kt}\alpha_0[x^2(t) + 2x(t)p_1(t)x(t - \tau_1(t)) + 2x(t)p_2(t)x(t - \tau_2(t))] \\
&\quad + p_1^2(t)x^2(t - \tau_1(t)) + p_2^2(t)x^2(t - \tau_2(t)) \\
&\quad + 2p_1(t)p_2(t)x(t - \tau_1(t))x(t - \tau_2(t)) \\
&\quad + 2e^{2Kt}\alpha_0\left[-\alpha(t)\frac{h(x)}{x}x^2(t) + x(t)b_1(t)\tanh x(t - \sigma_1(t))\right. \\
&\quad + x(t)b_2(t)\tanh x(t - \sigma_2(t)) - a(t)\frac{h(x)}{x}x(t)p_1(t)x(t - \tau_1(t)) \\
&\quad - a(t)\frac{h(x)}{x}x(t)p_2(t)x(t - \tau_2(t)) \\
&\quad + p_1(t)b_1(t)x(t - \tau_1(t))\tanh x(t - \sigma_1(t)) \\
&\quad + p_1(t)b_2(t)x(t - \tau_1(t))\tanh x(t - \sigma_2(t)) \\
&\quad + p_2(t)b_1(t)x(t - \tau_2(t))\tanh x(t - \sigma_1(t)) \\
&\quad + p_2(t)b_2(t)x(t - \tau_2(t))\tanh x(t - \sigma_2(t)) \\
&\quad + \alpha_1\sum_{i=1}^2 e^{2K(t+\tau_i)}x^2(t) - \alpha_1e^{2Kt}(1 - \mu_1)x^2(t - \tau_1(t)) \\
&\quad - \alpha_1e^{2Kt}(1 - \mu_2)x^2(t - \tau_2(t)) \\
&\quad + \frac{\alpha_2}{2K}e^{2Kt}\sum_{i=1}^2[e^{2K\tau_i} - 1]x^2(t) - \alpha_2e^{2Kt}\sum_{i=1}^2\int_{t-\tau_i}^t x^2(s)ds \\
&\quad + \alpha_3\sum_{i=1}^2 e^{2K(t+\sigma_i)}\tanh^2 x(t) - \alpha_3e^{2Kt}(1 - \mu_3)\tanh^2 x(t - \sigma_1(t)) \\
&\quad - \alpha_3e^{2Kt}(1 - \mu_4)\tanh^2 x(t - \sigma_2(t)) \\
&\quad + \frac{\alpha_4}{2K}e^{2Kt}\sum_{i=1}^2[e^{2K\sigma_i} - 1]\tanh^2 x(t) \\
&\quad - \alpha_4e^{2Kt}\sum_{i=1}^2\int_{t-\sigma_i}^t \tanh^2 x(s)ds.
\end{aligned}$$

Since

$$\tanh^2 x \leq x^2,$$

then

$$\begin{aligned}
\frac{dV(\cdot)}{dt} &\leq e^{2Kt}\left\{2K\alpha_0 - 2\alpha_0a(t)\frac{h(x)}{x} + \alpha_1\sum_{i=1}^2 e^{2K\tau_i} + \frac{\alpha_2}{2K}\sum_{i=1}^2(e^{2K\tau_i} - 1)\right. \\
&\quad \left.+ \alpha_3\sum_{i=1}^2 e^{2K\sigma_i} + \frac{\alpha_4}{2K}\sum_{i=1}^2(e^{2K\sigma_i} - 1)\right\}x^2(t)
\end{aligned}$$

$$\begin{aligned}
 &+ [4K\alpha_0 p_1(t) - 2\alpha_0 a(t) \frac{h(x)}{x} p_1(t)] x(t)(t - \tau_1(t)) \\
 &+ [4K\alpha_0 p_2(t) - 2\alpha_0 a(t) \frac{h(x)}{x} p_2(t)] x(t)(t - \tau_2(t)) \\
 &+ 2\alpha_0 b_1(t) x(t) \tanh x(t - \sigma_1(t)) \\
 &+ 2\alpha_0 b_2(t) x(t) \tanh x(t - \sigma_2(t)) \\
 &+ [2K\alpha_0 p_1^2(t) - \alpha_1(1 - \mu_1)] x^2(t - \tau_1(t)) \\
 &+ 4K\alpha_0 p_1(t) p_2(t) x(t - \tau_1(t)) x(t - \tau_2(t)) \\
 &+ 2\alpha_0 p_1(t) b_1(t) x(t - \tau_1(t)) \tanh x(t - \sigma_1(t)) \\
 &+ 2\alpha_0 p_1(t) b_2(t) x(t - \tau_1(t)) \tanh x(t - \sigma_2(t)) \\
 &+ [2K\alpha_0 p_2^2(t) - \alpha_1(1 - \mu_2)] x^2(t - \tau_2(t)) \\
 &+ 2\alpha_0 p_2(t) b_1(t) x(t - \tau_2(t)) \tanh x(t - \sigma_1(t)) \\
 &+ 2\alpha_0 p_2(t) b_2(t) x(t - \tau_2(t)) \tanh x(t - \sigma_2(t)) \\
 &- \alpha_3(1 - \mu_3) \tanh^2 x(t - \sigma_1(t)) \\
 &- \alpha_3(1 - \mu_4) \tanh^2 x(t - \sigma_2(t)) \\
 &- \alpha_2 \sum_{i=1}^2 \int_{t-\tau_i}^t x^2(s) ds \\
 &- \alpha_4 \sum_{i=1}^2 \int_{t-\sigma_i}^t \tanh^2 x(s) ds \}.
 \end{aligned}$$

Then,

$$\frac{dV(\cdot)}{dt} \leq \sum_{i=1}^2 \frac{1}{\tau_i} \int_{t-\tau_i}^t \xi_1^T(t, s) \Omega \xi_1(t, s) ds + \sum_{i=1}^2 \frac{1}{\sigma_i} \int_{t-\sigma_i}^t \xi_2^T(t, s) \Delta \xi_2(t, s) ds,$$

where

$$\xi_1(t, s) = [x(t), x(t - \tau_1(t)), x(t - \tau_2(t)), \tanh x(t - \sigma_1(t)), \tanh x(t - \sigma_2(t)), x(s)]^T$$

and

$$\xi_2(t, s) = [x(t), x(t - \tau_1(t)), x(t - \tau_2(t)), \tanh x(t - \sigma_1(t)), \tanh x(t - \sigma_2(t)), \tanh x(s)]^T.$$

From (4), we have $\frac{dV(\cdot)}{dt} < 0$, which implies that $V(\cdot) \leq V(0, x(0))$. In view of the (LF) $V(\cdot)$, we find

$$V(0, x(0)) = \alpha_0 [x(0) + \sum_{i=1}^2 p_i(0) x(-\tau_i(0))]^2 + \alpha_1 \sum_{i=1}^2 \int_{-\tau_i(0)}^0 e^{2K(s+\tau_i)} x^2(s) ds$$

$$\begin{aligned}
& + \alpha_2 \sum_{i=1}^2 \int_{-\tau_i}^0 \int_{\theta}^0 e^{2K(s-\theta)} x^2(s) ds d\theta + \alpha_3 \sum_{i=1}^2 \int_{-\sigma_i(0)}^0 e^{2K(s+\sigma_i)} \tanh^2 x(s) ds \\
& + \alpha_4 \sum_{i=1}^2 \int_{-\sigma_i}^0 \int_{\theta}^0 e^{2K(s-\theta)} \tanh^2 x(s) ds d\theta.
\end{aligned}$$

It is also obvious that

$$\begin{aligned}
\alpha_0 [x(0) + \sum_{i=1}^2 p_i(0)x(-\tau_i(0))]^2 & = \alpha_0 [x^2(0) + 2x(0) \sum_{i=1}^2 p_i(0)x(-\tau_i(0)) \\
& + (\sum_{i=1}^2 p_i(0)x(-\tau_i(0)))^2] \\
& = \alpha_0 [x^2(0) + 2x(0) \sum_{i=1}^2 p_i(0)x(-\tau_i(0)) \\
& + p_1^2(0)x^2(-\tau_1(0)) \\
& + 2p_1(0)x(-\tau_1(0))p_2(0)x(-\tau_2(0)) \\
& + p_2^2(0)x^2(-\tau_2(0))].
\end{aligned}$$

If we use the inequality

$$2|xy| \leq x^2 + y^2,$$

then

$$\begin{aligned}
\alpha_0 [x(0) + \sum_{i=1}^2 p_i(0)x(-\tau_i(0))]^2 & \leq \alpha_0 [x^2(0) + 2x^2(0) + p_1^2(0)x^2(-\tau_1(0)) \\
& + p_2^2(0)x^2(-\tau_2(0)) + p_1^2(0)x^2(-\tau_1(0)) \\
& + p_1^2(0)x^2(-\tau_1(0)) + p_2^2(0)x^2(-\tau_2(0)) \\
& + p_2^2(0)x^2(-\tau_2(0))] \\
& = \alpha_0 [3x^2(0) + 3p_1^2(0)x^2(-\tau_1(0)) \\
& + 3p_2^2(0)x^2(-\tau_2(0))].
\end{aligned}$$

In view of the assumption $\sum_{i=1}^2 p_i^2(t) \leq 1$, it follows that

$$\begin{aligned}
\alpha_0 [x(0) + \sum_{i=1}^2 p_i(0)x(-\tau_i(0))]^2 & \leq \alpha_0 [3x^2(0) + 3x^2(-\tau_1(0)) + 3x^2(-\tau_2(0))] \\
& \leq 9\alpha_0 \sup_{\theta \in [-r_i, 0]} |\phi(\theta)|^2, \\
\alpha_1 \sum_{i=1}^2 \int_{-\tau_i(0)}^0 e^{2K(s+\tau_i)} x^2(s) ds & \leq \alpha_1 \sum_{i=1}^2 e^{2K\tau_i} \int_{-\tau_i(0)}^0 \sup_{t \in [-\tau_i(0), 0]} e^{2Kt} x^2(t) ds
\end{aligned}$$

$$\begin{aligned}
 &= \alpha_1 \sum_{i=1}^2 e^{2K\tau_i} \sup_{t \in [-\tau_i(0), 0]} e^{2Kt} x^2(t) \tau_i(0) \\
 &\leq \alpha_1 \sum_{i=1}^2 e^{2Kr_i r_i} \sup_{t \in [-\tau_i(0), 0]} e^{2Kt} x^2(t) \\
 &\leq \alpha_1 \sum_{i=1}^2 e^{2Kr_i r_i} \sup_{\theta \in [-r_i, 0]} |\phi(\theta)|^2, \\
 \alpha_2 \sum_{i=1}^2 \int_{-\tau_i}^0 \int_{\theta}^0 e^{2K(s-\theta)} x^2(s) ds d\theta &\leq \alpha_2 \sum_{i=1}^2 \int_{-\tau_i}^0 \left[\sup_{s \in [\theta, 0]} x^2(s) \int_{\theta}^0 e^{2K(s-\theta)} ds \right] d\theta \\
 &= \alpha_2 \sum_{i=1}^2 \int_{-\tau_i}^0 \sup_{s \in [-\theta, 0]} x^2(s) \left[\frac{1}{2K} e^{-2K\theta} - \frac{1}{2K} \right] d\theta \\
 &\leq \frac{1}{2K} \alpha_2 \sum_{i=1}^2 \int_{-\tau_i}^0 \sup_{s \in [-\theta, 0]} x^2(s) e^{-2K\theta} d\theta \\
 &\leq \alpha_2 \sum_{i=1}^2 \sup_{\theta \in [-r_i, 0]} |\phi(\theta)|^2 \left[-\frac{1}{4K^2} + \frac{1}{4K^2} e^{2K\tau_i} \right] \\
 &\leq \frac{1}{4K^2} \alpha_2 \sum_{i=1}^2 e^{2Kr_i} \sup_{\theta \in [-r_i, 0]} |\phi(\theta)|^2, \\
 \alpha_3 \sum_{i=1}^2 \int_{-\sigma_i(0)}^0 e^{2K(s+\sigma_i)} \tanh^2 x(s) ds &\leq \alpha_3 \sum_{i=1}^2 e^{2K\sigma_i} \int_{-\sigma_i(0)}^0 e^{2Ks} x^2(s) ds \\
 &\leq \alpha_3 \sum_{i=1}^2 e^{2K\sigma_i} \int_{-\sigma_i(0)}^0 \sup_{t \in [-\sigma_i(0), 0]} e^{2Kt} x^2(t) ds \\
 &= \alpha_3 \sum_{i=1}^2 e^{2K\sigma_i} \sup_{t \in [-\sigma_i(0), 0]} e^{2Kt} x^2(t) \sigma_i(0) \\
 &\leq \alpha_3 \sum_{i=1}^2 e^{2Kr_i r_i} \sup_{t \in [-\sigma_i(0), 0]} e^{2Kt} x^2(t), \\
 \alpha_4 \sum_{i=1}^2 \int_{-\sigma_i}^0 \int_{\theta}^0 e^{2K(s-\theta)} \tanh^2 x(s) ds d\theta &\leq \alpha_4 \sum_{i=1}^2 \int_{-\sigma_i}^0 \left[\sup_{s \in [\theta, 0]} x^2(s) \int_{\theta}^0 e^{2K(s-\theta)} ds \right] d\theta \\
 &= \alpha_4 \sum_{i=1}^2 \int_{-\sigma_i}^0 \sup_{s \in [-\theta, 0]} x^2(s) \left[\frac{1}{2K} e^{-2K\theta} - \frac{1}{2K} \right] d\theta \\
 &\leq \frac{1}{2K} \alpha_4 \sum_{i=1}^2 \int_{-\sigma_i}^0 \sup_{s \in [-\theta, 0]} x^2(s) e^{-2K\theta} d\theta
 \end{aligned}$$

$$\begin{aligned}
&\leq \alpha_4 \sum_{i=1}^2 \sup_{\theta \in [-r_i, 0]} |\phi(\theta)|^2 \left[-\frac{1}{4K^2} + \frac{1}{4K^2} e^{2K\sigma_i} \right] \\
&\leq \frac{1}{4K^2} \alpha_4 \sum_{i=1}^2 e^{2Kr_i} \sup_{\theta \in [-r_i, 0]} |\phi(\theta)|^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
V(0, x(0)) &= \alpha_0 \left[x(0) + \sum_{i=1}^2 p_i(0)x(-\tau_i(0)) \right]^2 + \alpha_1 \sum_{i=1}^2 \int_{-\tau_i(0)}^0 e^{2K(s+\tau_i)} x^2(s) ds \\
&\quad + \alpha_2 \sum_{i=1}^2 \int_{-\tau_i}^0 \int_{\theta}^0 e^{2K(s-\theta)} x^2(s) ds d\theta + \alpha_3 \sum_{i=1}^2 \int_{-\sigma_i(0)}^0 e^{2K(s+\sigma_i)} \tanh^2 x(s) ds \\
&\quad + \alpha_4 \sum_{i=1}^2 \int_{-\sigma_i}^0 \int_{\theta}^0 e^{2K(s-\theta)} \tanh^2 x(s) ds d\theta \\
&\leq [9\alpha_0 + (\alpha_1 + \alpha_3) \sum_{i=1}^2 r_i e^{2Kr_i} + (\alpha_2 + \alpha_4) \frac{1}{4K^2} \sum_{i=1}^2 e^{2Kr_i}] \sum_{i=1}^2 \sup_{\theta \in [r_i, 0]} |\phi(\theta)|^2 \\
&\equiv M.
\end{aligned}$$

We can now write

$$\left| x + \sum_{i=1}^2 p_i(t)x(-\tau_i(t)) \right|^2 \leq M_1 e^{-2kt},$$

where $M_1 = \frac{M}{\alpha_0} > 0$. For $\forall \varepsilon \in (0, \min\{2K, -\frac{2}{r_i} \log |p_i(t)|\})$ and $v > 0$, the inequality $xy \leq vx^2 + \frac{1}{v}y^2$ for any $x, y \in R$ implies that

$$\begin{aligned}
e^{\varepsilon t} |x|^2 &\leq (1+v)e^{\varepsilon t} \left| x(t) + \sum_{i=1}^2 p_i(t)x(t-\tau_i(t)) \right|^2 + \frac{1+v}{v} e^{\varepsilon t} \sum_{i=1}^2 |p_i(t)x(t-\tau_i(t))|^2 \\
&\leq (1+v)M_1 + \frac{1+v}{v} \sum_{i=1}^2 |p_i(t)|^2 |x(t-\tau_i(t))|^2 e^{\varepsilon r_i} e^{\varepsilon(t-\tau_i(t))}.
\end{aligned}$$

And from $\forall \varepsilon \in (0, \min\{2K, -\frac{2}{r_i} \log |p_i(t)|\})$, we have $\sum_{i=1}^2 |p_i(t)|^2 e^{\varepsilon r_i} < 1$. Thus, if we choose $v > 0$ sufficiently large, then it follows that

$$\gamma = \frac{\sum_{i=1}^2 |p_i(t)|^2 (1+v) e^{\varepsilon r_i}}{v} < 1.$$

Therefore,

$$e^{\varepsilon t}|x|^2 \leq (1 + v)M_1 + \gamma \sum_{i=1}^2 |x(t - \tau_i(t))|^2 e^{\varepsilon(t - \tau_i(t))} \quad (\forall T \geq 0),$$

$$\sup_{0 \leq t \leq T} \{e^{\varepsilon t}|x(t)|^2\} \leq (1 + v)M_1 + \gamma \sup_{\theta \in [r_i, 0]} |\varphi(\theta)|^2 + \gamma \sup_{0 \leq t \leq T} \{e^{\varepsilon t}|x(t)|^2\}, \quad (i = 1, 2).$$

Consequently, we obtain

$$\sup_{0 \leq t \leq T} \{e^{\varepsilon t}|x|^2\} \leq \frac{(1 + v)M_1 + \gamma \sup_{\theta \in [r_i, 0]} |\varphi(\theta)|^2}{1 - \gamma}, \quad (i = 1, 2).$$

When $T \rightarrow +\infty$, we can find that

$$\sup_{0 \leq t \leq \infty} \{e^{\varepsilon t}|x(t)|^2\} \leq \frac{(1 + v)M_1 + \gamma \sup_{\theta \in [r_i, 0]} |\varphi(\theta)|^2}{1 - \gamma}, \quad (i = 1, 2).$$

Thus,

$$|x| \leq M_2 e^{-\alpha t},$$

where

$$M_2 = \sqrt{\frac{(1 + v)M_1 + \gamma \sup_{\theta \in [r_i, 0]} |\varphi(\theta)|^2}{1 - \gamma}} > 0, \alpha = \frac{\varepsilon}{2} > 0, \quad (i = 1, 2).$$

This ends the proof. □

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