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CONTINUOUS SPECTRUM OF STEKLOV NONHOMOGENEOUS ELLIPTIC PROBLEM

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Abstract. By applying two versions of the mountain pass theorem and Ekeland's variational principle, we prove three different situations of the existence of solutions for the following Steklov problem:

$$\begin{aligned} \Delta_{p(x)} u &= |u|^{p(x)-2}u \quad \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} &= \lambda |u|^{q(x)-2}u \quad \text{on } \partial\Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded smooth domain and $p, q \colon \overline{\Omega} \to (1, +\infty)$ are continuous functions.

Keywords: $p(x)$-Laplacian, Steklov problem, critical point theorem.

Mathematics Subject Classification: 35J48, 35J66.

1. INTRODUCTION

Motivated by the developments in elastic mechanics, electrorheological fluids and image restoration [3, 16, 18, 21, 22], the interest in variational problems and differential equations with variable exponents has grown in recent decades; see for example [4, 9, 11, 14]. We refer the reader to [1, 2, 5, 6, 19, 20] for developments in $p(x)$-Laplacian equations.

The aim of this article is to analyse the existence of solutions of the nonhomogeneous eigenvalue problem

$$\begin{aligned} \Delta_{p(x)} u &= |u|^{p(x)-2}u \quad \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} &= \lambda |u|^{q(x)-2}u \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded smooth domain, $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative on $\partial\Omega$, p is a continuous function on $\overline{\Omega}$. The main interest in studying such problems arises from the presence of the $p(x)$-Laplace operator $\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, which is a generalization of the classical p-Laplace operator $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ obtained in the case when p is a positive constant. Many authors have studied the inhomogeneous Steklov problems involving the p-Laplacian [13]. The authors have studied this class of inhomogeneous Steklov problems in the cases of $p(x) \equiv p = 2$ and of $p(x) \equiv p > 1$, respectively. In [4], the authors have studied the case $q(x) = p(x)$ for all $x \in \overline{\Omega}$, they proved that the existence of infinitely many eigenvalue sequences. Unlike the p-Laplacian case, for a variable exponent $p(x)$ ($\neq$ constant), there does not exist a principal eigenvalue and the set of all eigenvalues is not closed under some assumptions. Finally, they presented some sufficient conditions for the infimum of all eigenvalues which is zero and positive, respectively.

Here, problem (1.1) is stated in the framework of the generalized Sobolev space $X := W^{1,p(x)}(\Omega)$ for which some elementary properties are stated below.

By a weak solution for (1.1) we understand a function $u \in X$ such that

$$\int_{\Omega} |\nabla u|^{p(x)-2}\nabla u \nabla v \, dx + \int_{\Omega} |u|^{p(x)-2} uv \, dx - \lambda \int_{\partial\Omega} |u|^{q(x)-2} uv \, d\sigma = 0 \quad \text{for all } v \in X.$$

We point out that in the case when u is nontrivial, we say that $\lambda \in \mathbb{R}$ is an eigenvalue of (1.1) and u is called an associated eigenfunction.

Inspired by the works of Mihăilescu and Rădulescu [10, 14, 15, 17], we study (1.1) in three distinct situations.

This article consists of three sections. Section 2 contains some preliminary properties concerning the generalized Lebesgue-Sobolev spaces and an embedding result. The main results and their proofs are given in Section 3.

2. PRELIMINARIES

For completeness, we first recall some facts on the variable exponent spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$. For more details, see [7, 8]. Suppose that Ω is a bounded open domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $p \in C_+(\overline{\Omega})$, where

$$C_+(\overline{\Omega}) = \Big\{ p \in C(\overline{\Omega}) : \inf_{x \in \overline{\Omega}} p(x) > 1 \Big\}.$$

Denote by

$$p^- := \inf_{x \in \overline{\Omega}} p(x), \quad p^+ := \sup_{x \in \overline{\Omega}} p(x).$$

Define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega) = \Big\{ u \,\Big|\, u : \Omega \to \mathbb{R} \text{ is measurable and } \int_{\Omega} |u|^{p(x)} dx < +\infty \Big\},$$

with the norm

$$|u|_{p(x)} = \inf \left\{ \tau > 0 : \int\limits_{\Omega} \left| \frac{u}{\tau} \right|^{p(x)} dx \le 1 \right\}.$$

Define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm

$$\|u\| = \inf \left\{ \tau > 0 : \int\limits_{\Omega} \left(\left| \frac{\nabla u}{\tau} \right|^{p(x)} + \left| \frac{u}{\tau} \right|^{p(x)} \right) dx \le 1 \right\},$$

$$\|u\| = |\nabla u|_{p(x)} + |u|_{p(x)}.$$

We refer the reader to [6, 7] for the basic properties of the variable exponent Lebesgue and Sobolev spaces.

Lemma 2.1 ([8])**.** *Both $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ and $(W^{1,p(x)}(\Omega), \|\cdot\|)$ are separable and uniformly convex Banach spaces.*

Lemma 2.2 ([8])**.** *Hölder inequality holds, namely*

$$\int\limits_{\Omega} |uv| dx \le 2|u|_{p(x)}|v|_{p'(x)}, \quad u \in L^{p(x)}(\Omega), v \in L^{p'(x)}(\Omega),$$

where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

Lemma 2.3 ([8])**.** *Let $I(u) = \int\limits_{\Omega} (|\nabla u|^{p(x)} + |u|^{p(x)}) dx$. For $u \in W^{1,p(x)}(\Omega)$, we have:*

(i) $\|u\| < 1 (= 1, > 1) \Leftrightarrow I(u) < 1 (= 1, > 1)$,
(ii) $\|u\| \le 1 \Rightarrow \|u\|^{p^+} \le I(u) \le \|u\|^{p^-}$,
(iii) $\|u\| \ge 1 \Rightarrow \|u\|^{p^-} \le I(u) \le \|u\|^{p^+}$.

Lemma 2.4 ([7])**.** *Assume that the boundary of Ω possesses the cone property and $p \in C(\overline{\Omega})$ and $1 \le q(x) < p^*(x)$ for $x \in \overline{\Omega}$, then there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, where*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if} \quad p(x) < N, \\ +\infty, & \text{if} \quad p(x) \ge N. \end{cases}$$

Let $a : \partial\Omega \to \mathbb{R}$ be measurable. Define the weighted variable exponent Lebesgue space by

$$L^{p(x)}_{a(x)}(\partial\Omega) = \left\{ u \,\middle|\, u : \partial\Omega \to \mathbb{R} \text{ is measurable and } \int\limits_{\partial\Omega} |a(x)||u|^{p(x)} d\sigma < +\infty \right\},$$

with the norm

$$|u|_{(p(x),a(x))}=\inf\left\{\tau>0:\int\limits_{\partial\Omega}|a(x)|\left|\frac{u}{\tau}\right|^{p(x)}d\sigma\leq 1\right\},$$

where $d\sigma$ is the measure on the boundary. Then $L^{p(x)}_{a(x)}(\partial\Omega)$ is a Banach space. In particular, when $a\in L^{\infty}(\partial\Omega)$, $L^{p(x)}_{a(x)}(\partial\Omega)=L^{p(x)}(\partial\Omega)$.

Lemma 2.5 ([4]). *Let $\rho(u)=\int_{\partial\Omega}|a(x)||u|^{p(x)}d\sigma$. For $u\in L^{p(x)}_{a(x)}(\partial\Omega)$, we have:*

(i) $|u|_{(p(x),a(x))}\geq 1\Rightarrow |u|^{p^-}_{(p(x),a(x))}\leq\rho(u)\leq |u|^{p^+}_{(p(x),a(x))}$,
(ii) $|u|_{(p(x),a(x))}\leq 1\Rightarrow |u|^{p^+}_{(p(x),a(x))}\leq\rho(u)\leq |u|^{p^-}_{(p(x),a(x))}$.

For $A\subset\overline{\Omega}$, we set

$$p^-(A)=\inf_{x\in A}p(x),\quad p^+(A)=\sup_{x\in A}p(x).$$

Define

$$p^{\partial}(x)=(p(x))^{\partial}:=\begin{cases}\frac{(N-1)p(x)}{N-p(x)}, & \text{if}\quad p(x)<N,\\ \infty, & \text{if}\quad p(x)\geq N,\end{cases}$$

$$p^{\partial}_{r(x)}(x):=\frac{r(x)-1}{r(x)}p^{\partial}(x),$$

where $x\in\partial\Omega$, $r\in C(\partial\Omega,\mathbb{R})$ and $r(x)>1$.

Lemma 2.6 ([4]). *Assume that the boundary of Ω possesses the cone property and $p\in C(\overline{\Omega})$ with $p^->1$. Suppose that $a\in L^{r(x)}(\partial\Omega)$, $r\in C(\partial\Omega)$ with $r(x)>\frac{p^{\partial}(x)}{p^{\partial}(x)-1}$ for all $x\in\partial\Omega$. If $q\in C(\partial\Omega)$ and $1\leq q(x)<p^{\partial}_{r(x)}(x)$ for all $x\in\partial\Omega$, then there is a compact embedding $W^{1,p(x)}(\Omega)\hookrightarrow L^{q(x)}_{a(x)}(\partial\Omega)$. In particular, there is a compact embedding $W^{1,p(x)}(\Omega)\hookrightarrow L^{q(x)}(\partial\Omega)$, where $1\leq q(x)<p^{\partial}(x)$ for all $x\in\partial\Omega$.*

The Euler-Lagrange functional associated with (1.1) is defined as $\Phi_\lambda:X\to\mathbb{R}$,

$$\Phi_\lambda(u)=\int\limits_{\Omega}\frac{1}{p(x)}|\nabla u|^{p(x)}\,dx+\int\limits_{\Omega}\frac{1}{p(x)}|u|^{p(x)}\,dx-\lambda\int\limits_{\partial\Omega}\frac{1}{q(x)}|u|^{q(x)}d\sigma.$$

Standard arguments imply that $\Phi_\lambda\in C^1(X,\mathbb{R})$ and

$$\langle\Phi'_\lambda(u),v\rangle=\int\limits_{\Omega}|\nabla u|^{p(x)-2}\nabla u\nabla v\,dx+\int\limits_{\Omega}|u|^{p(x)-2}uv\,dx-\lambda\int\limits_{\partial\Omega}|u|^{q(x)-2}uvd\sigma$$

for all $u,v\in X$. Thus, the weak solutions of (1.1) coincide with the critical points of Φ_λ. If such a weak solution exists and is nontrivial, then the corresponding λ is an eigenvalue of problem (1.1).

Next, we write Φ'_λ as

$$\Phi'_\lambda = A - \lambda B,$$

where $A, B : X \to X'$ are defined by

$$\langle A(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\,dx + \int_\Omega |u|^{p(x)-2}uv\,dx,$$
$$\langle B(u), v\rangle = \int_{\partial\Omega} |u|^{q(x)-2}uv\,d\sigma.$$

Lemma 2.7 ([9])**.** *A satisfies condition (S^+), namely, $u_n \rightharpoonup u$ in X and $\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \le 0$, imply $u_n \to u$ in X.*

Remark 2.8. Noting that Φ'_λ is still of type (S^+). Hence, any bounded (PS) sequence of Φ_λ in the reflexive Banach space X has a convergent subsequence.

3. MAIN RESULTS AND PROOFS

Theorem 3.1. *Let $p, q \in C_+(\overline{\Omega})$. If*

$$q^+ < p^-, \tag{3.1}$$

then any $\lambda > 0$ is an eigenvalue for problem (1.1)*. Moreover, for any $\lambda > 0$ there exists a sequence (u_n) of nontrivial weak solutions for problem* (1.1) *such that $u_n \to 0$ in X.*

We want to apply the symmetric mountain pass lemma in [12].

Theorem 3.2 (Symmetric mountain pass lemma)**.** *Let E be an infinite dimensional Banach space and $I \in C^1(E, R)$ satisfy the following two assumptions:*

- (A1) *$I(u)$ is even, bounded from below, $I(0) = 0$ and $I(u)$ satisfies the Palais-Smale condition (PS), namely, any sequence u_n in E such that $I(u_n)$ is bounded and $I'(u_n) \to 0$ in E as $n \to \infty$ has a convergent subsequence.*
- (A2) *For each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u\in A_k} I(u) < 0$.*

Then, $I(u)$ admits a sequence of critical points u_k such that

$$I(u_k) < 0, u_k \neq 0 \text{ and } \lim_{k\to\infty} u_k = 0,$$

where Γ_k denote the family of closed symmetric subsets A of E such that $0 \notin A$ and $\gamma(A) \ge k$ with $\gamma(A)$ is the genus of A, i.e.,

$$\gamma(K) = \inf\left\{k \in \mathbb{N} : \text{there exists } h : K \to \mathbb{R}^k\backslash\{0\} \text{ such that } h \text{ is continuous and odd}\right\}.$$

We start with two auxiliary results.

Lemma 3.3. *The functional Φ_λ is even, bounded from below and satisfies the* (PS) *condition, and* $\Phi_\lambda(0) = 0$.

Proof. It is clear that Φ_λ is even and $\Phi_\lambda(0) = 0$. Since $q^+ < p^-$ and X is continuously embedded both in $L^{q^\pm}(\partial\Omega)$, there exists two positive constants $M_1, M_2 > 0$ such that

$$\int\limits_{\partial\Omega} |u|^{q^+} d\sigma \leq M_1 \|u\|^{q^+}, \quad \int\limits_{\partial\Omega} |u|^{q^-} d\sigma \leq M_2 \|u\|^{q^-} \quad \text{for all } u \in X.$$

According to the fact that

$$|u(x)|^{q(x)} \leq |u(x)|^{q^+} + |u(x)|^{q^-} \quad \text{for all } x \in \overline{\Omega}, \tag{3.2}$$

for all $u \in X$, we have

$$\begin{aligned} \Phi_\lambda(u) &\geq \frac{1}{p^+} \int\limits_{\Omega} \left(|\nabla u|^{p(x)} + |u|^{p(x)} \right) dx - \frac{\lambda M_1}{q^-} \|u\|^{q^+} - \frac{\lambda M_2}{q^-} \|u\|^{q^-} \\ &\geq \frac{1}{p^+} g(\|u\|) - \frac{\lambda M_1}{q^-} \|u\|^{q^+} - \frac{\lambda M_2}{q^-} \|u\|^{q^-}, \end{aligned}$$

where $g : [0, +\infty) \to \mathbb{R}$ is defined by

$$g(t) = \begin{cases} t^{p^+}, & \text{if} \quad t \leq 1, \\ t^{p^-}, & \text{if} \quad t > 1. \end{cases} \tag{3.3}$$

As $q^+ < p^-$, Φ_λ is bounded from below and coercive. It remains to show that the functional Φ_λ satisfies the (PS) condition to complete the proof. Let $(u_n) \subset X$ be a (PS) sequence of Φ_λ in X, that is,

$$\Phi_\lambda(u_n) \text{ is bounded and } \Phi'_\lambda(u_n) \to 0 \text{ in } X'. \tag{3.4}$$

Then, by the coercivity of Φ_λ, the sequence (u_n) is bounded in X. By the reflexivity of X, for a subsequence still denoted (u_n), we have

$$u_n \rightharpoonup u \quad \text{in } W^{1,p(x)}(\Omega), \; u_n \to u \quad \text{in } L^{p(x)}(\Omega), \quad \text{and} \quad u_n \to u \quad \text{in } L^{q(x)}(\partial\Omega).$$

Therefore,

$$\langle \phi'_\lambda(u_n), u_n - u \rangle \to 0 \quad \text{and} \quad \int\limits_{\partial\Omega} |u_n|^{q(x)-2} u_n (u_n - u) d\sigma \to 0.$$

Thus

$$\langle A(u_n), u_n - u \rangle := \int\limits_{\Omega} |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u) dx + \int\limits_{\Omega} |u_n|^{p(x)-2} u_n (u_n - u) dx \to 0.$$

According to the fact that A satisfies condition (S^+) (see [9]), we have $u_n \to u$ in $W^{1,p(x)}(\Omega)$. The proof is complete. □

Lemma 3.4. *For each $n \in \mathbb{N}^*$, there exists $H_n \in \Gamma_n$ such that*

$$\sup_{u \in H_n} \Phi_\lambda(u) < 0.$$

Proof. Let $v_1, v_2, \ldots, v_n \in C_0^\infty(\mathbb{R}^N)$ be such that

$$\overline{\{x \in \partial\Omega; v_i(x) \neq 0\}} \cap \overline{\{x \in \partial\Omega; v_j(x) \neq 0\}} = \emptyset \quad \text{if} \quad i \neq j$$

and

$$\operatorname{meas}(\{x \in \partial\Omega; v_i(x) \neq 0\}) > 0$$

for $i, j \in \{1, 2, \ldots, n\}$. Take $F_n = \operatorname{span}\{v_1, v_2, \ldots, v_n\}$. It is clear that $\dim F_n = n$ and

$$\int_{\partial\Omega} |v(x)|^{q(x)} d\sigma > 0 \quad \text{for all } v \in F_n \setminus \{0\}.$$

Denote $S = \{v \in W^{1,p(x)}(\Omega) : \ \|v\| = 1\}$ and $H_n(t) = t(S \cap F_n)$ for $0 < t \leq 1$. Obviously, $\gamma(H_n(t)) = n$ for all $t \in (0, 1]$.

Now, we show that, for any $n \in \mathbb{N}^*$, there exists $t_n \in (0, 1]$ such that

$$\sup_{u \in H_n(t_n)} \Phi_\lambda(u) < 0.$$

Indeed, for $0 < t \leq 1$, we have

$$\begin{aligned}
\sup_{u \in H_n(t)} \Phi_\lambda(u) &\leq \sup_{v \in S \cap F_n} \Phi_\lambda(tv) \\
&= \sup_{v \in S \cap F_n} \left\{ \int_\Omega \frac{t^{p(x)}}{p(x)} \Big(|\nabla v(x)|^{p(x)} + |v(x)|^{p(x)} \Big) dx - \lambda \int_{\partial\Omega} \frac{t^{q(x)}}{q(x)} |v(x)|^{q(x)} \, d\sigma \right\} \\
&\leq \sup_{v \in S \cap F_n} \left\{ \frac{t^{p^-}}{p^-} \int_\Omega \Big(|\nabla v(x)|^{p(x)} + |v(x)|^{p(x)} \Big) dx - \frac{\lambda t^{q^+}}{q^+} \int_{\partial\Omega} |v(x)|^{q(x)} \, d\sigma \right\} \\
&= \sup_{v \in S \cap F_n} \left\{ t^{p^-} \left(\frac{1}{p^-} - \frac{\lambda}{q^+} \frac{1}{t^{p^- - q^+}} \int_{\partial\Omega} |v(x)|^{q(x)} \, d\sigma \right) \right\}.
\end{aligned}$$

Since

$$m := \min_{v \in S \cap F_n} \int_{\partial\Omega} |v(x)|^{q(x)} \, d\sigma > 0,$$

we may choose $t_n \in (0, 1]$ which is small enough such that

$$\frac{1}{p^-} - \frac{\lambda}{q^+} \frac{1}{t_n^{p^- - q^+}} m < 0.$$

This completes the proof. □

Proof of Theorem 3.1. By Lemmas 3.3, 3.4 and Theorem 3.2, Φ_λ admits a sequence of nontrivial weak solutions $(u_n)_n$ such that for any n, we have

$$u_n \neq 0, \quad \Phi'_\lambda(u_n) = 0, \quad \Phi_\lambda(u_n) \leq 0, \quad \lim_{n\to\infty} u_n = 0. \tag{3.5}$$

□

Theorem 3.5. *Let $p, q \in C_+(\overline{\Omega})$. If*

$$q^- < p^- \quad \textit{and} \quad q^+ < p^\partial(x) \quad \textit{for all } x \in \overline{\Omega}, \tag{3.6}$$

then there exists $\lambda^* > 0$ such that any $\lambda \in (0, \lambda^*)$ is an eigenvalue for problem (1.1).

For applying Ekeland's variational principle we start with two auxiliary results.

Lemma 3.6. *There exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$ there exist $\rho, a > 0$ such that $\Phi_\lambda(u) \geq a > 0$ for any $u \in X$ with $\|u\| = \rho$.*

Proof. Since $q(x) < p^\partial(x)$ for all $x \in \overline{\Omega}$, it follows that X is continuously embedded in $L^{q(x)}(\partial\Omega)$. So, there exists a positive constant C_1 such that

$$|u|_{L^{q(x)}(\partial\Omega)} \leq C_1 \|u\| \quad \text{for all } u \in X. \tag{3.7}$$

Fix $\rho \in (0,1)$ such that $\rho < \frac{1}{C_1}$. Then relation (3.7) implies $|u|_{L^{q(x)}(\partial\Omega)} < 1$ for all $u \in X$ with $\|u\| = \rho$. Thus,

$$\int_{\partial\Omega} |u|^{q(x)} d\sigma \leq |u|^{q^-}_{L^{q(x)}(\partial\Omega)} \quad \text{for all } u \in X \text{ with } \|u\| = \rho. \tag{3.8}$$

Combining (3.7) and (3.8), we obtain

$$\int_{\partial\Omega} |u|^{q(x)} d\sigma \leq C_1^{q^-} \|u\|^{q^-} \quad \text{for all } u \in X \text{ with } \|u\| = \rho. \tag{3.9}$$

Hence, from (3.9) we deduce that for any $u \in X$ with $\|u\| = \rho$, we have

$$\begin{aligned}
\Phi_\lambda(u) &\geq \frac{1}{p^+} \int_\Omega \left(|\nabla u|^{p(x)} + |u|^{p(x)} \right) dx - \frac{\lambda}{q^-} \int_{\partial\Omega} |u|^{q(x)} d\sigma \\
&\geq \frac{1}{p^+} \|u\|^{p^+} - \frac{\lambda}{q^-} C_1^{q^-} \|u\|^{q^-} \\
&= \frac{1}{p^+} \rho^{p^+} - \frac{\lambda}{q^-} C_1^{q^-} \rho^{q^-} \\
&= \rho^{q^-} \left(\frac{1}{p^+} \rho^{p^+ - q^-} - \frac{\lambda}{q^-} C_1^{q^-} \right).
\end{aligned}$$

Putting

$$\lambda_* = \frac{\rho^{p^+ - q^-}}{2p^+} \frac{q^-}{c_1^{q^-}} \tag{3.10}$$

for any $u \in X$ with $\|u\| = \rho$, there exists $a = \rho^{p^+}/(2p^+)$ such that

$$\Phi_\lambda(u) \geq a > 0.$$

This completes the proof. □

Lemma 3.7. *There exists $\xi \in X$ such that $\xi \geq 0$, $\xi \neq 0$ and $\Phi_\lambda(t\xi) < 0$ for $t > 0$ small enough.*

Proof. Since $q^- < p^-$, there exists $\varepsilon_0 > 0$ such that

$$q^- + \varepsilon_0 < p^-.$$

Since $q \in C(\overline{\Omega})$, there exists an open set $A \subset \partial\Omega$ such that

$$|q(x) - q^-| < \varepsilon_0 \quad \text{for all} \quad x \in A.$$

Thus, we deduce that

$$q(x) \leq q^- + \varepsilon_0 < p^- \quad \text{for all } x \in A. \tag{3.11}$$

Take $\xi \in C_0^\infty(\mathbb{R}^N)$ such that $\overline{A} \subset \operatorname{supp}\xi$, $\xi(x) = 1$ for $x \in \overline{A}$ and $0 \leq \xi \leq 1$ in Ω. Without loss of generality, we may assume that $\|\xi\| = 1$, that is,

$$\int_\Omega \left(|\nabla\xi|^{p(x)} + |\xi|^{p(x)}\right) dx = 1. \tag{3.12}$$

By using (3.11), (3.12) and the fact

$$\int_A |\xi|^{q(x)} d\sigma = \operatorname{meas}(A)$$

for all $t \in (0,1)$, we obtain

$$\begin{aligned}
\Phi_\lambda(t\xi) &= \int_\Omega \frac{t^{p(x)}}{p(x)}\left(|\nabla\xi|^{p(x)} + |\xi|^{p(x)}\right) dx - \lambda \int_{\partial\Omega} \frac{t^{q(x)}}{q(x)}|\xi|^{q(x)} d\sigma \\
&\leq \frac{t^{p^-}}{p^-} \int_\Omega \left(|\nabla\xi|^{p(x)} + |\xi|^{p(x)}\right) dx - \frac{\lambda}{q^+} \int_{\partial\Omega} t^{q(x)}|\xi|^{q(x)} d\sigma \\
&\leq \frac{t^{p^-}}{p^-} - \frac{\lambda}{q^+} \int_A t^{q(x)}|\xi|^{q(x)} d\sigma \\
&\leq \frac{t^{p^-}}{p^-} - \frac{\lambda t^{q^- + \varepsilon_0}}{q^+} \operatorname{meas}(A).
\end{aligned}$$

Then, for any $t < \delta^{\frac{1}{p^- - q^- - \varepsilon_0}}$ with $0 < \delta < \min\{1, \lambda p^- \operatorname{meas}(A)/q^+\}$, we conclude that

$$\Phi_\lambda(t\xi) < 0.$$

The proof is complete. □

Proof of Theorem 3.5. By Lemma 3.6, we have

$$\inf_{\partial B_\rho(0)} \Phi_\lambda > 0, \tag{3.13}$$

where $\partial B_\rho(0) = \{u \in X : \|u\| = \rho\}$.

On the other hand, from Lemma 3.7, there exists $\xi \in X$ such that $\Phi_\lambda(t\xi) < 0$ for $t > 0$ small enough. Using (3.9), it follows that

$$\Phi_\lambda(u) \geq \frac{1}{p^+}\|u\|^{p^+} - \frac{\lambda}{q^-} C_1^{q^-} \|u\|^{q^-} \quad \text{for } u \in B_\rho(0).$$

Thus,

$$-\infty < \underline{c}_\lambda := \inf_{\overline{B_\rho(o)}} \Phi_\lambda < 0,$$

Let

$$0 < \varepsilon < \inf_{\partial B_\rho(0)} \Phi_\lambda - \inf_{\overline{B_\rho(0)}} \Phi_\lambda.$$

Then, by applying Ekeland's variational principle to the functional

$$\Phi_\lambda : \overline{B_\rho(0)} \to \mathbb{R},$$

there exists $u_\varepsilon \in \overline{B_\rho(0)}$ such that

$$\Phi_\lambda(u_\varepsilon) \leq \inf_{\overline{B_\rho(0)}} \Phi_\lambda + \varepsilon,$$

$$\Phi_\lambda(u_\varepsilon) < \Phi_\lambda(u) + \varepsilon\|u - u_\varepsilon\| \quad \text{for} \quad u \neq u_\varepsilon.$$

Since

$$\Phi_\lambda(u_\varepsilon) < \inf_{\overline{B_\rho(0)}} \Phi_\lambda + \varepsilon < \inf_{\partial B_\rho(0)} \Phi_\lambda,$$

we deduce that $u_\varepsilon \in B_\rho(0)$.

Now, define $I_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$ by

$$I_\lambda(u) = \Phi_\lambda(u) + \varepsilon\|u - u_\varepsilon\|.$$

It is clear that u_ε is an minimum of I_λ. Therefore, for $t > 0$ and $v \in B_1(0)$, we have

$$\frac{I_\lambda(u_\varepsilon + tv) - I_\lambda(u_\varepsilon)}{t} \geq 0$$

for $t > 0$ small enough and $v \in B_1(0)$, that is,

$$\frac{\Phi_\lambda(u_\varepsilon + tv) - \Phi_\lambda(u_\varepsilon)}{t} + \varepsilon\|v\| \geq 0$$

for t positive and small enough, and $v \in B_1(0)$. As $t \to 0$, we obtain

$$\langle \Phi'_\lambda(u_\varepsilon), v\rangle + \varepsilon\|v\| \geq 0 \quad \text{for all } v \in B_1(0).$$

Hence, $\|\Phi'_\lambda(u_\varepsilon)\|_{X'} \le \varepsilon$. We deduce that there exists a sequence $(u_n)_n \subset B_\rho(0)$ such that

$$\Phi_\lambda(u_n) \to \underline{c}_\lambda \quad \text{and} \quad \Phi'_\lambda(u_n) \to 0. \tag{3.14}$$

It is clear that (u_n) is bounded in X. By a standard argument and the fact A is type of (S^+), for a subsequence we obtain $u_n \to u$ in X as $n \to \infty$. Thus, by (3.14), we have

$$\Phi_\lambda(u) = \underline{c}_\lambda < 0 \quad \text{and} \quad \Phi'_\lambda(u) = 0 \quad \text{as } n \to \infty. \tag{3.15}$$

The proof is complete. □

Theorem 3.8. *Let* $p, q \in C_+(\overline{\Omega})$*. If*

$$p^+ < q^- \le q^+ < p^\partial(x) \quad \textit{for all } x \in \overline{\Omega}, \tag{3.16}$$

then for any $\lambda > 0$*, problem* (1.1) *possesses a nontrivial weak solution.*

We want to construct a mountain geometry, and first need two lemmas.

Lemma 3.9. *There exists* $\eta, b > 0$ *such that* $\Phi_\lambda(u) \ge b$ *for* $u \in X$ *with* $\|u\| = \eta$*.*

Proof. Since $q^+ < p^\partial$, in view the Theorem 3.2, there exists $M_1, M_2 > 0$ such that

$$|u|_{L^{q^+}(\partial\Omega)} \le M_1\|u\| \quad \text{and} \quad |u|_{L^{q^-}(\partial\Omega)} \le M_2\|u\|.$$

Thus, from (3.2) we obtain

$$\begin{aligned}
\Phi_\lambda(u) &\ge \frac{1}{p^+}\int_\Omega \Big(|\nabla u(x)|^{p(x)} + |u(x)|^{p(x)}\Big)dx - \frac{\lambda}{q^-}\left[(M_1\|u\|)^{q^+} + (M_2\|u\|)^{q^-}\right] \\
&\ge \frac{1}{p^+}g(\|u\|) - \frac{\lambda M_1^{q^+}}{q^-}\|u\|^{q^+} - \frac{\lambda M_2^{q^-}}{q^-}\|u\|^{q^-} \\
&= \begin{cases} \left(\frac{1}{p^+} - \frac{M_1^{q^+}}{q^-}\|u\|^{q^+-p^+} - \frac{\lambda M_2^{q^-}}{q^-}\|u\|^{q^--p^+}\right)\|u\|^{p^+} & \text{if } \|u\| \le 1, \\ \left(\frac{1}{p^+} - \frac{M_1^{q^+}}{q^-}\|u\|^{q^+-p^-} - \frac{\lambda M_2^{q^-}}{q^-}\|u\|^{q^--p^-}\right)\|u\|^{p^-} & \text{if } \|u\| > 1. \end{cases}
\end{aligned}$$

Since $p^+ < q^- \le q^+$, the functional $h : [0,1] \to \mathbb{R}$ defined by

$$h(s) = \frac{1}{p^+} - \frac{M_1^{q^+}}{q^-}s^{q^+-p^+} - \frac{\lambda M_2^{q^-}}{q^-}s^{q^--p^+}$$

is positive on the neighbourhood of the origin. So, the result of Lemma 3.9 follows. □

Lemma 3.10. *There exists* $e \in X$ *with* $\|e\| \ge \eta$ *such that* $\Phi_\lambda(e) < 0$*, where* η *is given in Lemma 3.9.*

Proof. Choose $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$ and $\varphi \neq 0$. For $t > 1$, we have

$$\Phi_\lambda(t\varphi) \leq \frac{t^{p^+}}{p^-} \int_\Omega \left(|\nabla\varphi(x)|^{p(x)} + |\varphi(x)|^{p(x)}\right) dx - \frac{\lambda t^{q^-}}{q^+} \int_{\partial\Omega} |\varphi(x)|^{q(x)} d\sigma.$$

Then, since $p^+ < q^-$, we deduce that

$$\lim_{t\to\infty} \Phi_\lambda(t\varphi) = -\infty.$$

Therefore, for $t > 1$ large enough, there is $e = t\varphi$ such that $\|e\| \geq \eta$ and $\Phi_\lambda(e) < 0$. This completes the proof. □

Lemma 3.11. *Let $p, q \in C_+(\overline{\Omega})$. Assume that $p^+ < q^-$. Then the functional Φ_λ satisfies the condition (PS).*

Proof. Let $(u_n) \subset X$ be a sequence such that $M := \sup_n \Phi_\lambda(u_n) < \infty$ and $\Phi'_\lambda(u_n) \to 0$ in X'. By contradiction suppose that

$$\|u_n\| \to +\infty \text{ as } n \to \infty \quad \text{and} \quad \|u_n\| > 1 \text{ for any } n.$$

Thus,

$$\begin{aligned}
M + 1 + \|u_n\| &\geq \Phi_\lambda(u_n) - \frac{1}{q^-}\langle \Phi'_\lambda(u_n), u_n\rangle \\
&= \int_\Omega \frac{1}{p(x)}\left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right) dx - \frac{1}{q^-}\int_\Omega \left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right) dx \\
&\quad + \lambda \int_{\partial\Omega} \left(\frac{1}{q^-} - \frac{1}{q(x)}\right) |u_n|^{q(x)}\, d\sigma \\
&\geq \left(\frac{1}{p^+} - \frac{1}{q^-}\right) \int_\Omega \left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right) dx \\
&\geq \left(\frac{1}{p^+} - \frac{1}{q^-}\right) \|u_n\|^{p^-}.
\end{aligned}$$

Since $p^+ < q^-$, this contradicts the fact that $p^- > 1$. So, the sequence (u_n) is bounded in X and similar arguments as those used in the proof of Lemma 3.4 completes the proof. □

Proof of Theorem 3.8. From Lemmas 3.9 and 3.10 we deduce that

$$\max(\Phi_\lambda(0), \Phi_\lambda(e)) = \Phi_\lambda(0) < \inf_{\|u\|=\eta} \Phi_\lambda(u) =: \beta.$$

By Lemma 3.11 and the mountain pass theorem, we deduce the existence of critical points u of Φ_λ associated with the critical value given by

$$c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \Phi_\lambda(\gamma(t)) \geq \beta, \tag{3.17}$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0 \text{ and } \gamma(1) = e\}.$$

This completes the proof. □

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