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Q-functional applications

Abstract The Q-measure indicate a weak* limit of the barycenter of a sequence of Borel measurable functions. In this paper, we will look only at Q-functional. Q-functional is defined by Q- measure, it is useful in the field of optimization. Computational results for Q-functional are presented and compared with Young functional. The obtained analytical results demonstrate relative error in Q-functional is lesser compared to Young functional.

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1. Introduction Sequences of bounded functions that are oscillatory and concentrated in nature often arise in many practical problems. For example, in the non-convex optimization problem when the integrand is the above-described sequence, the classical minimizer does not exist(cf.[2]). In this case, the minimizing sequence, which minimizes the integral, oscillates rapidly. In addition, it strongly diverges but remains uniformly bounded. Moreover, even the weak* convergent sub-sequences of such bounded sequence oscillate rapidly around the weak* limit.

To overcome this problem (cf.[12]), the generalized limits of such sequences are conceptualized by enlarging the space of functions to the measure spaces. The idea is to assign the limit, not as a *usual* function but as a probability measure-valued function, referred as *Young measure* . It is successfully used to capture oscillatory behavior of sequences of function; however, it fails to represent some case of concentration property of the sequence (cf.[12]). Some oscillatory sequences with concentration property in a bounded domain are constructed in [13]. Geometrical interpretation of Young measure is introduced in [11].

The major motivation of this work is to find a functional value using Q-measure. The Q- measure is weakly stable family of probability measure that can capture oscillatory and concentrating behavior of sequence of functions. It is first introduced by Jisha in[8]. This innovative concept of a new measure of a function is given by generating a sequence of functions through singular or regular perturbation. Q- measure application in PDE is discussed in this article. The application of Q- measure in PDE is discussed in [8]. In this manuscript discussed about Q-functional (QF), when Q-measure acts on a Caratheodory function and a numerical algorithm is given to compute it. It can be noticed that by increasing the sample size on uniform mesh, the QF value converges to an exact Q- functional value. In the calculus of variations or in the static problems in mechanics, the integral functionals of the form equation (8) acting on an appropriate function space play the fundamental role.

In section 2, we discuss the construction of the barycenter of a sequence of function is discussed. In section 3, the Q-functional for singular and regular perturb function and the relative error have been discussed. Detailed study of Q- functional values with respect to regular and singular perturb function with different types of mesh is provided in section 4.

2. Barycenter of sequence of functions Let A_i be a disjoint partition of $R = \prod_{i=1}^{n} [a_i, b_i]$ (see [1, 8]), and v_i is a given sequence of Borel measurable function in R and $\overline{u}_i \in L^1(A_i)$, for all A_i. Then we can define Barycenter of the sequence of n^{th} term of functions as equation (1)

$$\overline{u}_n(x) = \lim_{k\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{k} \frac{u_i \chi_{A_j}(x)}{n}. \tag{1}$$

where

$$u_i = \begin{cases} v_i & \text{if } x \in A_j \\ 0 & \text{if } x \notin A_j \end{cases} . \tag{2}$$

Barycenter of sequence of functions is

$$\overline{u}(x) = \lim_{n\to\infty} \overline{u}_n(x), \tag{3a}$$

$$= \lim_{n\to\infty} \lim_{k\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{k} \frac{u_i \chi_{A_j}(x)}{n}, \tag{3b}$$

The existence of Q- measure is discussed in [8].

Definition 2.1 Let $u_n \in L^2(K), K \subset \mathbb{R}$. The function $u(x)$ is said to be regularly perturbed with respect to $L^2(K)$ norm if it satisfied for all positive $\epsilon_n, n = 1, 2....$

$$||u(x - \epsilon_n) - u(x)|| \to 0 \text{ as } \epsilon_n \to 0 .$$

Otherwise it is said to be singularly perturbed.

2.1.1. Construction of sequence $u_n(x)$ from singularly perturbed function $u(x)$[8]

To construct a sequence of the function $u_n(x)$, we choose a sequence $\epsilon_n > 0$, $n = 1, 2, ...$ with $\sum_{i=1}^{\infty} \epsilon_i < \infty$ such that for $x \in K, c > 0$, $x \pm \epsilon_n + c \in K \subset \mathbb{R}$ and $u(x \pm \epsilon_n + c) \nrightarrow u(x)$
as ϵ_n tends to 0.

$$u_{2m+1}(x) = \begin{cases} u(x - \epsilon_n + c) & \text{if } x - \epsilon_n + c \in K \\ u(x) & \text{if } x - \epsilon_n + c \notin K, \end{cases} \tag{4}$$

$$u_{2m}(x) = \begin{cases} u(x + \epsilon_n + c) & \text{if } x + \epsilon_n + c \in K \\ u(x) & \text{if } x + \epsilon_n + c \notin K, \end{cases} \tag{5}$$

where $m = 1, 2, 3, ..., n$.

2.1.2. Construction of the sequence $u_n(x)$ from regularly perturbed function $u(x)$

In this case, we choose a sequence $\epsilon_n > 0$ with $\sum_{i=1}^{\infty} \epsilon_i < \infty$ such that for $x \in K$, $x \pm \epsilon_n \in K$, define the sequences $u_n(x)$.

$$u_{2m+1}(x) = \begin{cases} u(x - \epsilon_n) & \text{if } x - \epsilon_n \in K \\ u(x) & \text{if } x - \epsilon_n \notin K, \end{cases} \tag{6}$$

$$u_{2m}(x) = \begin{cases} u(x + \epsilon_n) & \text{if } x + \epsilon_n \in K \\ u(x) & \text{if } x + \epsilon_n \notin K. \end{cases} \tag{7}$$

Where $m = 1, 2, 3...n$.

We applied average concept $\overline{u}_n(x)$ of $u_n(x)$ in Young measure concept. We remarks that if $\overline{u}(x) = u(x)$ in a bounded domain, Q-measure is equal to Young measure almost everywhere. One of the main advantage of Q-measure is that it's possible to find the Q- measure in the unbounded domain provided Borel measurable function lies in $L^2(K), K \subseteq \mathbb{R}$.

To construct the sequences of function in equation (6) and (7) is useful for various fields especially calculus of variation[9], signals analysis problems and atmospheric sciences.

Remark 2.2

- Assume $u(x)$ be a function, which is not in the form of $u_n(x)$. While constructing $u_n(x)$ in (6) and (7), infer that Q measure is the weak* measurable map of the average of symmetric perturbation(or symmetric disturbance) of the function $u(x)$.

- We also remarks that Q-measure corresponding to a regularly perturbed piece-wise continuously differentiable function is equal to Young measure. In that case, both Young functional (YF) and Q-functional values are equal.

3. Q- functional related to certain class of sequence of function

Table 1: Young functional values with uniform mesh size at $N = 10000, a = 0, b = 2, c = 1, d = 4.$ and $H(x, y)$ is the Caratheodory function.

Sl.No.	$u(x)$	$H(x,y)$	Sample size	YF	Relative error $= \frac{E-YF}{E}$	E=exact solution
1	$(x-1)^2$	xy^4	100	1.0664	0.27983	0.83333
			1000	0.9286	0.1143245	
			10000	0.8316	0.0020760	
2	$(x-1)^2$	$x+y$	100	8.3534	0.0172470	8.5
			1000	8.5102	0.0012000	
			10000	8.5141	0.0016588	
3	$x+1$	$x+y^7$	100	1436.8	0.0161051	1237.5
			1000	1334.1	0.0780600	
			10000	1234.5	0.0024242	
4	$x+1$	y^{10}	100	18930	0.2163520	24156.27273
			1000	23585	0.0236491	
			10000	24465	0.0127804	
5	$(x-1)^2$	y^{10}	100	0.1049	0.2656999	0.142857142
			1000	0.1318	0.0773999	
			10000	0.1472	0.0346000	
6	$x+1$	y^5	100	185.1772	0.017460	182
			1000	186.6034	0.025293	
			10000	183.2184	0.006695	
7	$(x-1)^2$	y^5	100	0.1589	0.417366	0.272727
			1000	0.2511	0.0792991	
			10000	0.2629	0.036032	

In the elasticity theory, we need to find the minimum of the energy functional of the considered body. Generally, the functional is of the form

$$J(u) = \int_{\Omega} H(x, u(x), \nabla u(x))dx \tag{8}$$

where Ω is an elastic body under consideration, u denotes the displacement and H represents the density of the internal energy of Ω. The crucial point in this problem is the energy functionals are not often quasiconvex with respect to the third variable. This is the reason, the functional J is bounded from below does not attain its infimum. Moreover, elements of any minimizing sequence of J are the functions that rapidly oscillate. This oscillating nature of the minimizing sequences shows itself as *micro structure* that can be observed in nature. It turns out that in the case of non(quasi)convex variational problems, the Q-functional concept is more useful.

In this section, Q functional is introduced and compared with the Young functional in the case of continuously differentiable function with non-zero derivative in a given domain. The Q-functional is computed by extending the algorithm given by [5, 6, 7, 10] for computing Young's functional using the Monte-Carlo method.

In the table 1, the Young functional values are obtained by using simulation algorithm 1, the Young functional values using simple Mont Carlo simulation. The corresponding algorithm and required details are given in the appendix. From the table 1, it can be observed that error increases with the increase in the power of y. However, with the increase in sample size of the numerical value in Young functional converge to the exact Young functional value.

Example 3.1 (Convergence of error for YF with $u(x) = (x-1)^2$) In the following figure 1 the young functional values of the function $u(x) = (x-1)^2, x \in [0,2]$ and $H(x,y) = xy^4, y \in [0,4]$ are plotted for different sample size. These values show that error tends to zero with the increase in the sample size, however, compared to Q functional it exhibits more fluctuation.

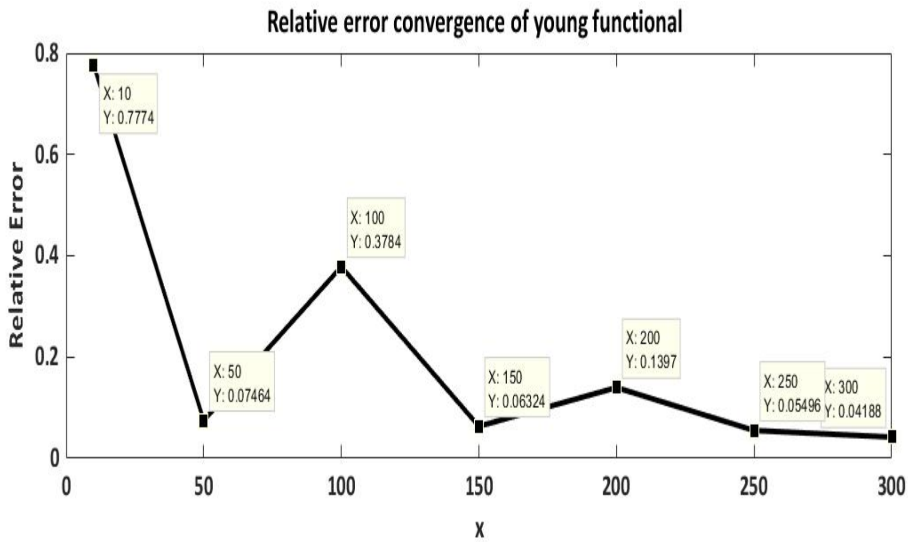

Figure 1: Representation of the relative error convergence of YF

3.1. Computation of Q functional

Definition 3.2 (Generalized Q- measure generated by the sequence) Let u_k be the sequence of continuously differentiable Borel measurable functions with non zero derivatives in a given domain and $\overline{u}_k(x)$ takes from the equation(1). We says, that a family $\nu = (\nu_x)_{x\in\Omega}$ of probability measures is a Generalized Q- measure generated by the sequence u_k if for any Caratheodory function $H : \Omega \times \mathbb{R}^3 \to \mathbb{R} \cup \{\infty\}$,

$$\int_\Omega H(x, \overline{u}_k(x))dx \to \int_\Omega \int_K H(x, y)d\nu_x(y)dx, \text{ as } k \to \infty \tag{9}$$

The procedure to compute the Q functional using Monte-Carlo method is given in following Algorithm 1 (compute QF value). This algorithm can be viewed as an extension of the procedure [6] for computing Young functional.

Algorithm 1 Compute QF value

procedure QF VALUE(u)
    Set $k = 1$
    input $u(x), H(x, y), a, b, c, d$
    Choose sample size N
    Divide the interval in to n equal disjoint interval $[a_i, b_i], i = 1, 2, ..., n+1$
    $y = \overline{u_k}(t)$
    for each integer t **do**
        $t = Random((a_i, b_i))$
        $y = f(t)$
        $z[k] = INT(H(x, y), x, c, d)$
    end for
    $z = (z[1], z[2], ..., z[n])$
    QF=mean(sample)
    Return QF
end procedure

From the comparison of Q functional and Young functional values indeed the following result. In Table 2, the computed Q-functional values (MQF) using Monte Carlo based Algorithm 1 is given. It can be seen from the given results that the computed values of Q-functional converge the analytical values with the increase of sample size. It is also evident from Table 1 and Table 2 that the relative error in the computation of Q functional is less compared to the relative error in computing the Young functional. We notice that in all test problems the endpoints of the intervals are included in the computation of Q-functional.

4. Detailed study of Q- functional values with respect to regular and singular perturb function with different types of mesh

In this section discussing the Q-functional values with respect to regular and singular perturb function with uniform, monotone and unequal mesh are given.

Sl.No.	$u(x)$	$H(x,y)$	Sample size	MQF	Relative error	Exact value of QF
1	$(x-1)^2$	xy^4	100	0.8861	0.06332	0.833
			1000	0.8372	0.00464	
			10000	0.8313	0.00244	
2	$(x-1)^2$	$x+y$	100	8.5201	0.002365	8.5
			1000	8.5020	0.000235	
			10000	8.5020	0.000235	
3	$x+1$	$x+y^7$	100	1287.5	0.0404	1237.5
			1000	1241.1	0.00291	
			10000	1236.3	0.00097	
4	$x+1$	y^{10}	100	25685	0.066328	24156.27
			1000	24256	0.00413	
			10000	24120	0.001502	
5	$(x-1)^2$	y^{10}	100	0.1663	0.1641	0.1428
			1000	0.1445	0.0115	
			10000	0.1418	0.0074	
6	$x+1$	y^5	100	187.1504	0.0283	182
			1000	182.3978	0.002	
			10000	181.8982	0.000559	
7	$(x-1)^2$	y^5	100	0.2979	0.0923	0.2727
			1000	0.2743	0.00577	
			10000	0.2718	0.003399	

Table 2: Q functional values for regular perturb functions with uniform mesh at $N = 10000, a = 0, b = 2, c = 1,$ and $d = 4$.

4.1. Comparison of the Q-functional values by regular perturb function with different types of mesh

Let us consider the function $u(x) = (x-1)^2$ & $H(x,y) = xy^4$ and exact solution of QF is 0.83333 from equation (13). Using the above algorithm, the sample size and different mesh on the MQF values.

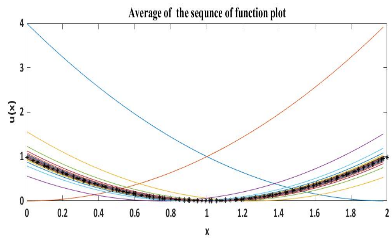

Figure 2: Black star graph is the average of function using Data 1.

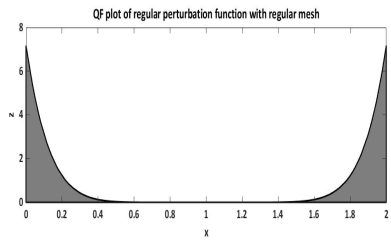

Figure 3: Total area represent the Q-functional value of the data 1

1. **Q-functional value of Regular perturbation function with uniform mesh:**
   From the figure 4, it is evident that mesh with equal size compared to mesh with the unequal size has a less relative error even though the sample size is increased. As we increase, the sample size, the relative

error converges fastly to zero and has less fluctuation. In compared to Young functional error analysis, Q- functional fastly converges to zero.

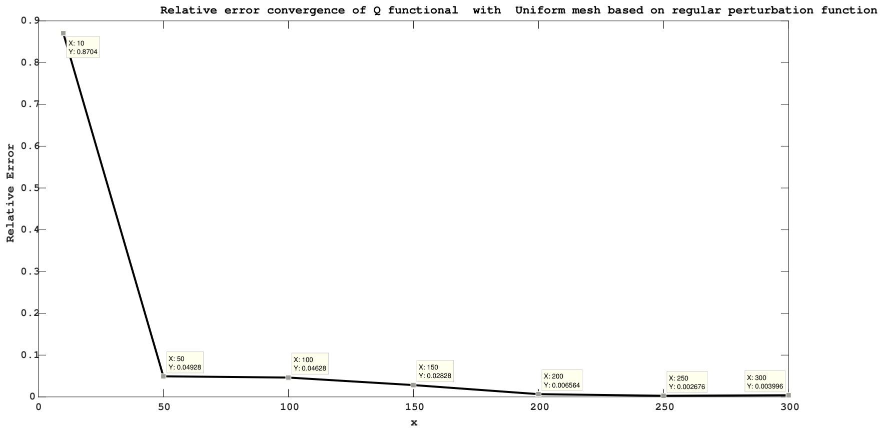

Figure 4: Representation of the relative error in different sample size

2. **Regular perturb function with Unequal mesh size**
The nonuniform mesh has also less error as compared to monotonic mesh. An increasing sample size, error converges to zero as depicted in figure 5. In QF plot, total shaded area denotes the Q- functional value. In this case, the error has less fluctuation as shown in figure 5.

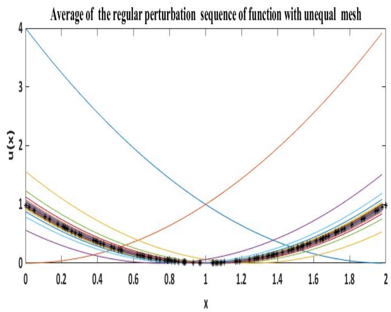

Figure 5: Black color star graph denote the average of the functions using Data 1.

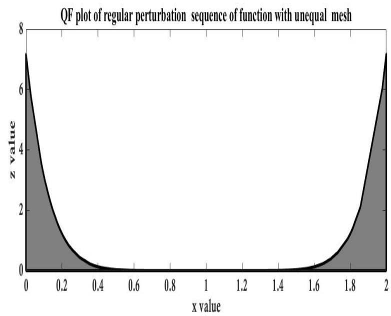

Figure 6: QF plot using for Data 1.

3. **Regular perturb function with Monotonic mesh**
From the given three error analysis graph(figure 1, figure 4 and figure 7), we can observe that the mesh with equal length has a less relative error. On compared to mesh with unequal size, monotonic mesh has a more relative error. The error is due to randomness, but the errors can be controlled by the characteristic function in the definition of $\overline{u}(x)$.

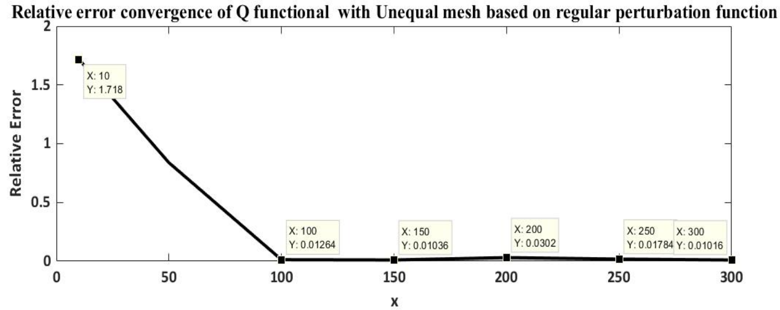

Figure 7: Represents the relative error of different sample size with monotonic mesh.

Therefore, mesh with equal length has a less relative error. The following error graph 10, has more fluctuation as compared to previous graphs(figure 5).

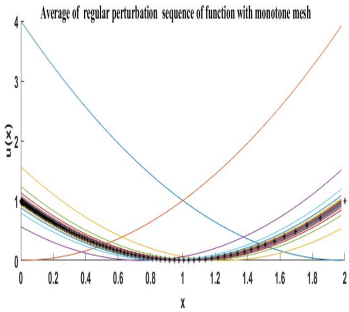

Figure 8: Black color $_*$ graph denote the average graph using interpolation for Data 1.

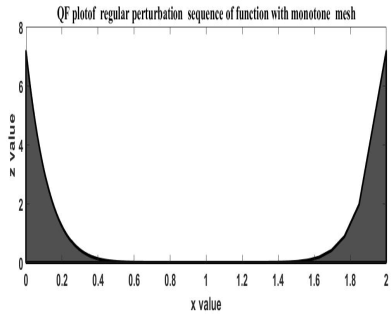

Figure 9: QF plot using for Data 1

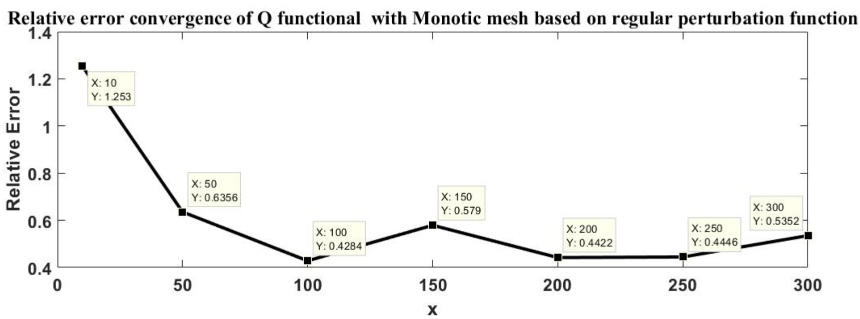

Figure 10: Represent the relative error of different sample size with monotonic mesh

In the next paragraph, we discuss the Q-functional for singular perturb function.

4.2. Q-functional for Singular perturb function
For construction of the table 3, we consider the sequence $u_n(x)$ from singular perturbed function $u(x)$, choose a sequence $\epsilon_n = \frac{1}{n^2} > 0$, $n = 1, 2, ...$ with $\sum_{i=1}^{\infty} \epsilon_i = \frac{1}{n^2} < \infty$ such that for $x \in [0, 2]$,
$c = 0.01 > 0$, $x \pm \epsilon_n \in [0, 2] \subset \mathbb{R}$ and $u(x - \epsilon_n) \nrightarrow u(x)$ as $\epsilon_n$tends to 0.

$$u_{2m+1}(x) = \begin{cases} u(x - \epsilon_n + 0.01) & \text{if } x - \epsilon_n + 0.01 \in [0, 2] \\ u(x) & \text{if } x - \epsilon_n + 0.01 \notin [0, 2] \end{cases}, \quad (10)$$

$$u_{2m}(x) = \begin{cases} u(x + \epsilon_n + 0.01) & \text{if } x + \epsilon_n + 0.01 \in [0, 2] \\ u(x) & \text{if } x + \epsilon_n + 0.01 \notin [0, 2] \end{cases}, \quad (11)$$

where, $m = 1, 2, 3, \cdots, n$. Using this sequence of the function, we calculate $\overline{u}$ and compute the Q-functional values from equation (9).

Sl.No.	$u(x)$	$H(x,y)$	Sample size	MQF	Relative error	Exact solution of QF
1	$(x-1)^2$	xy^4	100	0.8886	0.06154	0.837089
			1000	0.8309	0.007393	
			10000	0.8373	0.0025	
2	$(x-1)^2$	$x+y$	100	8.5307	0.003599	8.5
			1000	8.5142	0.001668	
			10000	8.5125	0.001468	
3	$x+1$	$x+y^7$	100	1266.9	0.023576	1237.032
			1000	1224.7	0.01007	
			10000	1220.3	0.01371	
4	$x+1$	y^{10}	100	24981	0.029735	24244.99
			1000	23726	0.02187	
			10000	23601	0.02729	
6	$x+1$	y^5	100	185.5748	0.017306	182.3633
			1000	181.0008	0.00753	
			10000	181.8982	0.00256	

Table 3: Q- functional value with uniform mesh at $N = 10000, a = 0, b = 2, c = 1,$ and $d = 4$.

Table 4.2 represents the Q- functional table corresponding to the singular perturbed function of u. The boundary points of u have been included in all the cases. This Table 3 shows more error compared to the Q- functional corresponding to regular perturb function (see Table 2). On comparing Table 1 and Table 2, the Table 3 shows more error. It is evident from Table 3, as we increase number of points, the error does not strictly decreases(refer data 3 and 4 in Table 3). If the caratheodory function $H(x, y)$ is linear in y, then Q- functional in singular perturbation has less error.

4.2.1. Comparison of the Q- functional values in different types of mesh with Singularly perturb function.

In this subsection, we discuss the affects of different types of mesh on the Q-functional value. We consider the function $u(x) = (x-1)^2$ & $H(x,y) = xy^4$ and exact value of QF is 0.837089. The total area of the shaded region denotes the Q functional value.

We compare the average graph of singular perturb and regular perturb function. The singular perturb function has more relative error as well as more scattering from $\overline{u}$.

1. **Singular perturb function with equal mesh**
   On compared to other tables of the singular perturbation case, the uniform mesh has a less relative error. As the sample size increases the relative error decreases at different mesh cases Also, the convergence of the error graph to zero is not uniform in all the cases. If the sample size is more than 10000, it will converge slowly to zero in all the three cases.

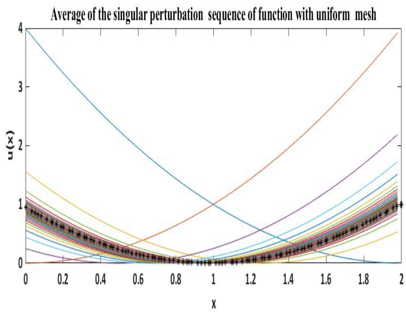

Figure 11: Black color $_*$ graph denote the average of the sequence of functions for Data 1 from table 3 using sample size 100 points

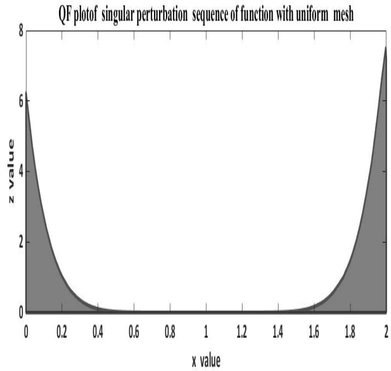

Figure 12: QF plot using for Data 1 in table 6 using 100 points

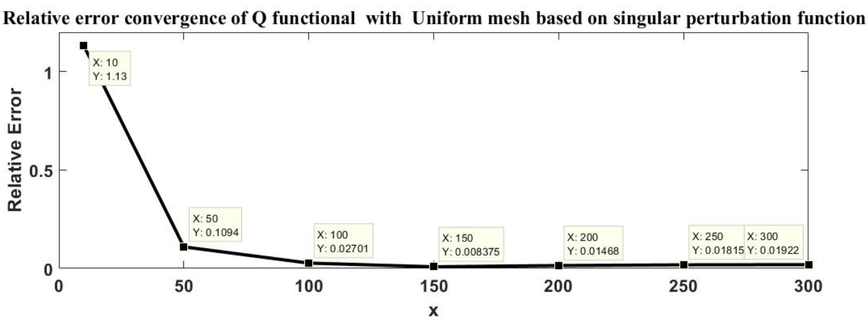

Figure 13: represent the relative error of different sample size with uniform mesh

In the Figure 5, we represent Q- functional value with uniform mesh and error graph 13 shows less fluctuation.

2. **Singular perturb function with unequal mesh**
In figure 16, due to the randomness of variable x, error analysis graph exhibits more fluctuations. As the increase the sample size, it restricts the probability of choosing the points x_i. As compared to monotonic mesh, it converges fastly, provided that the remains sample size above 10000.

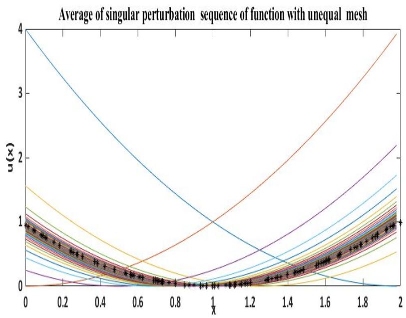

Figure 14: Black color $*$ graph denote the average of the function of Data 1 from table 6 using 100 points

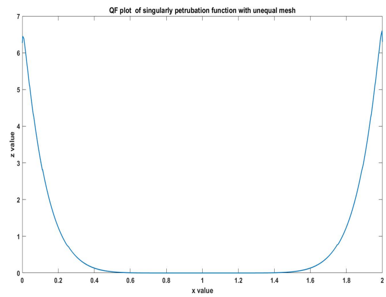

Figure 15: QF plot using for Data 1 in Table 3 using 100 points

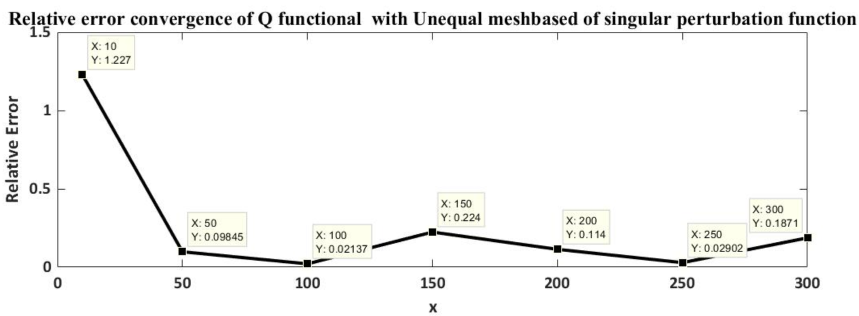

Figure 16: Above figure represent the relative error of different sample size with unequal non monotonic mesh

3. **Singular perturb function with Monotonic mesh**
Let $u(x) = (x-1)^2$, $x \in [0,2]$. Q-functional value of Singular perturb function and regular perturb function with monotonic mesh case indeed the following result. It is clear that Singular perturb function and regular perturb function Q-functional value have more relative error and more fluctuations. It is due to the randomness of $x_i \in [0,2]$, where $i = 1, 2, 3...n$.

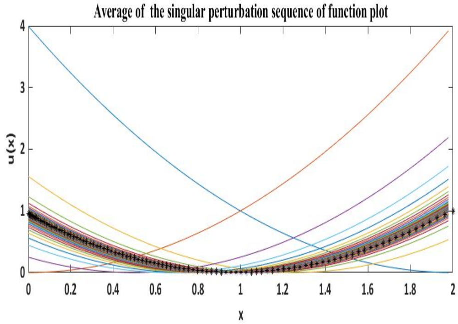

Figure 17: Black color $*$ graph denotes the average of the sequence of a function using interpolation for Data 1 from table 6 using 100 points

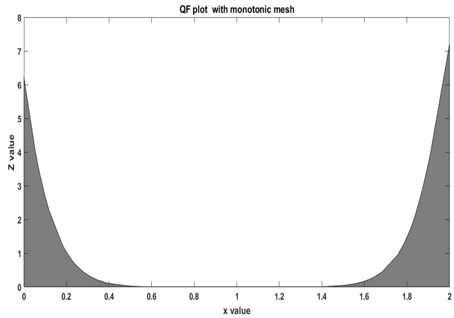

Figure 18: QF plot using for Data 1 in table 3 using 100 points

From the available tables and following figure 19, we can conclude that the value of Q- functional in singular and regular case with monotonic mesh size have more error.

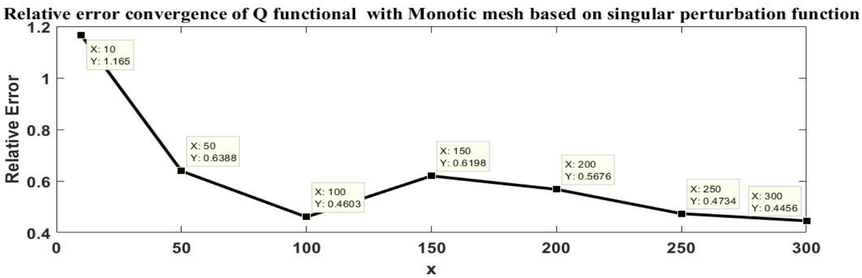

Figure 19: Representation of the relative error of different sample size with monotonic mesh.

5. Summary and conclusions.

A notion of average for underlying sequences to define the Q-measure is given and also applied in atmosphere science. The Q-measure is used to define Q-functional and an algorithm is given to compute it. In general, it is difficult to find out the value of Q-functional value analytically. Both analytical and computed results are show that relative error in Q-functional values is less as compared to Young functional. The best convergence of Q-functional is obtained in the case of uniform intervals. We can use modified Mont Carlo simulation it will help to reduced error compared to the simple Monte-Carlo method.

This study has future research directions in the calculus of variations or in the static problems in mechanics.

Appendices

A. Young functional

Grzybowski and Puchala (2017, 2018) defined a function $u : [a,b) \to K \subset \mathbb{R}$, where $[a,b)$ is non degenerate interval in R with values in a compact set K. Let u^e indicate the periodic extension of u to the entire real line with the period $T = b - a$. Define, a sequence (u_k) by $u_k(x) = u^e(kx)$. The sequence (u_k) will be a rapidly oscillated sequence with uniform representation u, and denoted as $ROSU(u)$. The domain K of the function u_k need not be equivalent to $[a,b)$ where $[a,b)$ is the domain K of the function u. In any neighborhood of $x \in K$ the behavior of u_k as k tends to infinity, is exactly the same. This means that the classical Young measure generated by u_k is homogeneous. Recall from probability theory, that random variable U correspond to the function u is uniformly distributed on the interval $[a,b)$, provided its probability distribution has a density which equals $\frac{1}{(b-a)}$.

We recall the following theorem as given in Grzybowski and Puchała (2018):

THEOREM A.1 *[3] Let Young measure ν' of the Borel measurable function $f : \Omega \subset \mathbb{R}^d \to K \subset \mathbb{R}^l$. Then ν' is the probability distribution corresponding to the random variable $Y = f(U)$, where U has a uniform distribution on K.*

DEFINITION A.2 (CLASSICAL YOUNG MEASURE GENERATED BY THE SEQUENCE) Let the family probability measures $\nu = (\nu_x)_x \in K$ is a classical Young measure[1] generated by the sequence of oscillated Borel functions (u_k) if for any Caratheodory function $H : \Omega \times \mathbb{R}^3 \to \mathbb{R} \cup \{\infty\}$ following holds.

$$\int_\Omega H(x, u_k(x))dx \to \int_\Omega \int_K H(x,y)d\nu_x(y)dx, \quad \text{as } k \to \infty. \tag{12}$$

EXAMPLE A.3 Let $u(x) = (x-1)^2, x \in [0,2]$ and Caratheodory function $H(x,y) = xy^4$, $y \in [0,4]$ exact value of Young functional and Q- functional are same because by construction of $\overline{u}$ strongly converge to u in the case of regular perturbation.

$$\begin{aligned} YF(H) &= \int_\Omega \int_K H(x,y)d\nu(y)dx, && (13)\\ &= \int_0^2 \int_1^4 xy^4 g(y)\ dy\ dx, && (14)\\ &= \int_0^2 \int_1^4 xy^4 \frac{1}{2\sqrt{y}}\ dy\ dx, \text{ where } g(y) \text{ is the density function} && (15)\\ &= 0.83333. && (16) \end{aligned}$$

[1]The term classical use to represent generalized Q-measure.

The following algorithm has been described in [6] for computing Young functional value.

Algorithm 1 Compute YF value

procedure YF VALUE(u)
  Set $k = 1$
  input $u(x), H(x, y)$
  for each integer t **do**
    $t = Random((a, b))$
    $y = f(t)$
    $z[k] = INT(H(x, y), x, c, d)$
  end for
  $z = (z[1], z[2], ..., z[n])$
  YF=mean(sample)
  Return YF
end procedure

Figure 20: Algorithm for Young functional

Supplementary Materials: All data used in our experiments have been produced with MATLAB random number generators and no external data sets have been used. The data sets generated and analyzed during the current study are available from the corresponding author on reasonable request.

Conflicts of Interest: Conflicts of Interest The authors declare that they have no confict of interest.

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Zastosowania Q-funkcjonałów.
CR Jisha

Abstract Miara Q wyznacza słabą* granicę barycentrum ciągu funkcji borelowskich. W tym artykule przyjrzymy się tylko funkcjonałom Q. Q-funkcjonalność jest definiowana przez miarę Q i jest przydatna w zastosowaniu do zadań optymalizacji. Przedstawiono wyniki obliczeń dla funkcjonału Q i porównano je z funkcjonałem Younga. Otrzymane wyniki analityczne pokazują, że błąd względny w funkcjonale Q jest mniejszy w porównaniu z funkcjonałem Younga.

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