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SYNCHRONIZATION OF AN UNCERTAIN DUFFING OSCILLATOR WITH HIGHER ORDER CHAOTIC SYSTEMS

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The problem of practical synchronization of an uncertain Duffing oscillator with a higher order chaotic system is considered. Adaptive control techniques are used to obtain chaos synchronization in the presence of unknown parameters and bounded, unstructured, external disturbances. The features of the proposed controllers are compared by solving Duffing–Arneodo and Duffing–Chua synchronization problems.

Keywords: chaos synchronization, adaptive control, Duffing oscillator.

1. Introduction

Chaos synchronization has been intensively studied in the last 30 years. The problem is interesting as an important basic research issue and offers wide application possibilities. Secure information transmission and synchronization of biological systems (such as neuron lattices) are well-known fields of implementation. An extensive review of chaos synchronization techniques developed till 2010 can be found in the works of Zhang *et al*. (2009) or Andrievskii and Fradkov (2004). Primary results were limited to synchronization of two similar chaotic systems with perfectly known parameters. But in real-life applications, the parameters of physical systems are perturbed by external factors and cannot be known exactly. Therefore, adaptive control techniques became most important for synchronization of chaotic systems and are used in numerous references (e.g., Wang and Fan, 2015; Sundarapandian, 2011; 2010; Wang *et al.*, 2015; Chang *et al.*, 2009; Hua and Guan, 2004; Hua *et al.*, 2005).

In this contribution, a simple but very important chaotic system is considered—the Duffing oscillator. For many years, the Duffing oscillator has been investigated intensively as a benchmark of a chaotic system which is able to demonstrate all phenomena of chaos. Practical oscillating systems that exhibit Duffing-like behaviour occur in many areas: MEMS (Rhoads *et al*., 2008), laser techniques (Hofmann *et al*., 2012), wireless power harvesters (Wang and Mortazawi, 2016), and many others. It is assumed that all parameters of the chaotic system considered are unknown and that a bounded external disturbance is present.

The control aim is to synchronize the response system with another, higher order chaotic one. The problem of synchronization of chaotic systems possessing different structures, different models and a different number of state variables is interesting *per se* and may lead to important applications. For example, numerous biological systems (such as circulatory and respiratory systems) behave in a synchronous way, although they are quite different. The problem of synchronization of a Duffing oscillator with a Chua system was reported by Femat and Solís-Perales (2002), who considered a known parameter case with the solution depending on the drive system parameters. The authors stress that "*synchronization of different chaotic systems is a hard task if we think that: (i) initial conditions of master and slave systems are different and unknown, (ii) topological and geometrical properties of different chaotic systems are quite distinct and (iii) unrelated chaotic systems have strictly different time evolution*." Here, the problem is made more challenging by assuming unknown parameters and external disturbances.

The main contribution of the paper is development and comparison of three simple adaptive controllers for the chaos synchronization problem. The controllers are based on well-known adaptive control techniques.

The first one is a standard adaptive controller for the multidimensional system with disturbances. This approach utilizes a control Lyapunov function selected for a linear model. The other two are based on the adaptive backstepping technique (Krstic *et al.*, 1995), including the filtering of the stabilizing function (Dong *et al.*, 2012). To demonstrate that the same controller can be used for different drive (master) systems, the experiments with Arneodo and Chua chaotic systems are reported.

2. Problem formulation

Consider the Duffing equation

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -px_2 - p_1x_1 - p_2x_1^3 + q\cos(\omega t) + u,\end{aligned} \tag{1}$$

where t is time, ω is external force frequency, q is its amplitude, and p, p_1, p_2 are real constants. The variable u represents the control is an external signal (force) added to the periodic excitation. For $p_1 > 0$, the Duffing oscillator can be interpreted as a forced oscillator with a spring whose restoring force is $R = -p_1x_1 - p_2x_1^3$. When $p_2 > 0$, we have a 'hardening spring', and when $p_2 < 0$, we have a 'softening spring', although this interpretation is valid only for small x. For $p_1 < 0$, the Duffing equation describes a trajectory of a point mass in a double well potential. It can be a model of a steel beam deflection in between two magnets.

It is assumed that the constant parameters of the oscillator are unknown, and only initial guesses of the approximated values are provided. An external, bounded disturbance $d(t)$ satisfying

$$|d(t)| \leq \delta \tag{2}$$

is supposed to affect the system. The disturbance may represent a non-periodical component of the external force or an inaccurate realization of the derived control. The bound δ must be estimated according to our knowledge about the disturbance nature and impact. Finally, the uncertain Duffing oscillator model may be represented as

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= \theta^T\phi + d(t) + u,\end{aligned} \tag{3}$$

where

$$\theta \in \mathbb{R}^4 \tag{4}$$

is the vector of unknown parameters and

$$\phi^T = \begin{bmatrix} -p_0x_2 & -p_{10}x_1 & -p_{20}x_1^3 & q_0\cos(\omega t)\end{bmatrix} \tag{5}$$

is composed of known functions, while $\begin{bmatrix} p_0 & p_{10} & p_{20} & q_0\end{bmatrix}$ are the initial guesses for $\begin{bmatrix} p & p_1 & p_2 & q\end{bmatrix}$. Such a representation of uncertainty allows us to balance all adaptive parameters.

The Duffing oscillator (1) is supposed to be a "response" (slave) system, i.e., a system to be synchronized with another chaotic one—a "drive system" (master). In this contribution, the drive system is supposed to be a higher order chaotic system, and it may possess a different model structure as well as distinct topological and geometrical properties. The control objective is to achieve reduced-order synchronization, i.e., all states of the response system should be synchronized, in some sense, with two selected states of the drive system. Such synchronization is possible only if the selected states of the drive system, let us say, x_{1d} and x_{2d}, fulfil the phase canonical equation $\dot{x}_{1d} = x_{2d}$.

Two examples of drive systems will be considered, although generalization to other chaotic systems is straightforward. The first drive system is a three-dimensional chaotic Arneodo one,

$$\begin{aligned}\dot{x}_{1d} &= x_{2d},\\ \dot{x}_{2d} &= x_{3d},\\ \dot{x}_{3d} &= -a_0x_{1d} - a_1x_{2d} - a_2x_{3d} + a_3x_{1d}^2,\end{aligned} \tag{6}$$

which generates a chaotic motion with parameters $a_0 = -7.5, a_1 = 3.8, a_2 = 1, a_3 = -1, x_{1d}(0) = 3$.

The second drive system is the Chua one. It was introduced as a model of a simple electronic circuit that consists of one linear resistor, two capacitors, one inductor, and one nonlinear resistor. The standard state variables are the capacitor voltages and the inductor current, but the circuit equations can be also written in the following, standard, dimensionless form:

$$\begin{aligned}\dot{x}_{1s} &= \gamma_1\{x_{2s} - (\gamma_3+1)x_{1s}\\ &\quad - 0.5(\gamma_4 - \gamma_3)[|x_{1s}+1| - |x_{1s}-1|]\},\\ \dot{x}_{2s} &= x_{1s} - x_{2s} + x_{3s},\\ \dot{x}_{3s} &= -\gamma_2x_{2s}.\end{aligned} \tag{7}$$

A discussion of equivalent models is presented by Pospsil *et al.* (2000). The parameters in (7) are chosen as $\gamma_1 = 10.00, \gamma_2 = 14.87, \gamma_3 = -0.68, \gamma_4 = -1.27$. The change of variables $x_{1d} = x_{3s}$, $x_{2d} = -\gamma_2x_{2s}$, $x_{3d} = x_{1s}$ provides the system equations

$$\begin{aligned}\dot{x}_{1d} &= x_{2d},\\ \dot{x}_{2d} &= -\gamma_2x_{1d} - x_{2d} - \gamma_2x_{3d},\\ \dot{x}_{3d} &= \gamma_1\{-x_{3d} - \frac{1}{\gamma_2}x_{2d} - (\gamma_3x_{3d}\\ &\quad + 0.5(\gamma_4 - \gamma_3)[|x_{3d}+1| - |x_{3d}-1|])\},\end{aligned} \tag{8}$$

which are used as a drive system for a Duffing oscillator. Both the drive systems belong to the general class of n-dimensional chaotic systems with state variables x_d

described by the equations

$$\begin{aligned} \dot{x}_{1d} &= x_{2d}, \\ \dot{x}_{2d} &= f_2(x_d), \\ &\vdots \\ \dot{x}_{nd} &= f_n(x_d). \end{aligned} \tag{9}$$

Sample plots of the Duffing oscillator and the drive system trajectories are presented in Figs. 1–3.

3. Adaptive synchronization

Consider the tracking errors $e_1 = x_1 - x_{1d}$ and $e_2 = x_2 - x_{2d}$, $e = \begin{bmatrix} e_1 & e_2 \end{bmatrix}^T$. As $\dot{x}_{1d} = x_{2d}$, from (3) it follows that

$$\begin{aligned} \dot{e}_1 &= e_2, \\ \dot{e}_2 &= \theta^T \phi + d(t) + u - \dot{x}_{2d}. \end{aligned} \tag{10}$$

Let us denote by $\hat{\theta}$ the adaptive parameters and define the parameter estimation error by

$$\tilde{\theta} = \theta - \hat{\theta}. \tag{11}$$

The proposed control is

$$u = a_1 e_1 + a_2 e_2 - \hat{\theta}^T \phi + u_d + \dot{x}_{2d}. \tag{12}$$

The parameters a_1 and a_2 are selected so that the matrix

$$A = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} \tag{13}$$

is stable and u_d is an additional control component to be designed soon. The application of the proposed control results in

$$\dot{e} = Ae + \begin{bmatrix} 1 \\ \tilde{\theta}^T \phi \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (d(t) + u_d). \tag{14}$$

As the matrix A is stable, for any positive definite matrix Q there exists a positive definite matrix P, which is the solution of the Lyapunov equation

$$A^T P + PA = -Q. \tag{15}$$

The matrix P is used to construct the Lyapunov function

$$V(e, \tilde{\theta}) = \frac{1}{2}(e^T P e + \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}), \tag{16}$$

where the matrix of design parameters Γ is positive definite. The derivative of the Lyapunov function along the system trajectories can be transformed as follows:

$$\begin{aligned} \dot{V} = &-\frac{1}{2} e^T Q e + \tilde{\theta}^T \{(e^T P \begin{bmatrix} 0 \\ 1 \end{bmatrix}) \phi - \Gamma^{-1} \frac{\mathrm{d}}{\mathrm{d}t} \hat{\theta}\} \\ &+ e^T \begin{bmatrix} 0 \\ 1 \end{bmatrix} (\mathrm{d}(t) + u_d). \end{aligned} \tag{17}$$

If the additional control is not applied ($u_d = 0$), then a robust adaptive law must be used. Among many possibilities, such as σ-modification or projection, the so-called e-σ modification is applied (Ioannou and Sun, 1989):

$$\frac{\mathrm{d}}{\mathrm{d}t} \hat{\theta} = \Gamma\{(e^T P \begin{bmatrix} 0 \\ 1 \end{bmatrix}) \phi - \|e\| \, \sigma \hat{\theta}\}, \tag{18}$$

where $\sigma > 0$ is a design parameter. The adaptive law yields

$$\dot{V} = -\frac{1}{2} e^T Q e + e_2 d(t) + \|e\| \, \sigma \tilde{\theta}^T \hat{\theta}. \tag{19}$$

Making use of the following inequalities:

$$e_2 d(t) \leq |e_2| \, \delta \leq \|e\| \, \delta, \tag{20}$$

$$\tilde{\theta}^T \hat{\theta} = \frac{1}{2} \left[-\|\tilde{\theta}\|^2 + \|\theta\|^2 - \|\hat{\theta}\|^2 \right], \tag{21}$$

$$-\frac{1}{2} e^T Q e \leq -\frac{1}{2} \lambda_{\min}\{Q\} \, \|e\|^2, \tag{22}$$

where $\lambda_{\min}\{Q\}$ denotes the minimal eigenvalue of Q and $k := \frac{1}{2}\lambda_{\min}\{Q\}$, we get

$$\begin{aligned} \dot{V} \leq &- \|e\| \left\{ k \, \|e\| - \delta + \frac{1}{2}\sigma\|\tilde{\theta}\|^2 - \frac{1}{2}\sigma\|\theta\|^2 \right. \\ &\left. + \frac{1}{2}\sigma\|\hat{\theta}\|^2 \right\} \\ \leq &-\|e\|\{k\|e\| - \delta + \frac{1}{2}\sigma\|\tilde{\theta}\|^2 - \frac{1}{2}\sigma\|\theta\|^2\}. \end{aligned} \tag{23}$$

Therefore, $\dot{V} \leq 0$ if $k\|e\| - \delta + \frac{1}{2}\sigma\|\tilde{\theta}\|^2 - \frac{1}{2}\sigma\|\theta\|^2 \geq 0$. The derivative of the Lyapunov function is negative outside the set

$$D_e := \{e : \; \|e\| \leq \frac{1}{k}(\delta + \frac{1}{2}\sigma \, \|\theta\|^2)\} \tag{24}$$

in spite of $\|\tilde{\theta}\|$, and outside the set

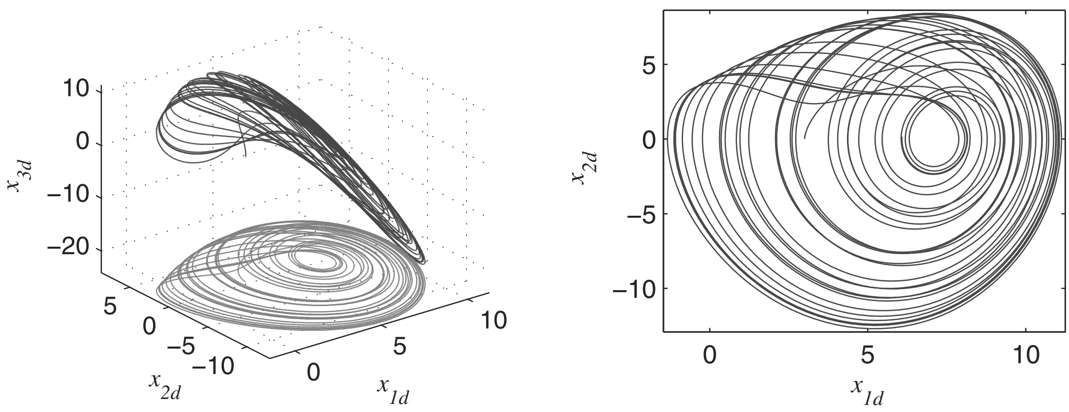
$$D_\theta := \{\tilde{\theta} : \; \|\tilde{\theta}\|^2 \leq \frac{2}{\sigma}(\delta + \frac{1}{2}\sigma \, \|\theta\|^2)\} \tag{25}$$

in spite of $\|e\|$.

From the well-known theorem due to LaSalle and Leftschetz (1961) it follows that the trajectories e and $\tilde{\theta}$ are uniformly, ultimately bounded (UUB) (Khalil, 2015). The sets D_e and D_θ which ultimately limit the evolution of e and $\tilde{\theta}$ are bounded (as $\delta, \sigma, \|\theta\|$ are bounded constants); moreover, the design parameter $k = \frac{1}{2}\lambda_{\min}\{Q\}$ can be used to reduce the volume of the limit set D_e.

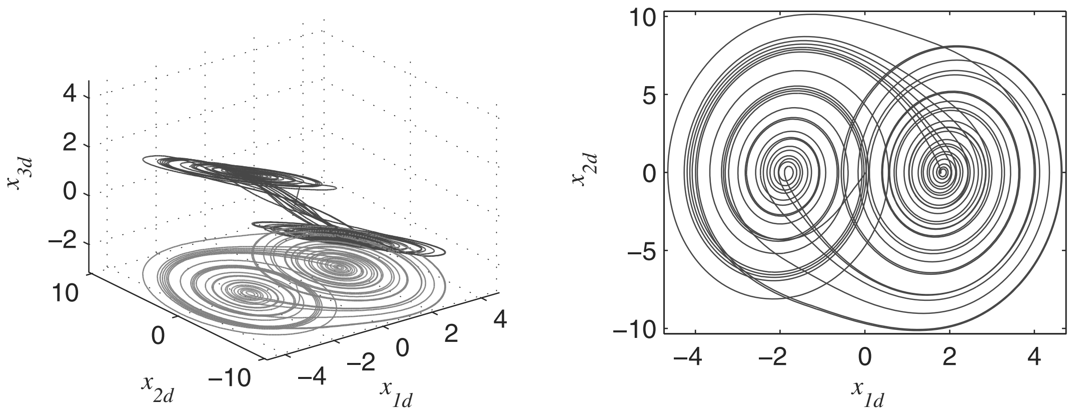
If we decide to cope with the disturbance $d(t)$ directly, an additional control term must be applied, such that $e_2(d(t) + u_d)$ is negative. If the selected control is

$$u_d = -\delta \, \mathrm{sign}(e_2), \tag{26}$$



(a) (b)

Fig. 1. Trajectories of the Arneodo system (a) and the canonical projection (b).



(a) (b)

Fig. 2. Trajectories of the Chua system (a) and the canonical projection (b).

we get

$$\begin{aligned} e_2(d(t) + u_d) &= e_2(d(t) - \delta \operatorname{sign}(e_2)) \\ &= -|e_2|[\delta + \operatorname{sign}(e_2)d(t)] \leq 0. \end{aligned} \tag{27}$$

The control u_d is singular if $e_2 = 0$; hence chattering can be expected. To obtain Lyapunov stability, it is sufficient to apply the regular adaptive law

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\theta} = \Gamma\phi(e^T P \begin{bmatrix} 0 \\ 1 \end{bmatrix}) \tag{28}$$

which, together with the control u_d, provides

$$\dot{V} = -\frac{1}{2}e^T Q e. \tag{29}$$

From the LaSalle–Yoshizawa theorem (Krstic *et al.*, 1995; Khalil, 2015) it follows that all trajectories are globally bounded and $e \to 0$.

4. Adaptive backstepping control

Adaptive backstepping is a systematic, recursive design procedure. The design process is conducted step by step, and each stage concerns a one-dimensional system. For the first step, the tracking error

$$z_1 = e_1 = x_1 - x_{1d} \tag{30}$$

is considered. The motion of e_1 is given by

$$\dot{z}_1 = \dot{x}_1 - \dot{x}_{1d} = x_2 - x_{2d}. \tag{31}$$

Let α denote the desired value of the “virtual control” x_2 to be defined. The change of coordinates

$$z_2 = x_2 - \alpha \tag{32}$$

yields

$$\dot{z}_1 = z_2 + \alpha - x_{2d}. \tag{33}$$

Therefore, selecting

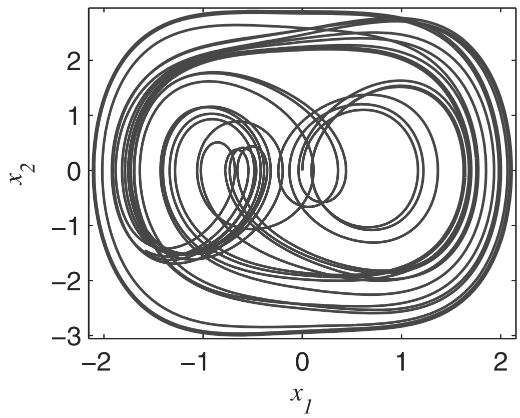


Fig. 3. Trajectories of the Duffing oscillator.

$$\alpha = -K_1 z_1 + x_{2d}, \tag{34}$$

where $K_1 > 0$ is the design parameter, results in

$$\dot{z}_1 = z_2 - K_1 z_1, \tag{35}$$

so it stabilizes the system if $z_2 \to 0$. The dynamics of z_2 are given by

$$\dot{z}_2 = \theta^T \phi + d(t) + u - \dot{\alpha}, \tag{36}$$

where θ and ϕ are defined in (4) and (5).

The second step of adaptive backstepping requires calculation of the derivative of the stabilizing function $\dot{\alpha}$. The way of obtaining such a derivative depends on the availability of state variables of the master system.

4.1. Available state variables of the drive system. The derivative $\dot{\alpha}$ is known and available for the control system as

$$\dot{\alpha} = -K_1 \dot{z}_1 + \dot{x}_{2d} = -K_1(x_2 - x_{2d}) + \dot{x}_{2d}. \tag{37}$$

The proposed control law is

$$u = -\hat{\theta}^T \phi + \dot{\alpha} - K_2 z_2 - z_1 + u_d, \tag{38}$$

where $K_2 > 0$ is the design parameter and u_d is an additional control component to be designed later. The application of control (38) yields

$$\dot{z}_2 = \tilde{\theta}^T \phi + d(t) - K_2 z_2 - z_1 + u_d. \tag{39}$$

The Lyapunov function

$$V(z_1, z_2, \tilde{\theta}) = \frac{1}{2}(z_1^2 + z_2^2 + \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}), \tag{40}$$

with a positive-definite matrix Γ, is applied to derive the adaptive law. The derivative of the Lyapunov function is

$$\begin{aligned} \dot{V} =& z_1(z_2 - K_1 z_1) + z_2(\tilde{\theta}^T \phi + d(t) \\ & - K_2 z_2 - z_1 + u_d) - \tilde{\theta}^T \Gamma^{-1} \frac{\mathrm{d}}{\mathrm{d}t}\hat{\theta} \\ =& - K_1 z_1^2 - K_2 z_2^2 + z_2(\mathrm{d}(t) + u_d) \\ & + \tilde{\theta}^T (z_2 \phi - \Gamma^{-1} \frac{\mathrm{d}}{\mathrm{d}t}\hat{\theta}). \end{aligned} \tag{41}$$

If the additional control is not applied ($u_d = 0$), then a robust adaptive law must be used. In much the same way as in (18), the so called e-σ-modification is applied, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\theta} = \Gamma\{z_2 \phi - \|z\| \sigma \hat{\theta}\}, \tag{42}$$

where $\sigma > 0$ is the design parameter and $z^T = \begin{bmatrix} z_1 & z_2 \end{bmatrix}$.

The adaptive law yields

$$\dot{V} = -K_1 z_1^2 - K_2 z_2^2 + z_2 d(t) + \|z\| \sigma \tilde{\theta}^T \hat{\theta}. \tag{43}$$

Making use of the inequalities

$$z_2 d(t) \leq |z_2| \delta \leq \|z\| \delta, \tag{44}$$

$$\tilde{\theta}^T \hat{\theta} = \frac{1}{2}\left[-\|\tilde{\theta}\|^2 + \|\theta\|^2 - \|\hat{\theta}\|^2\right], \tag{45}$$

$$- K_1 z_1^2 - K_2 z_2^2 \leq -K \|z\|^2, \tag{46}$$

$K = \min\{K_1, K_2\}$, leads to

$$\dot{V} \leq -\|z\| \left\{K \|z\| - \delta + \frac{1}{2}\sigma\|\tilde{\theta}\|^2 - \frac{1}{2}\sigma \|\theta\|^2\right\}. \tag{47}$$

Therefore, $\dot{V} < 0$ if $K \|z\| - \delta + \frac{1}{2}\sigma\|\tilde{\theta}\|^2 - \frac{1}{2}\sigma \|\theta\|^2 > 0$.

The derivative of the Lyapunov function is negative outside the set

$$\|z\| \leq \frac{1}{K}\left(\delta + \frac{1}{2}\sigma \|\theta\|^2\right) \tag{48}$$

irrespective of $\|\tilde{\theta}\|$, and for

$$\|\tilde{\theta}\|^2 > \frac{2}{\sigma}\left(\delta + \frac{1}{2}\sigma \|\theta\|^2\right) \tag{49}$$

irrespective of$\|z\|$.

It follows (similarly to (24) and (25)) from the previously cited theorem due to LaSalle and Leftschetz, that the trajectories z and $\tilde{\theta}$ are uniformly, ultimately bounded (UUB) and the design parameter K can be used to reduce the size of the limit set for z.

If we decide to cope with the disturbance $d(t)$ directly, an additional control term, such that $z_2(d(t)+u_d)$ is negative, must be applied. One can select

$$u_d = -\delta \,\mathrm{sign}(z_2) \tag{50}$$

to get

$$z_2(d(t)+u_d) = z_2(d(t)-\delta\mathrm{sign}(z_2)) \\ = -|z_2|[\delta+\mathrm{sign}(z_2)d(t)] \le 0. \tag{51}$$

The control u_d is singular if $z_2 = 0$, so chattering can be expected. If the control u_d is applied, the regular adaptive law

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\theta} = \Gamma z_2 \phi \tag{52}$$

provides

$$\dot{V} \le -K_1 z_1^2 - K_2 z_2^2. \tag{53}$$

From the LaSalle–Yoshizawa theorem it follows that all trajectories are globally uniformly bounded and $z \to 0$.

4.2. Unavailable state variables of the drive system.

If only the first two state variables of the drive system are available, a filter is used to calculate the derivative $\dot{\alpha}$ of $\alpha = -K_1 z_1 + x_{2d}$. The first-order filter

$$\dot{\beta} = -\Omega(\beta-\alpha) \tag{54}$$

with the state variable β and the design parameter $\Omega > 0$ assures that, when the filter transient is over, $\beta \approx \alpha$, and hence $\dot{\alpha} \approx -\Omega(\beta-\alpha)$. It is well known that, if $|\dot{\alpha}| < c < \infty$ and $\beta(0) = \alpha(0)$, then for all $t > 0$ we have $|\beta(t)-\alpha(t) \le c/\Omega$. Therefore, if $\rho(t) := \beta(t)-\alpha(t)$, it can be assumed that, for some $\epsilon > 0$,

$$|\rho| \le \epsilon, \quad \forall t > 0. \tag{55}$$

Define

$$z_{2f} := x_2 - \beta = x_2 - \alpha + \alpha - \beta = z_2 - \rho. \tag{56}$$

This leads to

$$\dot{z}_1 = z_{2f} + \rho - K_1 z_1. \tag{57}$$

The dynamics of the "filtered error" z_{2f} are described by

$$\dot{z}_{2f} = \dot{x}_2 - (\dot{\beta}) = \theta^T\phi + d(t) + u + \Omega(\beta-\alpha), \tag{58}$$

so the control

$$u = -\hat{\theta}\phi - \Omega(\beta-\alpha) - K_2 z_{2f} - z_1 + u_d \tag{59}$$

implies

$$\dot{z}_{2f} = \tilde{\theta}^T\phi + d(t) - K_2 z_{ef} - z_1 + u_d. \tag{60}$$

In this case, the Lyapunov function is

$$V(z_1, z_{2f}, \tilde{\theta}) = \frac{1}{2}(z_1^2 + z_{2f}^2 + \tilde{\theta}^T\Gamma^{-1}\tilde{\theta}). \tag{61}$$

The derivative of the Lyapunov function is

$$\begin{aligned}\dot{V} &= z_1(z_{2f}+\rho-K_1 z_1) \\ &+ z_{2f}\left(\tilde{\theta}^T\phi + d(t) - K_2 z_{ef} - z_1 + u_d\right) \\ &- \tilde{\theta}^T\Gamma^{-1}\frac{d}{dt}\hat{\theta} \\ &= -K_1 z_1^2 - K_2 z_{2f}^2 + z_1\rho + z_{2f}(d(t)+u_d) \\ &+ \tilde{\theta}^T\left(z_2\phi - \Gamma^{-1}\frac{d}{dt}\hat{\theta}\right).\end{aligned} \tag{62}$$

The "disturbance" $\rho(t)$ is not matched, so the control u_d is not able to compensate the component $z_1\rho$. Therefore, $u_d = 0$ and the following robust adaptive law is considered:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\theta} = \Gamma\{z_{2f}\phi - \|z_f\|\,\sigma\hat{\theta}\}, \tag{63}$$

where $\sigma > 0$ is a design parameter and $z_f^T = \begin{bmatrix} z_1 & z_{2f}\end{bmatrix}$. The selected adaptive law yields

$$\dot{V} = -K_1 z_1^2 - K_2 z_{2f}^2 + z_1\rho + z_{2f}d(t) + \|z_f\|\,\sigma\tilde{\theta}^T\hat{\theta}. \tag{64}$$

The inequality (45) and

$$\begin{aligned} z_1\rho + z_{2f}d(t) &\le |e_1|\epsilon + |z_{2f}|\delta \\ &\le \|z_f\|\,\mu, \quad \mu = \max\{\epsilon,\delta\}\end{aligned} \tag{65}$$

allow us to write

$$\dot{V} \le -\|z_f\|\left\{K\,\|z_f\| - \mu + \frac{1}{2}\sigma\|\tilde{\theta}\|^2 - \frac{1}{2}\sigma\,\|\theta\|^2\right\}, \tag{66}$$

where $K = \min\{K_1, K_2\}$. In much the same way, from the cited theorem due to LaSalle and Leftschetz it follows that the trajectories $z_f, \tilde{\theta}$ are uniformly, ultimately bounded (UUB) and the design parameter can be used to reduce the size of the limit set for z_f.

Of course, the application of the additional control component

$$u_d = -\delta\,\mathrm{sign}(z_{2f}) \tag{67}$$

compensates the term $z_{2f}(d(t)+u_d)$ and thus helps us to stabilize the system, but in this case the robust adaptive law is obligatory because of the component $z_1\rho$ in the Lyapunov function derivative $\dot{V}$.

5. Comparison of the proposed controllers

All presented controllers were compared operating with the Duffing oscillator as the response (slave) system and the Arneodo and Chua systems as the drive (master) system—all of them were presented in Section 2. The initial conditions for the adaptive parameters were selected 20% higher or lower than the real values. The disturbance was $d(t) = \cos(2\omega t)$.

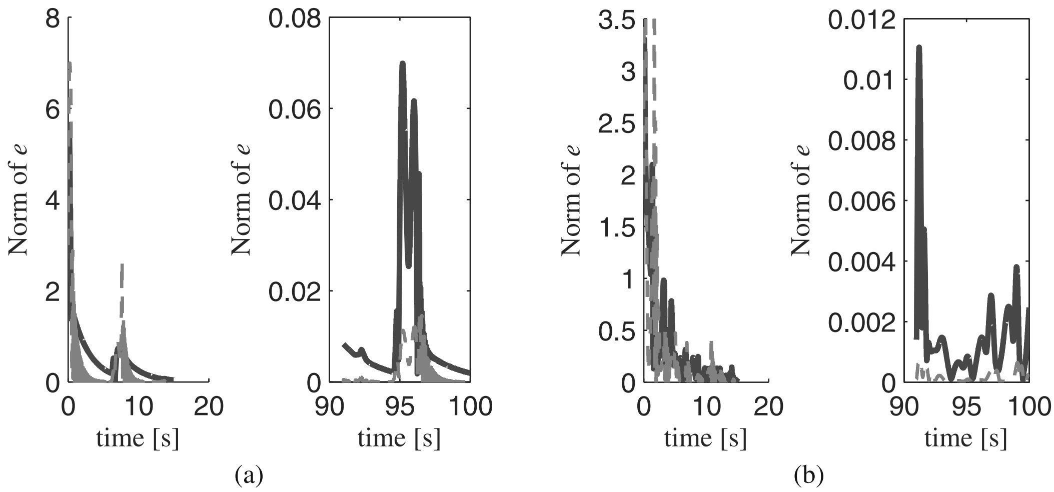


Fig. 4. Norm of the tracking error $\|e\|$ while following the Arneodo system (a) and the Chua system (b), without the disturbance: $d(t) = 0, \sigma = 0, u_d = 0$. The adaptive controller—a solid line, the backstepping controller—a dashed line.

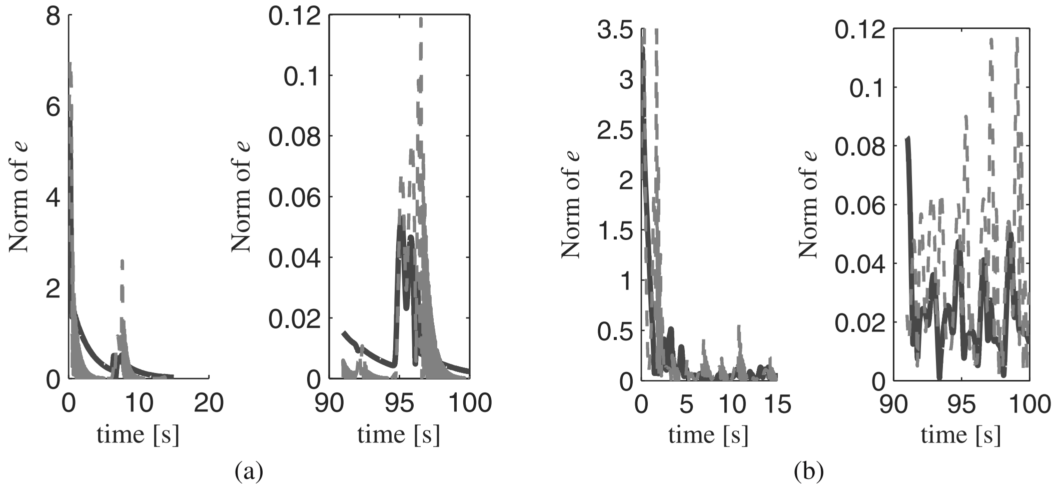


Fig. 5. Norm of the tracking error $\|e\|$ while following the Arneodo system (a) and the Chua system (b); the disturbance is active and only the robust adaptive law is used for compensation: $d(t) \neq 0, \sigma = 0, u_d = 0$. The adaptive controller—a solid line, the backstepping controller—a dashed line.

Parameters a_1 and a_2 of the adaptive controller were selected to place the eigenvalues of matrix A at $s_1 = -5, s_2 = -5$. The remaining parameters of this controller were $Q = 5\mathbf{I}(2), \Gamma = 10\mathbf{I}(4)$, with $\mathbf{I}(n)$ being the n-dimensional identity matrix.

Parameters of the backstepping controller were selected to obtain comparable dynamics. Therefore, parameters K_1, K_2 were chosen to place the eigenvalues of the matrix $\left[\begin{smallmatrix} -K_1 & 1 \\ -1 & -K_2 \end{smallmatrix}\right]$ (which describes the linear part of the error dynamics) in the same positions as in the case of the adaptive controller: $s_1 = -5, s_2 = -5$. To get approximately the same speed of adaptation, the adaptive law weights are reduced: $\Gamma = \mathbf{I}(4)$, and the robust adaptive law was applied with $\sigma = 0.1$. The filter parameter is $\Omega = 300$.

The comparison of the adaptive controller and the backstepping one is presented in Figs. 4–6. The same controller was applied to synchronize the Duffing oscillator with the Arneodo and Chua systems. Both the controllers are almost equivalent. The backstepping controller offers a slightly faster response during the initial period of time and slightly smaller steady-state errors. Both controllers offer fast tracking and are able to keep a small tracking error forever. The adaptive controller from Section 3 operates directly on the tracking error e, while the backstepping controller, described in Section 4.1, utilizes the error z, which is a linear transformation of e. Tuning the backstepping controller is more "informative" and straightforward, as the roles of each design parameters are clearly visible.

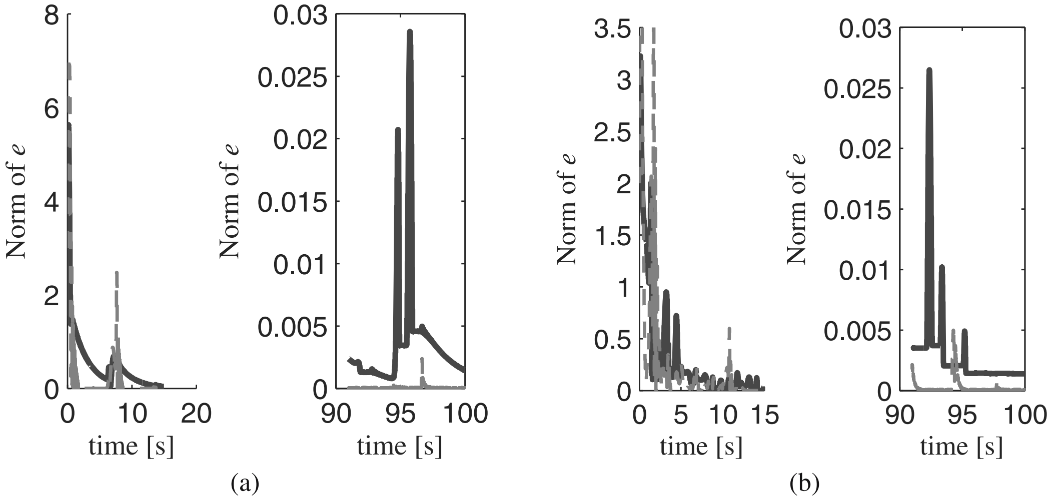


Fig. 6. Norm of the tracking error $\|e\|$ while following the Arneodo system (a) and the Chua system (b) with the disturbance compensated by the robust control law: $d(t) \neq 0, \sigma = 0, u_d \neq 0$. The adaptive controller—a solid line, the backstepping controller—a dashed line.

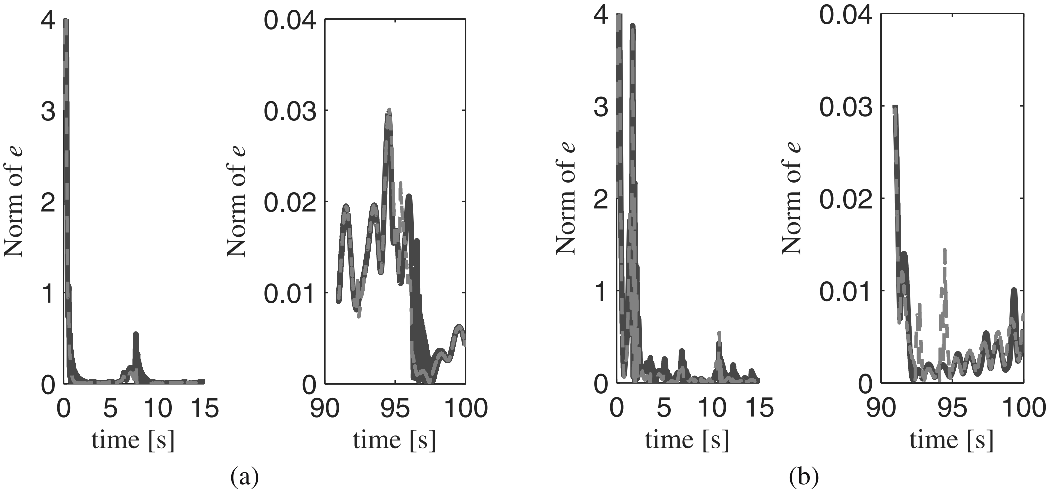


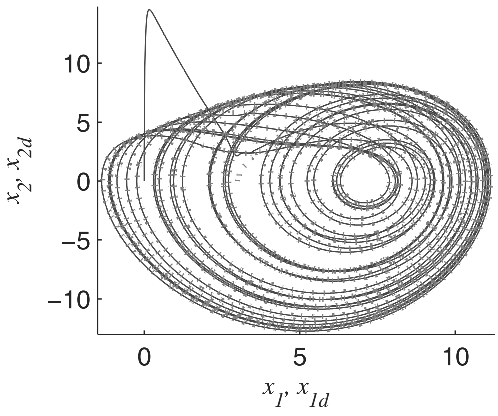
Fig. 7. Norm of the tracking error $\|e\|$ while following the Arneodo system (a) and the Chua system (b) with disturbance—a dashed line and without disturbance—a solid line. The adaptive backstepping controller described in Section 4.2.

The effects of the controller presented in Section 4.2 are demonstrated in Fig. 7. In spite of the unavailable derivative $\dot{x}_{2d}$, the closed-loop system is able to track the Arneodo or Chua system trajectories with sufficient accuracy. Of course, the use of the filter results in higher errors than in the case of known state variables. Finally, the trajectories of the response and the drive system, for the worst case observed during the experiments, are presented in Fig. 8.

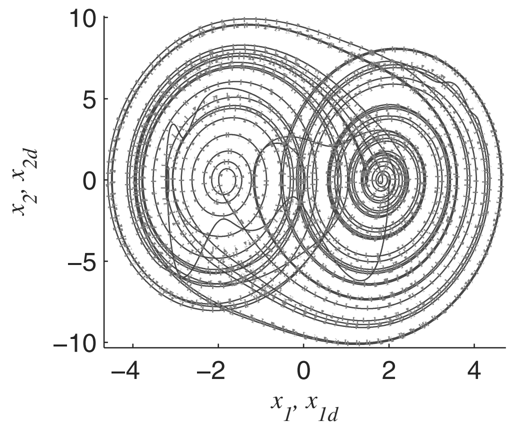
6. Conclusions

All the presented controllers were able to synchronize an uncertain Duffing oscillator with a canonical-plane projection of a higher order chaotic system. The same controller can be used with different drive (master) systems—it was tested with the Arneodo and Chua ones. It is not required to know the drive system parameters, and only approximate values of the parameters of the Duffing oscillator are necessary to propose starting values of the adaptive parameters. The closed-loop system performs well in the presence of an unstructured and bounded disturbance. The application of the switching control component compensating the disturbance is evidently beneficial, although the chattering of the control signal is possible. Any standard smooth approximation of the sign function may be used to eliminate the chattering.

The same adaptive techniques (as presented here) may be used to synchronize a chaotic Duffing



(a)



(b)

Fig. 8. Trajectory of the Duffing oscillator (solid) following the drive system (dotted): the Arneodo system (a) and the Chua system (b). The worst case observed.

oscillator with a stable limit cycle (chaos suppression) or a non-chaotic oscillator with a chaotic system (chaotification) (Kabziński, 2010). The state constraints may also be taken into account, as done by Kabziński (2016).

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