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Convex linear combination of the controllability pairs for linear systems*

by

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Abstract: The convex linear combination of the controllability pairs of linear continuous-time linear systems is defined and its properties are discussed. The main result is obtained using pure algebraic methods. In the illustrative examples different cases of linear convex combinations are analyzed.

Keywords: convex linear combination, controllability, linear system

1. Introduction

The notion of controllability of linear control systems was introduced by Kalman (Kalman, 1963) during the IFAC Congress in 1960 in Moscow. This notion and the related concepts are the basic concepts of the modern mathematical control theory (see Kaczorek, 1993; Kailath, 1980; Klamka, 1981, 2018). They have been also extended to positive linear systems (Kaczorek, 2008, 2010, 2018). In particular, the decomposition of the positive linear system has been addressed in Kaczorek (2018). It is well known that from the topological point of view, controllability of linear systems constitutes a generic property of the systems. Therefore, a randomly chosen linear dynamical system is controllable. Moreover, the class of uncontrollable linear system is closed. If a dynamical system is uncontrollable, then it can be decomposed into the controllable and uncontrollable parts. In this paper, the concept of convex linear combination of the controllability pairs of linear continuous-time finite dimensional systems is investigated.

The paper is organized as follows. Section 1 contains a brief description of the problem and the relationships to the previous results are pointed out.

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In Section 2 the basic definition and theorem, concerning controllability, are recalled. The notion of the convex linear combination of the linear controllable systems and the main result of the paper are presented in Section 3. In the proof of the main result pure algebraic methods, based on linear algebra, are used. Moreover, the subsequent Section 4 contains illustrative examples, in which different convex linear combination of the controllable and uncontrollable pairs of linear control systems are investigated. Finally, concluding remarks and a set of open problems are given in Section 5.

2. Mathematical model and basic definitions

The main purpose of this article is to present a compact review over the existing algebraic controllability and observability conditions mainly for linear, finite-dimensional, continuous-time and time-invariant control systems.

Controllability and observability are fundamental concepts in modern mathematical control theory. They are qualitative properties of control systems and are of particular importance in mathematical control theory. Systematic study of controllability and observability was started at the beginning of the 1960s, when the theory of controllability and observability, based on the description in the form of state space for both time-invariant and time-varying linear control systems was elaborated. Many dynamical systems display the property that the control does not affect the complete state of the dynamical system but only a part of it. On the other hand, very often, in real industrial processes it is possible to observe only a certain part of the complete state of the dynamical system. Therefore, it is very important to determine whether or not control and observation of the complete state of the dynamical system are possible. Roughly speaking, controllability generally means that it is possible to steer a dynamical system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. Controllability plays an essential role in the development of the modern mathematical control theory. There are important relationships between different types of controllability, observability and stabilizability for linear control systems. Controllability is also strongly connected with the theory of minimal realization of linear time-invariant control systems. Moreover, it should be pointed out that there exists a formal duality between the concepts of controllability and observability.

The literature of the subject contains many different definitions of controllability, which strongly depend on the type of dynamical control system considered. The main purpose of this article is to present a compact review over the existing algebraic controllability and observability conditions mainly for linear continuous-time and time-invariant control systems. It should be pointed out that for linear control systems controllability and observability conditions have pure algebraic forms and are rather easily computable. These conditions require verification of the rank conditions for the suitably defined constant controllability and observability matrices.

In the theory of linear time-invariant dynamical control systems the most popular and the most frequently used mathematical model is given by the following linear ordinary differential state equation:

$$x'(t) = Ax(t) + bu(t) \tag{1}$$

where $x(t) \in R^n$ is the state vector, $u(t) \in R$ is an input scalar admissible control, A is a real nxn-dimensional matrix, and b is a real n–dimensional vector.

It is well known that for a given initial state $x(0) \in R^n$ and control $u(t) \in R$, $t \geq 0$, there exists a unique solution $x(t;x(0),u) \in R^n$ of the state equation (1), having the following form

$$x(t;x(0),u) = \exp(At)x(0) + \int_0^t \exp(A(t-s))bu(s)ds. \tag{2}$$

Now, let us recall the most popular and most frequently used fundamental definition of controllability for linear control systems with constant coefficients.

Definition 1 (Klamka, 1981, 2018) *Dynamical system (1) is said to be controllable if for every initial condition $x(0)$ and every vector $x^1 \in R^n$, there exist a finite time t_1 and control $u(t) \in R$, $t \in [0,t_1]$, such that $x(t_1;x(0),u) = x^1$.*

This definition requires only of the system (1) that it can be steered from any initial state $x(0)$ to any final state x^1 within a finite time interval. It should be pointed out that the trajectory of the system (1) in the time interval $[0,t_1]$ is not specified. Furthermore, there are no constraints imposed on the control or the state variable.

In order to formulate easily the computable algebraic controllability criteria let us introduce the so called controllability matrix W, defined as follows:

$W = [b, Ab, A^2b, ..., A^{n-1}b]$.

The controllability matrix W is an $n \times n$-dimensional constant matrix, which depends only on system parameters.

Now, let us recall the necessary and sufficient controllability conditions.

Theorem 1 (Klamka, 1981, 2018) *The dynamical system (1) is controllable if and only if*

$$rank W = n.$$

Corollary 1 (Klamka, 1981, 2018) *The dynamical system (1) is controllable if and only if the $n \times n$-dimensional symmetric matrix WW^T is nonsingular.*

Since the controllability matrix W does not depend on time t_1, then from Theorem 1 and Corollary 1 it directly follows that, in fact, controllability of dynamical system does not depend on the length of the control interval. However, this is not true for dynamical systems with constraints on the state variables or admissible controls.

Let P be an $n \times n$ constant nonsingular transformation matrix and let us define the equivalence transformation in the state space R^n, given by $z(t) = Px(t)$. Then, the differential state equation (1) becomes

$$z'(t) = Jz(t) + gu(t) \tag{3}$$

where matrix $J = PAP^{-1}$, and vector $g = Pb$.

Dynamical systems (1) and (3) are said to be equivalent and many of their properties are invariant under the nonsingular equivalence transformations. For example, nonsingular transformation preserves the controllability property.

Among the different nonsingular transformations, we shall use in this paper the nonsingular transformation, which leads to the following nxn-dimensional matrix A^c and the n-dimensional vector b^c:

$$A^c = PAP^{-1} = \begin{bmatrix} 0,0,0, & \cdots & 0,-a_{1,0} \\ 1,0,0, & \ldots & 0,-a_{1,1} \\ 0,1,0, & \cdots & 0,-a_{1,2} \\ \ddots & & \\ 0,0,0, & \ldots 1, & -a_{1,n-1} \end{bmatrix} \quad - \quad b^c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{4}$$

Suppose that the dynamical system (1) is controllable, then the dynamical system remains controllable after the equivalence transformation. This is natural and intuitively clear, because an equivalence transformation changes only the basis of the finite dimensional state space. Therefore, we have the following corollary:

COROLLARY 2 *Controllability is invariant under any equivalence transformation*

$$z(t) = Px(t).$$

Since controllability of a dynamical system is preserved under any equivalence transformation, then it is possible to obtain a simpler controllability criterion by transforming the differential state equation (1) into a special form (3), (4).

It should be pointed out that if we transform the dynamical system (1) into the Jordan canonical form, then controllability can be determined very easily, almost by inspection (see Kalman, 1963, for more details).

REMARK 1 *It should be pointed out that for system (1), controllability does not depend on the length of time interval [0,t_1] and it depends only on system parameters. Hence, the necessary and sufficient conditions for controllability are purely algebraic.*

It is well nown that the controllability of linear systems is a generic property of the systems. A randomly chosen dynamical system is controllable. Moreover, the class of uncontrollable linear system is closed. If a dynamical system is uncontrollable, then it can be decomposed into the controllable and uncontrollable parts.

The structure of the set of controllable systems (1) is rather complicated, but it is possible to formulate a general result, which is given below.

COROLLARY 3 (KLAMKA, 1981) *The set of controllable systems (1) is open and dense in the set of all systems with the same dimensions of matrices A and vectors b.*

3. Controllability of the convex linear combination

Now, let us introduce the linear convex combination of linear finite-dimensional dynamical systems, given in the form (1).

For the linear control systems $(A_1,\ b_1)$ and $(A_2,\ b_2)$ we define their linear convex combination $(A(q),b(q))$ as follows:

$$A(q) = (1-q)A_1 + qA_2,\ b(q) = (1-q)b_1 + qb_2,\ 0 \le q \le 1 \tag{5}$$

where q is a real number.

Now, using the general formula (4), we introduce for the pairs of matrices (A_1,b_1) and (A_2,b_2) the following notations:

$$A_1^c = P_1A_1P_1^{-1} = \begin{bmatrix} 0,0,0, & \cdots & 0,-a_{1,0} \\ 1,0,0, & \ldots & 0,-a_{1,1} \\ 0,1,0, & \cdots & 0,-a_{1,2} \\ \ddots & & \\ 0,0,0, & \ldots 1, & -a_{1,n-1} \end{bmatrix} \quad b_1^c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$A_2^c = P_2A_2P_2^{-1} = \begin{bmatrix} 0,0,0, & \cdots & 0,-a_{2,0} \\ 1,0,0, & \ldots & 0,-a_{2,1} \\ 0,1,0, & \cdots & 0,-a_{2,2} \\ \ddots & & \\ 0,0,0, & \ldots 1, & -a_{2,n-1} \end{bmatrix} \quad b_2^c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The main result of the present paper is stated in the following theorem, which gives the sufficient condition for controllability of convex combination.

THEOREM 2 *Let the pairs $(A_1,\ b_1)$ and $(A_2,\ b_2)$ of linear control systems be controllable. Then their linear convex combination*

$A(q) = (1-q)A_1 + qA_2$, $b(q) = (1-q)b_1 + qb_2$, $0 \le q \le 1$

is also controllable for all $0 \le q \le 1$, *if* $P = P_1 = P_2$ *is a nonsingular matrix.*

Proof Taking into account the assumption $P = P_1 = P_2$ and formula (5) we have

$$A(q) = PA(q)P^{-1} = (1-q)PA_1^cP^{-1} + qPA_2^cP^{-1} =$$

$$= (1-q)\begin{bmatrix} 0,0,0, & \cdots & 0,-a_{1,0} \\ 1,0,0, & \dots & 0,-a_{1,1} \\ 0,1,0, & \cdots & 0,-a_{1,2} \\ \ddots & & \\ 0,0,0, & \dots 1, & -a_{1,n-1} \end{bmatrix} + q\begin{bmatrix} 0,0,0, & \cdots & 0,-a_{2,0} \\ 1,0,0, & \dots & 0,-a_{2,1} \\ 0,1,0, & \cdots & 0,-a_{2,2} \\ \ddots & & \\ 0,0,0, & \dots 1, & -a_{2,n-1} \end{bmatrix}$$

$$= \begin{bmatrix} 0,0,0, & \cdots & 0,-a_0 \\ 1,0,0, & \dots & 0,-a_1 \\ 0,1,0, & \cdots & 0,-a_2 \\ \ddots & & \\ 0,0,0, & \dots 1, & -a_{n-1} \end{bmatrix},$$

$$b(q) = (1-q)\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + q\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where a_k = (1-q)$a_{1,k}$+$qa_{2,k}$ for k = 0,1,2,...,n-1.

Upon computing the controllability matrix and its rank we obtain rank[$b(q)$,

$$A(q)b(q), \dots, A^{n^{-1}}(q)b(q)] = rank\begin{bmatrix} 1,0,0, & \cdots & 0 \\ 0,1,0, & \dots & 0 \\ 0,0,1, & \cdots & 0 \\ \ddots & & \\ 0,0,0,\dots\dots,1 & & \end{bmatrix} = n.$$

Therefore, the Kalman controllability condition is satisfied and the convex linear combination of the pairs is controllable for all q, $0 \le q \le 1$. □

4. Examples

In the here presented examples different cases of linear convex combinations of linear control systems are discussed.

Example 1 *Let us consider a system, characterized by the following matrix and vector:*

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \epsilon R^{nxn} : \quad b \quad = \quad \begin{bmatrix} b_1 \\ 0 \end{bmatrix} \epsilon R^{nxm},$$

where b_1 is an n-dimensional vector, A_1 is an $n_1 x\, n_1$ matrix, A_2 is an $n_1 x\, n_2$ matrix, and A_3 is an $n_2 x\, n_2$ matrix, with $n_1 + n_2 = n$.

This system is uncontrollable for all values of the entries of the submatrices A_1, A_2, A_3, and vector b_1. This follows directly from the equality:

$$rank\ [b,\ Ab, \ldots, A^{n-1}b] = rank \begin{bmatrix} b_1 & A_1 b_1 & \ldots & A_1^{n-1} b_1 \\ 0 & 0 & \ldots & 0 \end{bmatrix} < n.$$

Now, let us consider two uncontrollable systems:

$$A_i = \begin{bmatrix} A_{i1} & A_{i2} \\ 0 & A_{i3} \end{bmatrix} \epsilon R^{nxn};\ b_i = \begin{bmatrix} b_{i1} \\ 0 \end{bmatrix} \ for\ i = 1, 2. \tag{6}$$

Here, A_{i_1} is an $n_1 x\, n_1$ dimensional matrix, A_{i_2} is an $n_1 x\, n_2$ dimensional matrix, and A_{i_3} is an $n_2 x\, n_2$ dimensional matrix, where $n_1 + n_2 = n$.

Next, let us consider their convex linear combinations:

$$\begin{aligned} A(q) &= (1-q)A_1 + qA_2 = \begin{bmatrix} (1-q)\, A_{11} + qA_{21} & (1-q)\, A_{12} + \mathrm{q}A_{22} \\ 0 & (1-q)\, A_{13} + \mathrm{q}A_{23} \end{bmatrix} \\ b(q) &= (1-q)b_1 + qb_2 = \begin{bmatrix} (1-q)\, b_{11} + qb_{21} \\ 0 \end{bmatrix}. \end{aligned} \tag{7}$$

In order to simplify our considerations let us introduce the following notations:

$$M = [(1-q)\, A_{11} + qA_{21}]$$
$$N = [(1-q)\, b + qb_{21}].$$

Using the well known Kalman controllability condition for the convex linear combination (7) we obtain

$$\begin{aligned} & rank\, [b(q), A(q)b(q), \ldots, A^{n-1}b(q)] = \\ & = rank \begin{bmatrix} N & MN & M^2N & \ldots & M^{n-1}N \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} < n. \end{aligned}$$

This example shows that if a pair of systems contains uncontrollable systems, then their convex linear combination (7) is also uncontrollable for $0 \leq q \leq 1$.

Example 2 *Consider the controllable pair*

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{8}$$

and the uncontrollable pair

$$A_2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{9}$$

The convex linear combination of the pairs (8) and (9) has the forms

$$A(q) = (1-q)A_1 + qA_2 = \begin{bmatrix} q & 1-q \\ 1+q & 2-q \end{bmatrix},$$
$$b(q) = (1-q)b_1 + qb_2 = \begin{bmatrix} 1-q \\ q \end{bmatrix} \tag{10}$$

where $0 \leq q \leq 1$.

Using (10) and the Kalman controllability condition we obtain

$$rank[b(q), A(q)b(q)] = rank \begin{bmatrix} 1-q & 2q(1-q) \\ q & 1+2q-2q^2 \end{bmatrix} < 2$$

for $q = 1$ *and* $1 + 2q - 4q^2 = 0$
since

$$det \begin{bmatrix} 1-q & 2q(1-q) \\ q & 1+2q-2q^2 \end{bmatrix} = (q-1)(1+2q-4q^2) = 0. \tag{11}$$

Therefore, the convex linear combination of the controllable pair (8) and uncontrollable pair (9) is uncontrollable for the positive root of the equation (11) equal to $q \quad = 0.25(\sqrt{5}+1) < 1$.

Generally, a convex combination of controllable and uncontrollable dynamical systems may be uncontrollable.

Example 3 *In general case, let the system, given by*

$$A_1 \quad = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \epsilon R^{nxn};\ b_1 = \begin{bmatrix} b_{11} \\ b_{12} \end{bmatrix} \tag{12}$$

be controllable,
and the system

$$A_2^{'} = \begin{bmatrix} A_{11}^{'} & A_{12}^{'} \\ 0 & A_{22}^{'} \end{bmatrix} \epsilon R^{nxn};\ b_2^{'} = \begin{bmatrix} b_{21}^{'} \\ 0 \end{bmatrix} \tag{13}$$

be uncontrollable.

The convex linear combinations of (12) and (13) have the forms

$$A(q) = (1-q)A_1 + qA_2^{'} = \begin{bmatrix} (1-q)\,A_{11} + qA_{11}^{'} & (1-q)\,A_{12} + qA_{2}^{;} \\ (1-q)A_{21} & (1-q)\,A_{22} + qA_{22}^{'} \end{bmatrix} \epsilon R^{nxn} \tag{14}$$

$$B(q) = (1-q)B_1 + qB_2 = \begin{bmatrix} (1-q) + B_{11} + qB_{21}^{'} \\ (1-q)B_{12} \end{bmatrix} \epsilon R^{nxm}. \tag{15}$$

Since in this case the controllability matrix is square, therefore we have

$$det[b(q), A(q)b(q), \ldots, A^{n-1}(q)b(q)] = a_n q^n + a_{n-1}q^{n-1} + \cdots + a_1 q + a_0.$$

Let the equation

$$a_n q^n + a_{n-1}q^{n-1} + \cdots + a_1 q + a_0 = 0$$

have in the interval [0,1] the real roots:

$$0<q_1 < q_2 < \cdots < q_{k_{-1}}<q_k < 1;\ k \leq n. \tag{16}$$

Then the convex linear combination of (12) and (13) is uncontrollable for parameters

$0<q_1<q_2<\ldots<q_{k_{-1}}<q_k<1$.

If the pair (12) is controllable and the pair (13) is uncontrollable, then their convex linear combination is uncontrollable for at most n *values of parameter* q*, given by (16).*

Example 4 *Let us consider two controllable control systems:*

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad b_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad b_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The corresponding canonical forms and transformation matrices are as follows:

$$A_1^c = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad b_1^c = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad P_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A_2^c = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \quad b_2^c = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad P_2 = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

$$A\,(q) = (1-q)\,A_1^c\,(q) + qA_2^c = (1-q)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + q\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1-q \\ 1 & q \end{bmatrix}.$$

Similarly

$$b\,(q) = (1-q)\,b_1^c + qb_2^c = (1-q)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + q\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$rank\ [b\,(q)\,, A\,(q)\,b\,(q)] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2\,.$$

Remark 2 *Therefore, this example shows that the linear convex combination of dynamical systems may be controllable also for different nonsingular transformation matrices. Hence, Theorem 2 is only sufficient, but not necessary condition for controllability of a linear convex combination.*

5. Concluding remarks

The algebraic approach to the structure of the set of linear dynamical systems with the same dimension of state space was presented in this paper. Using controllability matrix the convex linear combination of the controllability pairs of linear continuous-time linear systems was defined and discussed.

In this paper only the most popular linear control systems are discussed. However, the considerations can be extended in many directions. For example, controllability of convex linear combinations of discrete-time linear systems and fractional order linear continuous-time and fractional discrete-time linear systems may be considered.

Moreover, since observability of linear control systems is a dual concept with respect to controllability, then quite similar results may be obtained for the convex combination of the observability pairs.

Finally, it should be pointed out that the concept of convex combination is considered in theory of quantum systems. For example, convex combination of quantum channels and its properties have been recently studied in the paper by Jagadish, Srikanth and Patruccione (2020).

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