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Dedicated to the Memory of Professor Zdzisław Kamont

ABOUT SIGN-CONSTANCY OF GREEN'S FUNCTIONS FOR IMPULSIVE SECOND ORDER DELAY EQUATIONS

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Abstract. We consider the following second order differential equation with delay

$$\begin{cases}(Lx)(t) \equiv x''(t) + \sum_{j=1}^{p} b_j(t)x(t-\theta_j(t)) = f(t), \quad t \in [0,\omega],\\ x(t_j) = \gamma_j x(t_j - 0), x'(t_j) = \delta_j x'(t_j - 0), \quad j = 1, 2, \ldots, r.\end{cases}$$

In this paper we find necessary and sufficient conditions of positivity of Green's functions for this impulsive equation coupled with one or two-point boundary conditions in the form of theorems about differential inequalities. By choosing the test function in these theorems, we obtain simple sufficient conditions. For example, the inequality $\sum_{i=1}^{p} b_i(t)\left(\frac{1}{4}+r\right) < \frac{2}{\omega^2}$ is a basic one, implying negativity of Green's function of two-point problem for this impulsive equation in the case $0 < \gamma_i \leq 1$, $0 < \delta_i \leq 1$ for $i = 1, \ldots, p$.

Keywords: impulsive equations, Green's functions, positivity/negativity of Green's functions, boundary value problem, second order.

Mathematics Subject Classification: 34K10, 34B37, 34A40, 34A37, 34K48.

1. INTRODUCTION

Let us consider the following impulsive equation:

$$(Lx)(t) \equiv x''(t) + \sum_{j=1}^{p} b_j(t)x(t-\theta_j(t)) = f(t), \quad t \in [0,\omega], \tag{1.1}$$

$$x(t_j) = \gamma_j x(t_j - 0), \quad x'(t_j) = \delta_j x'(t_j - 0), \quad j = 1, 2, \ldots, r, \tag{1.2}$$

$$0 = t_0 < t_1 < t_2 < \ldots < t_r < t_{r+1} = \omega,$$

$$x(\zeta) = 0, \quad \zeta < 0, \tag{1.3}$$

where $f, b_j : [0,\omega] \to \mathbb{R}$ are summable functions and $\theta_j : [0,\omega] \to [0,+\infty)$ are measurable functions for $j = 1, 2, \ldots, p$. p and r are natural numbers, γ_i and δ_i are real positive numbers.

For equation (1.1) we consider the following variants of boundary conditions:

$$x(0) = \alpha_0, \quad x'(0) = \beta_0, \tag{1.4}$$

$$x(\omega) = \alpha_0, \quad x'(\omega) = \beta_0, \tag{1.5}$$

$$x(0) = \alpha_0, \quad x(\omega) = \beta_0. \tag{1.6}$$

Differential equations with impulses has attracted the attention of many researchers. Note the monographs [3, 5, 22, 26–28, 30], in which problems of existence, uniqueness and stability are considered.

Note that one possible approach to study impulsive equations is the theory of generalized differential equations allowing researchers to consider systems with continuous and discontinuous solutions in the framework of the delay equations (see, for example [2, 12, 13, 16, 21, 29, 31, 32]). In the works [17, 22, 23, 26–28, 30, 34], impulsive ordinary differential equations were considered. Let us assume that all trajectories of solutions to non-impulsive ordinary differential equations are known. In this case, impulses imply only choosing the trajectory between the points of impulses, but we stay on a trajectory of a corresponding solution of a non-impulsive equation between t_i and t_{i+1}.

In the case of an impulsive equation with delay it is not true anymore. That is why properties of delay impulsive equations can be quite different. Oscillation/nonoscillation and stability of delay differential equations are considered in [1, 6–9, 35]. Delay impulsive differential equations of second order are considered concerning stabilization by impulses in [15, 36]. For second order delay differential equations we succeeded to find only the paper [33] where their nonoscillation is studied. There are almost no results about boundary value problems for impulsive differential equations of high orders. Note that second order ordinary impulsive differential equations are considered in [4, 17, 34]. The Dirichlet boundary value problem is studied in [24] and the generalized Dirichlet problem in [14, 18, 24]. For delay differential equations we note only the paper [10].

In this paper we develop the approach of [9]. This approach is based on the analysis of Green's functions of auxiliary impulsive equations. Note that for first order functional differential equations these Green's functions for nonlocal boundary value problems are constructed in [11]. We construct Green's functions for various auxiliary boundary value problems for second order impulsive equations with delay. Our approach is based on a reduction of the impulsive boundary value problem to an integral equation and then the corresponding Krasnoselskii's theorems about estimates of the spectral radius are used. On this basis, we obtain theorems on differential inequalities allowing us to make the conclusion about sign constancy of Green's functions. Choosing the test functions, we get conditions of positivity/negativity of the Green's functions.

2. CONSTRUCTION OF THE CAUCHY AND GREEN'S FUNCTIONS TO AUXILIARY IMPULSIVE EQUATIONS

It is known that the solution of equation (1.1) with the homogenous initial conditions

$$x(0) = 0, \quad x'(0) = 0 \tag{2.1}$$

can be represented in the form

$$x(t) = \int_0^t C(t,s) f(s) ds, \quad t \in [0, \omega]. \tag{2.2}$$

The kernel $C(t,s)$ (defined in the zone $0 \leq s \leq t \leq \omega$) is called the Cauchy function of equation (1.1)–(1.3). It can be noted that the Cauchy function $C(t,s)$ as a function of t for fixed $s \in [0,\omega)$ satisfies the problem

$$(Lx)(t) \equiv x''(t) + \sum_{j=1}^{p} b_j(t) x(t - \theta_j(t)) = 0, \quad t \in [s, \omega], \tag{2.3}$$

$$x(t_j) = \gamma_j x(t_j - 0), \quad x'(t_j) = \delta_j x'(t_j - 0), \quad j = k, \ldots, r, \tag{2.4}$$

$$t_{k-1} < s < t_k < \ldots < t_r < t_{r+1} = \omega,$$

$$x(\zeta) = 0, \quad \zeta < 0. \tag{2.5}$$

Denote by $C_0(t,s)$ the Cauchy function of the equation $x''(t) = f(t)$ with impulses (1.2). For every fixed s, the Cauchy function $C_0(t,s)$ satisfies the problem

$$\begin{cases} x''(t) = 0, & t \in [s, \omega], \\ x(s) = 0, & x'(s) = 1, \end{cases} \tag{2.6}$$

with impulses (1.2).

Denote by $G_0(t,s)$ the Green's function of the problem

$$\begin{cases} x''(t) = f(t), & t \in [0, \omega], \\ x(0) = 0, & x(\omega) = 0, \end{cases} \tag{2.7}$$

with impulse (1.2).

Let us consider the following equation:

$$\begin{cases} x''(t) = f(t), & t \in [0, \omega], \\ x(t_j) = \gamma_j x(t_j - 0), & j = 1, 2, \ldots, r, \\ x'(t_j) = \delta_j x'(t_j - 0), & j = 1, 2, \ldots, r, \end{cases} \tag{2.8}$$

with initial condition (1.4).

The general solution of this equation in the interval $t \in [t_i, t_{i+1})$ can be represented as follows:

$$x(t) = \alpha_i + \beta_i (t - t_i) + \int_{t_i}^{t} (t-s) f(s) ds, \tag{2.9}$$

where $\alpha_i = x(t_i), \beta_i = x'(t_i)$.

From (2.9) and the impulse conditions, we get a recursive formula for solution in the intervals $t \in [t_i, t_{i+1})$ for $i = 1, 2, \ldots, r$:

$$\begin{cases} x(t) = x(0) + x'(0)t + \int_0^t (t-s) f(s) ds, & t \in [0, t_1], \\ x(t) = \alpha_i + \beta_i (t - t_i) + \int_{t_i}^t (t-s) f(s) ds, & t \in [t_i, t_{i+1}), \\ \alpha_i = \gamma_i \big[\alpha_{i-1} + \beta_{i-1} (t_i - t_{i-1}) + \int_{t_{i-1}}^{t_i} (t_i - s) f(s) ds\big], \\ \beta_i = \delta_i \big[\beta_{i-1} + \int_{t_{i-1}}^{t_i} f(s) ds\big], \end{cases} \tag{2.10}$$

Let us build the Cauchy function $C_0(t,s)$ of the problem (2.8), (1.4), for example, in the case $r = 3$. First, let us build the solution of the equation for each interval.

For $t \in [0, t_1)$,

$$x(t) = \alpha_0 + \beta_0 t + \int_0^t (t-s) f(s) ds.$$

For $t \in [t_1, t_2)$,

$$\begin{aligned} x(t) &= \alpha_1 + \beta_1 (t - t_1) + \int_{t_1}^{t} (t-s) f(s) ds = \gamma_1 \left[\alpha_0 + \beta_0 t_1 + \int_0^{t_1} (t_1 - s) f(s) ds\right] + \\ &\quad + \delta_1 \left[\beta_0 + \int_0^{t_1} f(s) ds\right] (t - t_1) + \int_{t_1}^{t} (t-s) f(s) ds = \\ &= \alpha_0 \gamma_1 + \beta_0 (\gamma_1 t_1 + \delta_1 (t - t_1)) + \\ &\quad + \int_0^{t_1} \left[\gamma_1 (t_1 - s) + \delta_1 (t - t_1)\right] f(s) ds + \int_{t_1}^{t} (t-s) f(s) ds, \end{aligned}$$

$$\begin{aligned} \alpha_2 = \gamma_2 \Bigg[&\alpha_0 \gamma_1 + \beta_0 (\gamma_1 t_1 + \delta_1 (t_2 - t_1)) + \int_0^{t_1} \left[\gamma_1 (t_1 - s) + \delta_1 (t_2 - t_1)\right] f(s) ds + \\ &+ \int_{t_1}^{t_2} (t_2 - s) f(s) ds \Bigg], \end{aligned}$$

$$\beta_2 = \delta_2 \left[\delta_1 \left(\beta_0 + \int_0^{t_1} f(s) ds\right) + \int_{t_1}^{t_2} f(s) ds\right].$$

For $t \in [t_2, t_3)$,

$$x(t) = \alpha_2 + \beta_2 (t - t_2) + \int\limits_{t_2}^{t} (t-s) f(s) ds =$$

$$= \gamma_2 \Bigg[\alpha_0 \gamma_1 + \gamma_1 \beta_0 t_1 + \delta_1 \beta_0 (t_2 - t_1) + + \int\limits_{0}^{t_1} [\gamma_1 (t_1 - s) + \delta_1 (t_2 - t_1)] f(s) ds + + \int\limits_{t_1}^{t_2} (t_2 - s) f(s) ds \Bigg] +$$

$$+ \delta_2 \left[\delta_1 \left(\beta_0 + \int\limits_{0}^{t_1} f(s) ds \right) + \int\limits_{t_1}^{t_2} f(s) ds \right] (t - t_2) +$$

$$+ \int\limits_{t_2}^{t} (t-s) f(s) ds =$$

$$= \gamma_2 \gamma_1 \alpha_0 + \beta_0 (\gamma_2 \gamma_1 t_1 + \gamma_2 \delta_1 (t_2 - t_1) + \delta_2 \delta_1 (t - t_2)) +$$

$$+ \int\limits_{0}^{t_1} [\gamma_2 \gamma_1 (t_1 - s) + \gamma_2 \delta_1 (t_2 - t_1) + \delta_2 \delta_1 (t - t_2)] f(s) ds +$$

$$+ \int\limits_{t_1}^{t_2} [\gamma_2 (t_2 - s) + \delta_2 (t - t_2)] f(s) ds +$$

$$+ \int\limits_{t_2}^{t} (t-s) f(s) ds,$$

$$\alpha_3 = \gamma_3 \Bigg[\gamma_2 \gamma_1 \alpha_0 + \beta_0 (\gamma_2 \gamma_1 t_1 + \gamma_2 \delta_1 (t_2 - t_1) + \delta_2 \delta_1 (t_3 - t_2)) + + \int\limits_{0}^{t_1} [\gamma_2 \gamma_1 (t_1 - s) + \gamma_2 \delta_1 (t_2 - t_1) + \delta_2 \delta_1 (t_3 - t_2)] f(s) ds + + \int\limits_{t_1}^{t_2} [\gamma_2 (t_2 - s) + \delta_2 (t_3 - t_2)] f(s) ds + + \int\limits_{t_2}^{t_3} (t_3 - s) f(s) ds \Bigg],$$

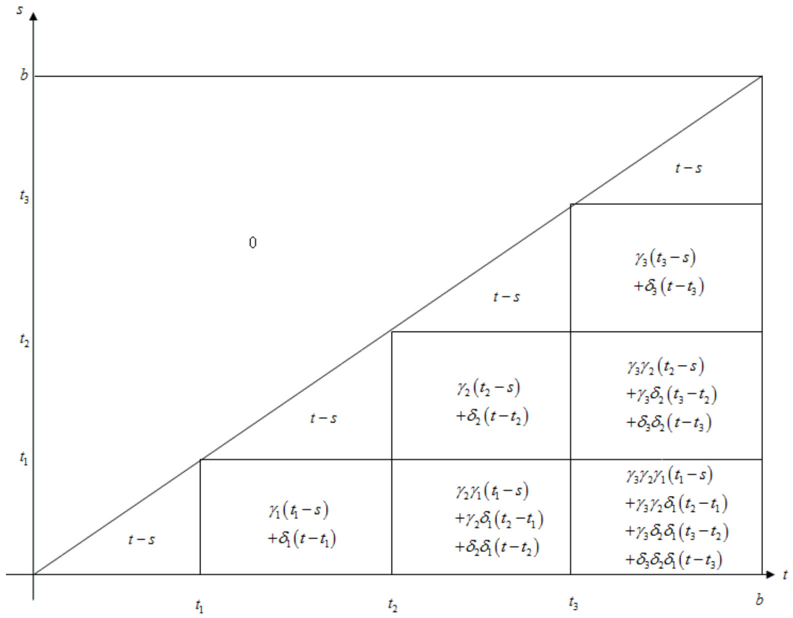
$$\beta_3 = \delta_3 \left[\delta_2 \delta_1 \beta_0 + \delta_2 \delta_1 \int\limits_{0}^{t_1} f(s) ds + \delta_2 \int\limits_{t_1}^{t_2} f(s) ds + \int\limits_{t_2}^{t_3} f(s) ds \right].$$

For $t \in [t_3, \omega]$,

$$
\begin{aligned}
x(t) &= \alpha_3 + \beta_3 (t - t_3) + \int_{t_2}^{t} (t - s) f(s) ds = \\
&= \gamma_3 \Bigg[\gamma_2 \gamma_1 \alpha_0 + \beta_0 \left(\gamma_2 \gamma_1 t_1 + \gamma_2 \delta_1 (t_2 - t_1) + \delta_2 \delta_1 (t_3 - t_2) \right) + \\
&\qquad + \int_{0}^{t_1} \left[\gamma_2 \gamma_1 (t_1 - s) + \gamma_2 \delta_1 (t_2 - t_1) + \delta_2 \delta_1 (t_3 - t_2) \right] f(s) ds + \\
&\qquad + \int_{t_1}^{t_2} \left[\gamma_2 (t_2 - s) + \delta_2 (t_3 - t_2) \right] f(s) ds + \\
&\qquad + \int_{t_2}^{t_3} (t_3 - s) f(s) ds \Bigg] + \\
&\quad + \delta_3 \left[\delta_2 \delta_1 \beta_0 + \delta_2 \delta_1 \int_{0}^{t_1} f(s) ds + \delta_2 \int_{t_1}^{t_2} f(s) ds + \int_{t_2}^{t_3} f(s) ds \right] (t - t_3) + \\
&\quad + \int_{t_2}^{t} (t - s) f(s) ds = \\
&= \gamma_3 \gamma_2 \gamma_1 \alpha_0 + \\
&\quad + \beta_0 (\gamma_3 \gamma_2 \gamma_1 t_1 + \gamma_3 \gamma_2 \delta_1 (t_2 - t_1) + \\
&\quad + \gamma_3 \delta_2 \delta_1 (t_3 - t_2) + \delta_3 \delta_2 \delta_1 (t - t_3)) + \\
&\quad + \int_{0}^{t_1} [\gamma_3 \gamma_2 \gamma_1 (t_1 - s) + \gamma_3 \gamma_2 \delta_1 (t_2 - t_1) + \gamma_3 \delta_2 \delta_1 (t_3 - t_2) + \\
&\quad + \delta_3 \delta_2 \delta_1 (t - t_3)] f(s) ds + \\
&\quad + \int_{t_1}^{t_2} \left[\gamma_3 \gamma_2 (t_2 - s) + \gamma_3 \delta_2 (t_3 - t_2) + \delta_3 \delta_2 (t - t_3) \right] f(s) ds + \\
&\quad + \int_{t_2}^{t_3} \left[\gamma_3 (t_3 - s) + \delta_3 (t - t_3) \right] f(s) ds + \int_{t_3}^{t} (t - s) f(s) ds.
\end{aligned}
$$

After these calculations, we can describe the Cauchy function by distributing its values to zones in the plane of the variables t and s (see Table 1).

Table 1. The Cauchy function of impulsive equation (2.8)



For every number r of impulses, we get the following analytical representation of the Cauchy function $C_0(t,s)$ of the impulsive equation $x''(t) = f(t)$:

$$C_0(t,s) = \sum_{i=1}^{r}\sum_{j=0}^{i-1}\Big[\prod_{k=j+1}^{i}\gamma_k(t_{j+1}-s) + \sum_{l=j+2}^{i}\prod_{k=l}^{i}\gamma_k\prod_{k=j+1}^{l-1}\delta_k(t_l-t_{l-1}) + \\ + \prod_{k=j+1}^{i}\delta_k(t-t_i)\Big][H_{t_i}(t)-H_{t_{i+1}}(t)][H_{t_j}(s)-H_{t_{j+1}}(s)] + \\ + \sum_{i=0}^{r}H_s(t)(t-s)[H_{t_i}(t)-H_{t_{i+1}}(t)][H_{t_i}(s)-H_{t_{i+1}}(s)], \tag{2.11}$$

where $H_{t_i}(t)$ is the Heaviside function

$$H_{t_i}(t) = \begin{cases} 1, & t_i \le t, \\ 0, & t < t_i, \end{cases} \tag{2.12}$$

and the general solution for the boundary value problem (2.8), (1.4) can be represented in the form:

$$x(t) = \prod_{i=1}^{j}\gamma_i\alpha_0 + C_0(t,0)\beta_0 + \int_0^t C_0(t,s)f(s)ds, \quad t \in [t_j, t_{j+1}), \tag{2.13}$$

where the Cauchy function $C_0(t,s)$ of this problem is defined by (2.11) where $C_0(t,s) = 0$ for $t < s$, and $j = 0, 1, \dots, r$, where $t_0 = 0$ and $\prod_{i=1}^{j} \gamma_i = 1$ for $j = 0$. As we expect, this representation is analogous to the representation of the general solution for the first order obtained in [9]. Summarizing, we can formulate our results in the following lemma.

Lemma 2.1. *The general solution of the boundary value problem* (2.8), (1.4) *can be represented in the form:*

$$x(t) = U(t) + \int_0^t C_0(t,s) f(s) ds, \tag{2.14}$$

where the Cauchy function $C_0(t,s)$ *of this problem is defined by* (2.11) *with* $C_0(t,s) = 0$ *for* $t < s$ *and*

$$U(t) = \prod_{i=1}^{j} \gamma_i \alpha_0 + C_0(t,0)\beta_0, \quad t \in [t_j, t_{j+1}), \quad j = 0, 1, \dots, r, \quad t_0 = 0. \tag{2.15}$$

Let us build now the Green's function $P_0(t,s)$ of the problem (2.8) with (1.5). Its solution can be represented in the form

$$x(t) = \int_0^{\omega} P_0(t,s) f(s) ds. \tag{2.16}$$

It is clear that this problem is symmetrical with (2.8), (1.4). So if we write $\omega - t$ instead of t, we get the problem

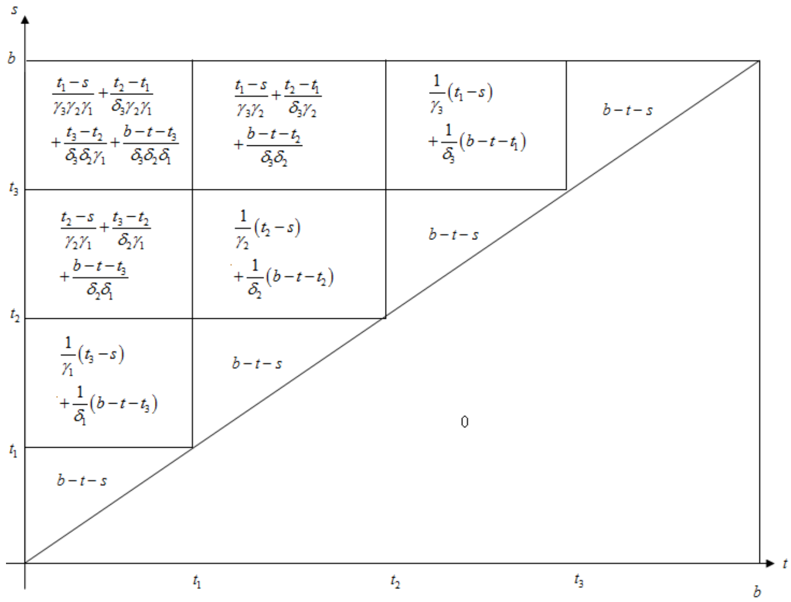
$$\begin{cases} x''(\omega - t) = f(\omega - t), & \omega - t \in [0, \omega], \\ x(\omega) = \alpha_0, \quad x'(\omega) = \beta_0, \end{cases} \tag{2.17}$$

and consequently $x(t) = U(\omega - t) + \int_0^t C_0(\omega - t, s) f(s) ds$ and $P_0(t,s) = C_0(\omega - t, s)$. Since we need to know the impulses in $\omega - t_j$, let us assume that there is the symmetry such that $t_j = \omega - t_{r-j}$. So we get a new representation of the impulses

$$\begin{cases} x(\omega - t_j) = \frac{1}{\gamma_{r-j}} x((\omega - t_j) - 0), & j = 1, 2, \dots, r, \\ x'(\omega - t_j) = \frac{1}{\delta_{r-j}} x'((\omega - t_j) - 0), & j = 1, 2, \dots, r. \end{cases} \tag{2.18}$$

After this calculation we obtain representation of $P_0(t,s)$ and we can describe Green's function $P_0(t,s)$ by distributing its values to zones in the plane of the variables t and s (see Table 2).

Table 2. The Green's function $P_0(t,s)$



Summarizing, we can formulate our results in the following lemma.

Lemma 2.2. *The general solution for the boundary value problem* (2.8) *with* (1.5) *can be represented in the form:*

$$x(t) = W(t) + \int_0^\omega P_0(t,s)f(s)ds, \quad t \in [0,\omega], \tag{2.19}$$

where the Green's function $P_0(t,s)$ *of this problem is*

$$P_0(t,s) = C_0(\omega - t, s), \quad t,s \in [0,\omega], \tag{2.20}$$

where the Cauchy function $C_0(t,s)$ *of this problem is* (2.11) *with* $C_0(t,s) = 0$ *for* $t < s$ *and*

$$W(t) = U(\omega - t), \quad t \in [0,\omega]. \tag{2.21}$$

Let us build now the Green's function $G_0(t,s)$ of the problem (2.8) with (1.6). This problem is similar to the first problem with the difference that we do not have $x'(0)$. Let us use the second boundary condition $x(\omega) = \beta_0$ in order to find a representation of $x'(0)$ through α_0 and β_0. From the general solution of the problem we get

$$x(\omega) = \beta_0 = \prod_{i=1}^{r} \gamma_i \alpha_0 + C_0(\omega,0)x'(0) + \int_0^\omega C_0(\omega,s)f(s)ds.$$

From here, we obtain

$$x'(0) = \frac{\beta_0 - \prod_{i=1}^{r} \gamma_i \alpha_0 - \int\limits_0^{\omega} C_0(\omega, s) f(s) ds}{C_0(\omega, 0)}$$

and the general solution for the boundary value problem (2.8) with (1.6) can be represented in the form:

$$x(t) = \prod_{i=1}^{r} \gamma_i \alpha_0 + C_0(t,0) \frac{\beta_0 - \prod_{i=1}^{r} \gamma_i \alpha_0}{C_0(\omega, 0)} + \int\limits_0^{\omega} \left[C_0(t,s) - C_0(\omega, s) \frac{C_0(t,0)}{C_0(\omega,0)} \right] f(s) ds. \tag{2.22}$$

Thus the Green's function $G_0(t,s)$ of this problem is

$$G_0(t,s) = C_0(t,s) - C_0(\omega,s) \frac{C_0(t,0)}{C_0(\omega,0)}. \tag{2.23}$$

Summarizing, we obtained the actual representation of $G_0(t,s)$ and formulate the following lemma.

Lemma 2.3. *The general solution for the boundary value problem* (2.8) *with* (1.6) *can be represented in the form:*

$$x(t) = V(t) + \int\limits_0^{\omega} G_0(t,s) f(s) ds, \quad t \in [0, \omega], \tag{2.24}$$

where the Green's function $G_0(t,s)$ of this problem is

$$G_0(t,s) = C_0(t,s) - C_0(\omega,s) \frac{C_0(t,0)}{C_0(\omega,0)}, \quad t \in [0,\omega], \tag{2.25}$$

where the Cauchy function $C_0(t,s)$ of this problem is (2.11) *with $C_0(t,s) = 0$ for $t < s$ and*

$$V(t) = \prod_{i=1}^{j} \gamma_i \alpha_0 + C_0(t,0) \frac{\beta_0 - \prod_{i=1}^{r} \gamma_i \alpha_0}{C_0(\omega,0)}, \quad t \in [t_j, t_{j+1}), \quad j = 0, 1, \ldots, r, \quad t_0 = 0. \tag{2.26}$$

3. POSITIVITY AND NEGATIVITY OF GREEN'S FUNCTIONS OF THE AUXILIARY IMPULSIVE EQUATION

In this section we prove positivity or negativity of Green's function for one and two-point impulsive problems with the auxiliary equation $x''(t) = f(t)$.

Lemma 3.1. *If $\gamma_k, \delta_k > 0, k = 1, \ldots, r$, then $C_0(t,s)$ is positive in $0 \le t \le s < \omega$.*

Proof. Let us assume that $t \in [t_i, t_{i+1})$ and $s \in [t_j, t_{j+1})$, where $i \geq j$, $i, j = 0, \ldots, r$, $t_0 = 0$. If $i = j$, then $C_0(t, s) = t - s \geq 0$. If $i > j$, then

$$C_0(t,s) = \prod_{k=j+1}^{i} \gamma_k (t_{j+1} - s) + \prod_{k=j+1}^{i} \delta_k (t - t_i) + \sum_{l=j+2}^{i-1} \prod_{k=l}^{i} \gamma_k \prod_{k=j+1}^{l-1} \delta_k (t_l - t_{l-1}) \geq 0,$$

since $t_l > t_{l-1}$, $l = 2, \ldots, r$, $t_{j+1} > s$, $t \geq t_i$ and $\gamma_k, \delta_k > 0$, $k = 1, \ldots, r$. Lemma 3.1 has been proven. □

Lemma 3.2. *If γ_k, $\delta_k > 0$, $k = 1, \ldots, r$, then $P_0(t, s)$ is positive for $(t, s) \in (0, \omega) \times (0, \omega), t < s$.*

Proof. From Lemma 3.1 we know that $C_0(t, s)$ is non-negative for every t and for almost every s. Since $t \in [0, \omega]$ implies that $(\omega - t) \in [0, \omega]$, we get, from the previous lemma, that $C_0(\omega - t, s)$ is non-negative for every t and for almost every s. Hence, $P_0(t, s) = C_0(\omega - t, s)$ is non-negative for every t and for almost every s. Lemma 3.2 has been proven. □

Remark 3.3. Let us assume that $t \in [t_i, t_{i+1})$ and $s \in [t_j, t_{j+1})$, where $i > j$. Then, we can write

$$C_0(t,s) = a_{i,j} + \left(\prod_{k=j+1}^{i} \delta_k \right) t - \left(\prod_{k=j+1}^{i} \gamma_k \right) s, \tag{3.1}$$

where

$$a_{i,j} = \sum_{l=j+1}^{i} \left(\prod_{k=l+1}^{i} \gamma_k \right) \left(\prod_{k=j+1}^{l-1} \delta_k \right) (\gamma_l - \delta_l)\, t_l. \tag{3.2}$$

Theorem 3.4. *If γ_k, $\delta_k > 0$, $k = 1, \ldots, r$, then $G_0(t, s)$ is negative for every $(t, s) \in (0, \omega) \times (0, \omega)$.*

Proof. Let us assume that $t \in [t_i, t_{i+1})$ and $s \in [t_j, t_{j+1})$, where $i > j$. So if we use the form (3.1), we get

$$C_0(t,0) = a_{i,0} + \left(\prod_{k=1}^{i} \delta_k \right) t, \tag{3.3}$$

$$C_0(\omega,s) = a_{r,j} + \left(\prod_{k=j+1}^{r} \delta_k \right) b - \left(\prod_{k=j+1}^{i} \gamma_k \right) s, \tag{3.4}$$

$$C_0(\omega,0) = a_{r,0} + \left(\prod_{k=1}^{r} \delta_k \right) b, \tag{3.5}$$

where $a_{i,j}$ is defined by (3.2). Now we can see that the assertion of our theorem is equivalent to the following inequality

$$C_0(t,s) C_0(\omega,0) < C_0(t,0) C_0(\omega,s),$$

which we have to prove.

From (3.1)–(3.5) above we get

$$C_0(t,s)C_0(\omega,0) = a_{i,j}a_{r,0} + a_{i,j}\Big(\prod_{k=1}^{r}\delta_k\Big)b + a_{r,0}\Big(\prod_{k=j+1}^{i}\delta_k\Big)t - a_{r,0}\Big(\prod_{k=j+1}^{i}\gamma_k\Big)s+ \\ +\Big(\prod_{k=1}^{r}\delta_k\Big)\Big(\prod_{k=j+1}^{i}\delta_k\Big)bt - \Big(\prod_{k=1}^{r}\delta_k\Big)\Big(\prod_{k=j+1}^{i}\gamma_k\Big)bs$$

and

$$C_0(t,0)C_0(\omega,s) = a_{i,0}a_{r,j} + a_{r,j}\Big(\prod_{k=1}^{i}\delta_k\Big)t + a_{i,0}\Big(\prod_{k=j+1}^{r}\delta_k\Big)b - a_{i,0}\Big(\prod_{k=j+1}^{i}\gamma_k\Big)s+ \\ +\Big(\prod_{k=j+1}^{r}\delta_k\Big)\Big(\prod_{k=1}^{i}\delta_k\Big)bt - \Big(\prod_{k=j+1}^{i}\gamma_k\Big)\Big(\prod_{k=1}^{i}\delta_k\Big)st.$$

Since

$$a_{i,j}a_{r,0} = \sum_{l_1=1}^{r}\sum_{l_2=j+1}^{i}\Big(\prod_{k=l_1+1}^{r}\gamma_k\Big)\Big(\prod_{k=1}^{l_1-1}\delta_k\Big)\Big(\gamma_{l_1}-\delta_{l_1}\Big)\times \\ \times\Big(\prod_{k=l_2+1}^{i}\gamma_k\Big)\Big(\prod_{k=j+1}^{l_2-1}\delta_k\Big)\Big(\gamma_{l_2}-\delta_{l_2}\Big)t_{l_1}t_{l_2}$$

and

$$a_{i,0}a_{r,j} = \sum_{l_1=1}^{i}\sum_{l_2=j+1}^{r}\Big(\prod_{k=l_1+1}^{i}\gamma_k\Big)\Big(\prod_{k=j+1}^{l_1-1}\delta_k\Big)\Big(\gamma_{l_1}-\delta_{l_1}\Big)\times \\ \times\Big(\prod_{k=l_2+1}^{r}\gamma_k\Big)\Big(\prod_{k=j+1}^{l_2-1}\delta_k\Big)\Big(\gamma_{l_2}-\delta_{l_2}\Big)t_{l_1}t_{l_2},$$

we can conclude that $C_0(t,s)C_0(\omega,0) < C_0(t,0)C_0(\omega,s)$. Hence, $G_0(t,s)$ is negative for every $(t,s) \in (0,\omega)\times(0,\omega)$. Theorem 3.4 has been proven. □

4. NEGATIVITY OF GREEN'S FUNCTION OF THE TWO-POINT IMPULSIVE PROBLEM

In this section we obtain necessary and sufficient conditions for negativity of Green's function to the two-point impulsive problem (1.1)–(1.3) with (1.6). By choosing the test function we will get a sufficient condition for its negativity. Define the following space of functions $x : [0, \omega] \to \mathbb{R}$:

$$D(t_1, t_2, \ldots, t_r) = \Big\{ x : x(t) = \prod_{i=1}^{j} \gamma_i \alpha_0 + C_0(t,0)\beta_0 + \int_0^t C_0(t,s) z(s) ds, t \in [t_j, t_{j+1}), \quad j = 0, 1, \ldots, r,\ t_0 = 0, \text{ for every } z \in L_{[0,\omega]}, \quad \gamma_i, \delta_i, \alpha_0, \beta_0 \in \mathbb{R},\ i = 1, \ldots, r \Big\}, \tag{4.1}$$

where $L_{[0,\omega]}$ is the space of summable functions.

It is clear that the functions from the space $D(t_1, t_2, \ldots, t_r)$ and their derivatives are absolutely continuous between the points t_i and t_{i+1}. We have the impulses defined by (1.2) at the points t_i and continuous from the right at the points t_i. Actually, the functions x with these properties define the space $D(t_1, t_2, \ldots, t_r)$.

We say that the function x is a solution of the impulsive equation (1.1), (1.2) if $x \in D(t_1, t_2, \ldots, t_r)$ and satisfies this equation.

Define the operator $K : D(t_1, t_2, \ldots, t_r) \to D(t_1, t_2, \ldots, t_r)$ by the equality

$$\begin{cases} (Kx)(t) = -\int_0^\omega G_0(t,s) \sum_{j=1}^{p} b_j(s) x(s - \theta_i(s)) ds, \\ x(\xi) = 0, \quad \xi < 0. \end{cases} \tag{4.2}$$

$G_0(t,s)$ is the Green's function of the problem (2.8) with (1.6). According to Theorem 3.4, we obtained $G_0(t,s) < 0$. Its spectral radius is denoted as $\rho(K)$.

Denote by $C(t,s)$ the Cauchy function of (1.1), (1.2) with (1.4), and $G(t,s)$ and $P(t,s)$ are the Green's functions of (1.1), (1.2) with (1.6) and (1.1), (1.2) with (1.5) respectively. Now we can formulate the following theorem.

Theorem 4.1. *Let $b_i \geq 0, \gamma_i > 0, \delta_i > 0$ and the function $h_i(t) = t - \theta_i(t)$ be such that $mes\{t : h_i(t) = const\} = 0$ for $i = 1, \ldots, r$. Then the following assertions* 1)–4) *are equivalent, each of which follows from* 5)*. If we assume that the function $v \in D(t_1, t_2, \ldots, t_r)$, satisfying assertion* 1)*, is such that*

$$v(\mu) - \prod_{i=1}^{r} \gamma_i v(\nu) \geq 0 \quad \textit{for all} \quad 0 \leq \nu < \mu \leq \omega, \tag{4.3}$$

then 5) *follows from* 1)*.*

1) *There exists a function $v \in D(t_1, t_2, \ldots, t_r)$ such that*

$$v(t) > 0, \quad \Psi(t) \equiv (Lv)(t) \leq 0,$$

$$v(\omega) - \int_0^\omega (Lv)(t) dt > 0, \quad v(\omega) - \prod_{i=1}^{r} \gamma_i v(0) \geq 0, \quad t \in (0, \omega).$$

2) *There exists a function* $u \in D(t_1, t_2, \ldots, t_r)$ *such that* $u(t) > 0, u(t) > (Ku)(t)$ *for* $t \in (0, \omega)$.
3) $\rho(K) < 1$.
4) *The boundary value problem* (1.1), (1.2) *with* (1.6) *is uniquely solvable and its Green's function* $G(t,s)$ *satisfies the inequality* $G(t,s) < 0, (t,s) \in (0,\omega) \times (0,\omega)$.
5) *The Cauchy function* $C(t,s)$ *of problem* (1.1), (1.2) *with* (1.4) *satisfies the inequality* $C(t,s) > 0, (t,s) \in (0,\omega) \times (0,\omega), t > s$.

Remark 4.2. It is clear that condition (4.3) is fulfilled if the function v is non-decreasing and $0 < \gamma_i \leq 1$.

Proof of Theorem 4.1. 1)⇒2) The function v satisfies the problem

$$(Lv)(t) = \Psi(t), \quad v(0) = \alpha_0, \quad v(\omega) = \beta_0, \tag{4.4}$$

where $\Psi(t) \leq 0$, $\alpha_0 \geq 0$, $\beta_0 \geq 0$. It is clear that

$$v''(t) + \sum_{j=1}^{p} b_j(t) v(t - \theta_j(t)) = \Psi(t). \tag{4.5}$$

Using the solution representation formula (2.22), we get

$$v(t) = -\int_0^\omega G_0(t,s) \sum_{j=1}^{p} b_j(s) v(s - \theta_j(s)) ds + \int_0^\omega G_0(t,s) \Psi(s) ds + V(t), \tag{4.6}$$

where $V(t)$ is defined by equality (2.26). According to Theorem 3.4, we have that $G_0(t,s) < 0$. This means that the operator K, defined by equality (4.2), is positive. Using this fact, from the condition $\Psi(t) \leq 0$, we get $\int_0^\omega G_0(t,s)\Psi(s)ds \geq 0$. The condition $v(\omega) - \prod_{i=1}^{r} \gamma_i v(0) \geq 0$ and positivity of the Cauchy function $C_0(t,s)$ established by Lemma 3.1 imply that $V(t) \geq 0$. It is clear now that we can set $u(t) \equiv v(t)$ in assertion 2.

The implication 1)⇒2) has been proven.

The proof of implication 2)⇒3) follows from Theorem 5.6 of the book [20, p.86] which can be formulated in the following convenient form for us:

Lemma 4.3. *If there exists a function* $u \in D(t_1, t_2, \ldots, t_m)$ *such that* $u(t) > 0$, $u(t) > (Ku)(t)$ *for* $t \in (0, \omega)$ *then* $\rho(K) < 1$.

3)⇒4) Let the right hand side $f(t)$ in (1.1) is nonpositive. We have to get that the solution $x(t)$ of (1.1), (1.2) with (1.6) is nonnegative. We assume that $x(0) = 0$, $x(\omega) = 0$, then $x(t)$ satisfies the integral equation

$$x(t) = (Kx)(t) + g(t), \tag{4.7}$$

where $g(t) = \int_0^\omega G_0(t,s) f(s) ds$. According to Theorem 3.4, we have $G_0(t,s) < 0$. Then the operator K is positive and $g(t) \geq 0$. The condition $\rho(K) < 1$ allows us to write

$$x(t) = (I - K)^{-1} g(t) = g(t) + Kg(t) + K^2 g(t) + \ldots \tag{4.8}$$

The positivity of the operator K implies $x(t) \geq 0$. Let us compare the solution of (1.1), (1.2), (1.6) with the function g which is the solution of impulsive problem (4.9), (1.2), where

$$\begin{cases} x''(t) = f(t), & t \in [0,\omega], \\ x(0) = 0, & x(\omega) = 0. \end{cases} \tag{4.9}$$

We get

$$x(t) - g(t) = Kg(t) + K^2 g(t) + \ldots \geq 0. \tag{4.10}$$

Theorem 3.4 claims that $G_0(t,s) < 0$ for $(t,s) \in (0,\omega) \times (0,\omega)$. Let us come back to (4.10)

$$0 \leq x(t) - g(t) = \int_0^\omega G(t,s)f(s)ds - \int_0^\omega G_0(t,s)f(s)ds = \int_0^\omega [G(t,s) - G_0(t,s)]f(s)ds. \tag{4.11}$$

It means that $G(t,s) \leq G_0(t,s) < 0$ for $(t,s) \in (0,\omega) \times (0,\omega)$.

4)⇒1) Let us set $f(t) \equiv -1, v(0) = 0, v(\omega) = 0$. We get the solution

$$v(t) = -\int_0^\omega G(t,s)ds, \tag{4.12}$$

which satisfies the conditions of assertion 1).

5)⇒1) Since $C(t,s) > 0$ for $(t,s) \in (0,\omega) \times (0,\omega), t > s$, we set

$$v(t) = C(t,0). \tag{4.13}$$

It is clear that v satisfies the conditions of assertion 1).

1)⇒5) Define an operator $K_{\nu\mu} : D\left(t_{k_\nu}, \ldots, t_{k_{\mu-1}}\right) \to D\left(t_{k_\nu}, \ldots, t_{k_{\mu-1}}\right)$, where $\nu < \mu \subseteq [0,\omega]$ and $\{t_{k_\nu}, \ldots, t_{k_{\mu-1}}\} = \{t_1, \ldots, t_r\} \cap [\nu,\mu]$, by the equality

$$(K_{\nu\mu}x)(t) = -\int_\nu^\mu G_0^{\nu\mu}(t,s) \sum_{j=1}^p b_j(s)x(s-\theta_i(s))\chi_{[\nu,\mu]}(s-\theta_i(s))ds, \tag{4.14}$$

where $\chi_{[\nu,\mu]}(t)$ is the characteristic function of the set $[\nu,\mu]$, i.e.

$$\chi_{[\nu,\mu]}(t) = \begin{cases} 1, & t \in [\nu,\mu], \\ 0, & t \notin [\nu,\mu]. \end{cases} \tag{4.15}$$

$G_0^{\nu\mu}(t,s)$ is the Green's function of the problem

$$\begin{cases} x''(t) = f(t), \quad t \in [\nu,\mu], \\ x(t_j) = \gamma_j x(t_j - 0), \quad x'(t_j) = \delta_j x'(t_j - 0), \quad j = i, i+1, \ldots, i+m, \quad m \geq 0, \\ x(\nu) = 0, \quad x(\mu) = 0, \end{cases} \tag{4.16}$$

where $t_i, \dots, t_{i+m}$ are the points of impulses inside the interval $[\nu, \mu]$. It is clear that there is only one point of impulse in the case $m = 0$.

The proof is based on the following assertion.

Lemma 4.4. *Let condition* (4.4) *be fulfilled. If* $\rho(K) < 1$, *then* $\rho(K_{\nu\mu}) < 1$ *for* $[\nu, \mu] \subseteq [0, \omega]$.

Proof. By implication 3)⇒4), problem (1.1), (1.2) with (1.6) is uniquely solvable and the Green's function $G(t, s)$ is negative for $(t, s) \in (0, \omega) \times (0, \omega)$.

The function

$$v(t) = -\int_0^\omega G(t, s) ds \tag{4.17}$$

is a positive solution of the boundary value problem

$$\begin{cases} (Lv)(t) = -1, \quad t \in [0, \omega], \\ v(t_j) = \gamma_j v(t_j - 0), \quad v'(t_j) = \delta_j v'(t_j - 0), \quad j = 1, 2, \dots, r, \\ v(\xi) = 0 \quad \text{for} \quad \xi < 0, \\ v(0) = 0, \quad v(\omega) = 0. \end{cases} \tag{4.18}$$

It is clear that

$$\begin{aligned} \Psi(t) \equiv v''(t) + \sum_{j=1}^{p} b_j(t) v(t - \theta_j(t)) \chi_{[\nu,\mu]}(t - \theta_j(t)) + \\ + \sum_{j=1}^{p} b_j(t) v(t - \theta_j(t))(1 - \chi_{[\nu,\mu]}(t - \theta_j(t))), \end{aligned} \tag{4.19}$$

and we get

$$\begin{aligned} v''(t) + \sum_{j=1}^{p} b_j(t) v(t - \theta_j(t)) \chi_{[\nu,\mu]}(t - \theta_j(t)) = \\ = \Psi(t) - \sum_{j=1}^{p} b_j(t) v(t - \theta_j(t))(1 - \chi_{[\nu,\mu]}(t - \theta_j(t))) \equiv \tilde{\Psi}(t). \end{aligned} \tag{4.20}$$

From here it is clear that $\tilde{\Psi}(t) \le 0, t \in [\nu, \mu]$. Let us use now implication 1)⇒3) on the interval $[\nu, \mu]$. We get $\rho(K_{\nu\mu}) < 1$. Lemma 4.4 has been proven. □

We continue the proof of implication 1)⇒5).

Let us assume the contrary. Then there exist $\nu < \mu$ such that $C(\mu, \nu) = 0$. In this case $u(t) = C(t, \nu)$ is a characteristic function of the operator $K_{\nu\mu}$, i.e. $\rho(K_{\nu\mu}) = 1$. But we get a contradiction with Lemma 4.3 which implies that $\rho(K_{\nu\mu}) < 1$.

Theorem 4.1 has been proven. □

Example 4.5. Let us now find an example of a function v satisfying condition 1) of Theorem 4.1. To this end, let us start with $v(t) = t(\omega - t)$ in the interval $t \in [0, t_1)$. The function v in the rest of the interval will be of the form

$$v(t) = v(t_i) + v'(t_i)(t - t_i) - (t - t_i)^2 \quad t \in [t_i, t_{i+1}), \quad i = 1, \dots, r, t_{r+1} = \omega, \tag{4.21}$$

where

$$\begin{cases} v(t_i) = \gamma_i v(t_i - 0), \\ v'(t_i) = \delta_i v'(t_i - 0). \end{cases} \tag{4.22}$$

Thus

$$\begin{cases} v(t) = t(\omega - t), & t \in [0, t_1), \\ v(t) = v(t_i) + v'(t_i)(t - t_i) - (t - t_i)^2, & t \in [t_i, t_{i+1}), \end{cases} \tag{4.23}$$

where $v(t_i)$ and $v'(t_i)$ can be presented in the form

$$\begin{cases} v(t_i) = t_1(\omega - t_1) \prod_{j=1}^{i} \gamma_j + \\ \qquad + \sum_{k=2}^{i} v'(t_k)(t_k - t_{k-1}) \prod_{j=k}^{i} \gamma_j - \sum_{k=2}^{i} (t_k - t_{k-1})^2 \prod_{j=k}^{i} \gamma_j, \\ v'(t_i) = (\omega - 2t_1) \prod_{j=1}^{i} \delta_j - 2 \sum_{k=2}^{i} (t_k - t_{k-1}) \prod_{j=k}^{i} \delta_j. \end{cases} \tag{4.24}$$

Let us assume that $v(t) > 0$ and substitute this $v(t)$ into assertion 1) of Theorem 4.1

$$-2 + \sum_{i=1}^{p} b_i(t) \max \left\{ \max_{i=1,2,\dots,r} v\left(\frac{v'(t_i)}{2} + t_i\right), \max_{i=0,1,\dots,r+1} v(t_i) \right\} \leq 0, \tag{4.25}$$

and we get the condition

$$\Omega_1 \sum_{i=1}^{p} b_i(t) \leq 2, \tag{4.26}$$

where

$$\Omega_1 = \max \left\{ \max_{i=1,2,\dots,r} v\left(\frac{v'(t_i)}{2} + t_i\right), \max_{i=0,1,\dots,r+1} v(t_i) \right\}. \tag{4.27}$$

In the case of the non-impulsive equation (1.1), the following classical sufficient condition for negativity of Green's function

$$\sum_{i=1}^{p} b_i(t) \leq \frac{8}{\omega^2}, \tag{4.28}$$

is known. In the case $t - \theta_i(t) \equiv \frac{\omega}{2}$ for $i = 1, \dots, p$, this condition cannot be improved for the nonimpulsive equation (1.1). If we take a sequence of the impulse points $t_i^k < t_{i+1}^k, i = 1, \dots, r, t_1^k \to \omega$ when $k \to \infty$, it is clear that these impulses could not influence essentially on condition (4.26) and consequently on (4.28), where the inequality is strong implies the positivity of Green's function. Therefore, we obtain the following corollary.

Corollary 4.6. *Assume that* $0 < \gamma_i \leq 1$, $0 < \delta_i \leq 1$ *for* $i = 1, \ldots, r$ *and*

$$\sum_{i=1}^{p} b_i \left(\frac{1}{4} + r\right) < \frac{2}{\omega^2}. \tag{4.29}$$

Then Green's function $G(t,s)$ *is nonegative in* $(0,w) \times (0,w)$.

From (4.26) for the nonimpulsive equation ($r = 0$), we get inequality (4.28).

5. POSITIVITY OF GREEN'S FUNCTION OF THE ONE-POINT IMPULSIVE PROBLEM

In this section we obtain necessary and sufficient conditions for positivity of Green's function to the one-point impulsive problem (1.1)–(1.3) with (1.5).

Define the operator $\tilde{K} : D(t_1, t_2, \ldots, t_r) \to D(t_1, t_2, \ldots, t_r)$ by the equality

$$\begin{cases} (\tilde{K}x)(t) = -\int_0^\omega P_0(t,s) \sum_{j=1}^{p} b_j(s) x(s - \theta_i(s)) ds, \\ x(\xi) = 0, \quad \xi < 0. \end{cases} \tag{5.1}$$

$P_0(t,s)$ is the Green's function of the problem (2.8) with (1.5). According to Lemma 3.2, $P_0(t,s) \geq 0$ and the operator $\tilde{K}$ is positive. Its spectral radius is denoted as $\rho(\tilde{K})$.

Theorem 5.1. *Let* $b_i \leq 0$, $\gamma_i > 0$, $\delta_i > 0$, $t_i = \omega - t_{r-i}$, $i = 1, \ldots, r$. *Then the following assertions are equivalent:*

1) *There exists a function* $v \in D(t_1, t_2, \ldots, t_r)$ *such that*

$$v(t) > 0, \quad \Psi(t) = (Lv)(t) \geq 0, \quad v'(w) \leq 0 \quad t \in [0, \omega].$$

2) *There exists a function* $u \in D(t_1, t_2, \ldots, t_r)$ *such that* $u(t) > 0, u(t) > (\tilde{K}u)(t)$ *for* $t \in [0, \omega]$.
3) $\rho(\tilde{K}) < 1$.
4) *The boundary value problem* (1.1), (1.2) *with* (1.5) *is uniquely solvable and its Green's function* $P(t,s)$ *is nonnegative for* $(t,s) \in [0,\omega) \times [0,\omega]$ *and satisfies the inequalities* $P(t,s) > 0$ *for* $0 < t < s < w$.

Proof. 1)⇒2) The function v satisfies the problem

$$(Lv)(t) = \Psi(t), \quad v(\omega) = \alpha_0, \quad v'(\omega) = \beta_0, \tag{5.2}$$

where $\Psi(t) \geq 0$, $\alpha_0 > 0, \beta_0 \leq 0$. It is clear that

$$v''(t) + \sum_{j=1}^{p} b_j(t) v(t - \theta_j(t)) = \Psi(t). \tag{5.3}$$

Using the solution representation formula (2.19), we get

$$v(t) = -\int_0^{\omega} P_0(t,s)\sum_{j=1}^{p} b_j(s)v(s-\theta_j(s))ds + \int_0^{\omega} P_0(t,s)\Psi(s)ds + W(t), \tag{5.4}$$

where $W(t)$ is defined by equality (2.21). According to Lemma 3.2, we have that $P_0(t,s) > 0$. This means that the operator $\tilde{K}$, defined by equality (5.1), is positive. Using this fact, from condition $\Psi(t) \geq 0$, we get $\int_0^{\omega} P_0(t,s)\Psi(s)ds \geq 0$. The positivity of the Cauchy function $C_0(t,s)$ established by Lemma 3.1 implies that $W(t) > 0$. It is clear now that we can set $u(t) \equiv v(t)$ in assertion 2. The implication 1)⇒2) has been proven.

The proof of implication 2)⇒3) follows from Theorem 5.6 of the book [20, p.86] which we formulated in the form of Lemma 4.3.

3)⇒4) Let the right hand side $f(t)$ in (1.1) be nonnegative. We have to get that the solution $x(t)$ of (1.1), (1.2) with (1.5) is nonnegative. We assume that $x(\omega) = 0$, $x'(\omega) = 0$. Then $x(t)$ satisfies the integral equation

$$x(t) = (\tilde{K}x)(t) + p(t), \tag{5.5}$$

where $p(t) = \int_0^{\omega} P_0(t,s)f(s)ds$. According to Lemma 3.2, we have $P_0(t,s) > 0$. Then the operator $\tilde{K}$ is positive and $p(t) \geq 0$. The condition $\rho(\tilde{K}) < 1$ allows us to write

$$x(t) = (I - \tilde{K})^{-1}p(t) = p(t) + \tilde{K}p(t) + \tilde{K}^2 p(t) + \dots \tag{5.6}$$

The positivity of $\tilde{K}$ implies $x(t) \geq 0$. Let us compare the solution of (1.1), (1.2) with (1.5) with the function p which is the solution of (5.7), (1.2), where

$$\begin{cases} x''(t) = f(t), \quad t \in [0,\omega], \\ x(\omega) = 0, \quad x'(\omega) = 0. \end{cases} \tag{5.7}$$

We get

$$x(t) - p(t) = \tilde{K}p(t) + \tilde{K}^2 p(t) + \dots \geq 0. \tag{5.8}$$

Lemma 3.2 claims that $P_0(t,s) > 0$ for $(t,s) \in (0,\omega) \times (0,\omega)$. Let us come back to (5.8):

$$\begin{aligned} 0 \leq x(t) - p(t) &= \int_0^{\omega} P(t,s)f(s)ds - \int_0^{\omega} P_0(t,s)f(s)ds = \\ &= \int_0^{\omega} [P(t,s) - P_0(t,s)]f(s)ds. \end{aligned} \tag{5.9}$$

This means that $P(t,s) \geq P_0(t,s) \geq 0$ for $(t,s) \in (0,\omega) \times (0,\omega)$.

4)⇒1) Let us set $f(t) \equiv 1, v(\omega) = 1, v'(\omega) = 0$. We get the solution

$$v(t) = \int_0^{\omega} P(t,s)ds + W(t), \tag{5.10}$$

where $W(t) > 0$, $W'(w) = 0$, which satisfies the conditions of assertion 1).

Theorem 5.1 has been proven. □

Example 5.2. Let us now find an example of a function v satisfying condition 1) of Theorem 5.1. To this end, let us start with $v(t) = e^{-\alpha t}$ in the interval $t \in [0, t_1)$. The function v in the rest of the intervals will be of the form

$$v(t) = c_i e^{-\alpha a_i t}, \quad t \in [t_i, t_{i+1}), \tag{5.11}$$

where

$$\begin{cases} v(t_i) = \gamma_i v(t_i - 0), \\ v'(t_i) = \delta_i v'(t_i - 0). \end{cases} \tag{5.12}$$

After some calculations we get that v is of the form

$$\begin{cases} v(t) = e^{-\alpha t}, & t \in [0, t_1), \\ v(t) = \prod_{j=1}^{i} \gamma_j e^{-\alpha \frac{\prod_{j=1}^{i} \delta_j}{\prod_{j=1}^{i} \gamma_j} t}, & t \in [t_i, t_{i+1}). \end{cases} \tag{5.13}$$

Let us substitute this $v(t)$ into assertion 1) of Theorem 5.1 and assume that $\delta_j > \gamma_j$, $j = 1, \ldots, r$,

$$\alpha^2 \frac{\left(\prod_{j=1}^{i} \delta_j\right)^2}{\left(\prod_{j=1}^{i} \gamma_j\right)^2} + \sum_{j=1}^{p} b_j(t) e^{\alpha \frac{\prod_{j=1}^{i} \delta_j}{\prod_{j=1}^{i} \gamma_j} \theta_j(t)} \geq 0. \tag{5.14}$$

Thus

$$\alpha^2 C^2 e^{-\alpha C \Theta} \geq \sum_{j=1}^{p} |b_j(t)|, \tag{5.15}$$

where

$$C = \min_{i=1,2,\ldots,r} \frac{\prod_{j=1}^{i} \delta_j}{\prod_{j=1}^{i} \gamma_j} \tag{5.16}$$

and

$$\Theta = \max_{t \in [0,\omega]} \max_{i=1,2,\ldots,r} \theta_j(t). \tag{5.17}$$

Denoting $F(\alpha) = \alpha^2 C^2 e^{-\alpha C \Theta}$, we can find its maximum:

$$F'(\alpha) = \left(2\alpha e^{-\alpha C \Theta} - \alpha^2 C \Theta e^{-\alpha C \Theta}\right) C^2 = \alpha \left(2 - C \Theta \alpha\right) e^{-\alpha C \Theta} C^2 \tag{5.18}$$

and $\alpha = \frac{2}{C\Theta}$ is a point of maximum. Thus

$$\frac{4}{\Theta^2} e^{-2} \geq \sum_{j=1}^{p} |b_j(t)| \tag{5.19}$$

is a sufficient condition for the positivity of Green's function $P(t, s)$ for the case $\delta_j > \gamma_j$, $j = 1, \ldots, r$.

Let us now assume the opposite, i.e. $\delta_j \le \gamma_j, j = 1, \dots, r$, and substitute this $v(t)$ into assertion 1) of Theorem 4.4

$$\alpha^2 \frac{\left(\prod_{j=1}^{i} \delta_j\right)^2}{\left(\prod_{j=1}^{i} \gamma_j\right)^2} + \sum_{j=1}^{p} b_j(t) e^{\alpha\theta_j(t)} \ge 0. \tag{5.20}$$

Thus

$$\alpha^2 C^2 e^{-\alpha\Theta} \ge \sum_{j=1}^{p} |b_j(t)|. \tag{5.21}$$

Denoting $F(\alpha) = \alpha^2 C^2 e^{-\alpha\Theta}$, we can find its maximum:

$$F'(\alpha) = \left(2\alpha e^{-\alpha\Theta} - \alpha^2 \Theta e^{-\alpha\Theta}\right) C^2 = \alpha\left(2 - \Theta\alpha\right) e^{-\alpha\Theta} C^2 \tag{5.22}$$

and $\alpha = \frac{2}{\Theta}$ is a point of maximum. Thus

$$\frac{4C^2}{\Theta^2} e^{-2} \ge \sum_{j=1}^{p} |b_j(t)| \tag{5.23}$$

is a sufficient condition for the positivity of Green's function $P(t, s)$ for the case $\delta_j \le \gamma_j, j = 1, \dots, r$.

For the non-impulsive equation (1.1) we have the inequality

$$\alpha^2 - \sum_{j=1}^{p} |b_j(t)| \, e^{\alpha\theta_j(t)} \ge 0$$

and for $p = 1$ and constant coefficients and $b_1(t) = b_1$, $\theta_1(t) = \theta$, we have $\alpha^2 e^{-\alpha\theta} \ge |b_1|$. Denoting $F(\alpha) = \alpha^2 e^{-\alpha\theta}$, we can find its maximum:

$$F'(\alpha) = (2\alpha - \alpha^2\theta) e^{-\alpha\theta} = \alpha(2 - \alpha\theta) e^{-\alpha\theta}$$

and $\alpha = \frac{2}{\theta}$ is a point of maximum $\theta\sqrt{b_1} \le \frac{2}{e}$. It is known that this inequality cannot be improved since the opposite inequality $\theta\sqrt{b_1} > \frac{2}{e}$ implies nonexistence of positive solution on the semiaxis [19]. It is clear that in the case of vanishing impulses, i.e. if we choose a sequence of impulses $\beta_i^k \longrightarrow 1$ when $k \to \infty$, inequality (5.15) cannot be improved.

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