ARCHIVES OF MECHANICS
Arch. Mech. **74** (2-3), 127–142, 2022, DOI: 10.24423/aom.4003

Non-simple elastic materials with double porosity structure

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This paper is concerned with a strain gradient theory of elastic materials that have a double porosity structure. Firstly, we present the basic equations and the boundary conditions of the nonlinear theory. Then, we derive the equations of the linear theory and present the constitutive equations for chiral materials. The theory is applied to study the deformation of a chiral cylinder. The materials with a double porosity are of interest in geophysics and in mechanics of bone.

Key words: strain-gradient theory, porous solids, chiral materials, torsion of a circular cylinder.

1. Introduction

The basic equations and the boundary conditions of the strain gradient theory of elastic solids were first established by Toupin [1, 2] and Mindlin [3]. This theory has been studied and extended in various works. The linear theory has been developed by Mindlin and Eshel [4]. The interest in the gradient theory of elasticity is stimulated by the fact that this theory is adequate to study problems related to size effects [5]. In this paper we present a strain gradient theory of elastic materials which have a double porosity structure: a macroporosity connected to pores in the body, and a microporosity related to fissures in the porous skeleton. The intended applications of the theories which describe the behavior of materials with double porosity are to geological materials and in mechanics of bone [6–11]. Various porous composites are chiral materials. It is known that the classical theory of porous elastic solids cannot be used to describe the behavior of chiral materials [12, 13]. The strain gradient theory of elasticity is an adequate tool to invest igate these materials [14–16]. The linear theory for porous elastic solids having a single porosity can be obtained as a special case of the theory established in [17] for the deformation of non-simple microstretch elastic solids. In the present paper we establish the nonlinear theory of non-simple elastic solids that have a double porosity structure. In the first part of the paper we use the results of [2, 3, 18, 19] to derive a nonlinear strain gradient theory of porous elastic solids. Then, we establish the basic equations of the linear theory and the constitutive equations for chiral materials. Finally, we apply the results to investigate the deformation of a chiral cylinder. We study the equilibrium of a beam subjected to extension and torsion. It is shown that the torsion of the cylinder produces a variation of porosities and an extension.

2. Nonlinear theory

We consider a body that at time t_0 occupies the bounded region B with the Lipschitz boundary ∂B. The boundary of the body consists of the union of a finite number of smooth surfaces, smooth curves (edges) and points (corners). We denote by C the union of the edges. The deformation of the body is referred to a rectangular Cartesian coordinate system. We assume that Latin subscripts (unless otherwise specified) are understood to range over the integers $(1, 2, 3)$, summation over repeated subscripts is implied and a subscript preceeded by a comma denotes partial differentiation with respect to the corresponding coordinate. A superposed dot denotes the material derivative with respect to the time t. We denote by X_K the coordinates of a typical particle in B, and suppose that the coordinates of this particle at time t are x_i. Thus,

$$x_i = x_i(X_K, t), \quad (X_K) \in B,\, t \in I, \tag{2.1}$$

where $I = [t_0, t_1)$ is a given interval of time. In what follows we assume the continuous differentiability of x_i with respect to the variables X_K and t as many times as required, and $\det(x_{i,K}) > 0$ on $B \times I$. We study the deformation of nonsimple elastic solids with a double porosity structure. We denote by ν_1 the volume fraction field corresponding to pores and by ν_2 the volume fraction field corresponding to fissures. We assume that ν_1 and ν_2 are sufficiently smooth for the ensuing analysis to be valid. Let us consider an arbitrary region ω of the body, bounded by a surface $\partial\omega$ at time t, and assume that Ω is the corresponding region at time t_0, with the surface $\partial\Omega$. Following [2, 17] we postulate an energy balance in the form

$$\begin{aligned}\int\limits_{\Omega} \rho(v_i\dot{v}_i + \kappa_1\dot{\nu}_1\ddot{\nu}_1 + \kappa_2\dot{\nu}_2\ddot{\nu}_2)\, dv + \int\limits_{\Omega} \rho\dot{e}\, dv \\ = \int\limits_{\Omega} \rho(f_i v_i + g\dot{\nu}_1 + l\dot{\nu}_2) + \int\limits_{\partial\Omega} (T_i v_i + M_{ji} v_{i,j} + \sigma\dot{\nu}_1 + \tau\dot{\nu}_2)\, da,\end{aligned} \tag{2.2}$$

for every part Ω of B and every time. In this relation we have used the following notation: ρ is the mass density at time t_0, e is the internal energy per unit mass, κ_1 and κ_2 are coefficients of inertia, f_i is the body force per unit mass, g is the extrinsic equilibrated body force per unit mass associated to macro pores, l is the extrinsic body forces per unit mass associated to fissures, T_i is the stress vector associated with the surface $\partial\omega$, but measured per unit area of the surface $\partial\Omega$, M_{ij} is the dipolar surface force associated with the surface $\partial\omega$ and measured per unit area of $\partial\Omega$, σ is the equilibrated stress corresponding to ν_1, associated with the surface $\partial\omega$ but measured per unit area of $\partial\Omega$, τ is the equilibrated stress corresponding to ν_2, associated with $\partial\omega$ and measured per unit area of the surface $\partial\Omega$, dv and da are the elements of volume and area in the reference configuration B, and $v_i = \dot{x}_i$. We use the method of GREEN and RIVLIN [20] to derive the equations of motion from the balance of energy and the invariance requirements under superposed rigid motion. We consider a motion of the body which differs from the given motion by a constant superposed rigid body translational velocity and suppose that the functions ρ, κ_1, κ_2, e, f_i, $\ddot{x}_i$, g, l, T_i, M_{ij}, σ and τ are not affected by such motion. From (2.2) we obtain the balance of linear momentum,

$$\int_{\Omega} \rho \dot{v}_i \, dv = \int_{\Omega} \rho f_i \, dv + \int_{\partial\Omega} T_i \, da. \tag{2.3}$$

This relation implies that

$$T_i = T_{Ki} n_K, \tag{2.4}$$

where T_{Ki} is the first Piola-Kirchhoff stress tensor, and n_K is the outward unit normal to $\partial\Omega$. From (2.3) and (2.4) we obtain the well-known equations

$$T_{Ki,K} + \rho f_i = \rho \ddot{x}_i. \tag{2.5}$$

With the help of (2.3) and (2.4), the energy balance becomes

$$\begin{aligned} \int_{\Omega} \rho(\dot{e} + \kappa_1 \dot{\nu}_1 \ddot{\nu}_1 + \kappa_2 \dot{\nu}_2 \ddot{\nu}_2) \, dv = & \int_{\Omega} [T_{Ki} v_{i,K} + \rho(g \dot{\nu}_1 + l \dot{\nu}_2)] \, dv \\ & + \int_{\partial\Omega} (M_{ji} v_{i,j} + \sigma \dot{\nu}_1 + \tau \dot{\nu}_2) \, da. \end{aligned} \tag{2.6}$$

If we apply this relation to a region which in the reference configuration was a tetrahedron bounded by coordinate planes through the point (X_K) and by a plane whose unit normal is n_K, then we obtain

$$(M_{ji} - P_{Kji} n_K) v_{i,j} + (\sigma - \sigma_K n_K) \dot{\nu}_1 + (\tau - \tau_K n_K) \dot{\nu}_2 = 0, \tag{2.7}$$

where P_{Kij} is the hyperstress tensor, and σ_K and τ_K are equilibrated stress vectors, associated with surfaces which were originally coordinate planes perpendicular to the X_K-axes through the point (X_L), measured per unit undeformed area. Then, using (2.7) in (2.6) and applying the resulting equation to an arbitrary region, we get

$$\rho\dot{e} = (T_{Ki} + P_{Lji,L}X_{K,j})v_{i,K} + P_{Kji}X_{L,j}v_{i,LK} + \sigma_K\dot{\nu}_{1,K} + \tau_K\dot{\nu}_{2,K} - \xi\dot{\nu}_1 - \zeta\dot{\nu}_2, \tag{2.8}$$

when the functions ξ and ζ satisfy the equations

$$\sigma_{K,K} + \xi + \rho g = \rho\kappa_1\ddot{\nu}_1, \qquad \tau_{K,K} + \zeta + \rho l = \rho\kappa_2\ddot{\nu}_2. \tag{2.9}$$

The functions ξ and ζ are characterized by constitutive equations. If we introduce the notation

$$M_{KLi} = P_{Kji}X_{L,j}, \quad S_{Ki} = T_{Ki} + M_{LKi,L}, \tag{2.10}$$

then Eq. (2.8) can be expressed as

$$\rho\dot{e} = S_{Ki}v_{i,K} + M_{KL,i}v_{i,KL} + \sigma_K\dot{\nu}_{1,K} + \tau_K\dot{\nu}_{2,K} - \xi\dot{\nu}_1 - \zeta\dot{\nu}_2. \tag{2.11}$$

The invariance requirements under superposed uniform rigid body angular velocity implied the following relations

$$S_{Ki}x_{j,K} + M_{KLi}x_{j,KL} = S_{Kj}x_{i,K} + M_{KLj}x_{i,KL}. \tag{2.12}$$

It is known [2] that $M_{KLi} = M_{LKi}$. The skew symmetric part of M_{KLi} makes no contribution to the work over any closed surface in $\overline{B}$.

We require constitutive equations for e, T_{Ki}, M_{KLi}, σ_K, τ_K, ξ and ζ and assume that these are functions of the variables $x_{i,K}$, $x_{i,KL}$, ν_1, ν_2, $\nu_{1,K}$, $\nu_{2,K}$ and X_L. We suppose that there is no internal constraint and that the constitutive functions are sufficiently smooth. From (2.11) we get

$$\begin{gathered} S_{Ki} = \frac{\partial W}{\partial x_{i,K}}, \quad M_{KLi} = \frac{\partial W}{\partial x_{i,KL}}, \quad \sigma_K = \frac{\partial W}{\partial \nu_{1,K}}, \\ \tau_K = \frac{\partial W}{\partial \nu_{2,K}}, \quad \xi = -\frac{\partial W}{\partial \nu_1}, \quad \zeta = -\frac{\partial W}{\partial \nu_2}, \end{gathered} \tag{2.13}$$

where $W = \rho_0 e$. Since W must be invariant under the superposed rigid-body motion it can be expressed as [2]

$$W = \widehat{W}(E_{KL}, G_{KLM}, \nu_1, \nu_2, \nu_{1,K}, \nu_{2,K}, X_N), \tag{2.14}$$

where the strain tensors E_{KL} and G_{KLM} are given by

$$2E_{KL} = x_{i,K}x_{i,L} - \delta_{KL}, \quad G_{KLM} = x_{i,K}x_{i,LM}, \tag{2.15}$$

and δ_{KL} is the Kronecker delta. In view of (2.10), (2.13)–(2.15), we obtain

$$
\begin{aligned}
S_{Ki} &= \frac{\partial\widehat{W}}{\partial E_{KL}}x_{i,L} + \frac{\partial\widehat{W}}{\partial G_{KLM}}x_{i,LM},\\
M_{KLi} &= \frac{\partial\widehat{W}}{\partial G_{MKL}}x_{i,M}, \qquad \sigma_K = \frac{\partial\widehat{W}}{\partial \nu_{1,K}},\\
\tau_K &= \frac{\partial\widehat{W}}{\partial \nu_{2,K}}, \qquad \xi = -\frac{\partial\widehat{W}}{\partial \nu_1}, \qquad \zeta = -\frac{\partial\widehat{W}}{\partial \nu_2},\\
T_{Ki} &= \frac{\partial\widehat{W}}{\partial E_{KL}}x_{i,L} + \frac{\partial\widehat{W}}{\partial G_{KLM}}x_{i,LM} - \left(\frac{\partial\widehat{W}}{\partial G_{MLK}}x_{i,M}\right)_{,L}.
\end{aligned}
\tag{2.16}
$$

We note that if S_{Ki} and M_{KLi} have the form (2.16), then the relations (2.12) are identically satisfied.

The basic equations consist of the equations of motion (2.5), the equations of equilibrated forces (2.10), the constitutive equations (2.16), and the geometrical equations (2.15).

For a given deformation $v_{i,j}, \dot{\nu}_1$ and $\dot{\nu}_2$ in (2.7) may be chosen arbitrarily and in view of constitutive equations we obtain

$$
M_{ji} = P_{Kji}n_K, \qquad \sigma = \sigma_K n_K, \qquad \tau = \tau_K n_K. \tag{2.17}
$$

Following [2,3] we have

$$
\int\limits_{\partial B}(T_i v_i + M_{ji}v_{i,j})\,da = \int\limits_{\partial B}(P_i v_i + R_i D v_i)\,da + \int\limits_{C}\Pi_i v_i\,ds, \tag{2.18}
$$

where

$$
\begin{aligned}
P_i &= T_{Ki}n_K - D_L(M_{KLi}n_K) + (D_P n_P)M_{KLi}n_K n_L,\\
R_i &= M_{KLi}n_K n_L, \qquad \Pi_i = \langle M_{KLi}n_K\gamma_L\rangle, \qquad \gamma_L = \varepsilon_{LKA}s_K n_A.
\end{aligned}
\tag{2.19}
$$

Here we have used the following notation: D_K is the surface gradient, $Df = f_{,K}n_K$, $\langle f\rangle$ denotes the difference in values of f as a given point on the edge is approached from either side, s_K are the components of the unit vector tangent to C, and ε_{KLM} is the alternating symbol. Let S_m $(m = 1, 2, \ldots, 8)$ be subsets of ∂B such that $\overline{S}_1 \cup S_2 = \overline{S}_3 \cup S_4 = \overline{S}_5 \cup S_6 = \overline{S}_7 \cup S_8 = \partial B$, $S_1 \cap S_2 = S_3 \cap S_4 = S_5 \cap S_6 = S_7 \cap S_8 = \emptyset$. We assume that the boundary conditions are

$$
\begin{aligned}
& x_i = \widetilde{x}_i \text{ on } \overline{S}_1 \times I, && Dx_i = \widetilde{d}_i \text{ on } \overline{S}_3 \times I,\\
& \nu_1 = \widetilde{\nu}_1 \text{ on } \overline{S}_5 \times I, && \nu_2 = \widetilde{\nu}_2 \text{ on } \overline{S}_7 \times I,\\
& P_i = \widetilde{P}_i \text{ on } S_2 \times I, && R_i = \widetilde{R}_i \text{ on } S_4 \times I,\\
& \sigma = \widetilde{\sigma} \text{ on } S_6 \times I, && \tau = \widetilde{\tau} \text{ on } S_8 \times I,\\
& \Pi_i = \widetilde{\Pi}_i \text{ on } C \times I,
\end{aligned}
\tag{2.20}
$$

where $\widetilde{x}_i, \widetilde{d}_i, \widetilde{\nu}_1, \widetilde{\nu}_2, \widetilde{P}_i, \widetilde{R}_i, \widetilde{\sigma}, \widetilde{\tau}$ and $\widetilde{\Pi}_i$ are prescribed functions.

The initial conditions are

$$
\begin{aligned}
&x_i(X_K,0) = x_i^0(X_K), \quad \dot{x}_i(X_K,0) = v_i^0(X_K),\\
&\nu_\alpha(X_K,0) = \nu_\alpha^0(X_K), \quad \dot{\nu}_\alpha(X_K,0) = \zeta_\alpha^0, \qquad (\alpha = 1,2),\ (X_K) \in B,
\end{aligned} \tag{2.21}
$$

where the functions $x_i^0, v_i^0, \nu_\alpha^0$ and ζ_α^0 are given. The problem consists in finding the functions x_i, ν_1 and ν_2 that satisfy Eqs. (2.15), (2.16), (2.5) and (2.10) on $B \times I$, the boundary conditions (2.20) and the initial conditions (2.21).

3. Linear theory

In this section we denote the material Cartesian coordinates by X_j and define the functions

$$
u_j = x_j - X_j, \qquad \varphi = \nu_1 - \nu_1^*, \qquad \psi = \nu_2 - \nu_2^*, \tag{3.1}
$$

where ν_1^* and ν_2^* are the volume fraction fields in the reference configuration. We suppose that ν_1^* and ν_2^* are prescribed constants. In the linear theory we assume that the functions u_i, φ and ψ have the form $u_j = \varepsilon u_j^*$, $\varphi = \varepsilon\varphi^*$, $\psi = \varepsilon\psi^*$, where ε is a constant small enough for squares and higher powers to be neglected, and u_i^*, φ^* and ψ^* are independent of ε. Now, the strain tensors (2.15) become

$$
e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \qquad \kappa_{ijk} = u_{k,ij}. \tag{3.2}
$$

The independent constitutive variables are e_{ij}, κ_{ijk}, φ, ψ, $\varphi_{,j}$ and $\psi_{,j}$. In the linear theory we assume that W is a quadratic form in these variables,

$$
\begin{aligned}
2W = {}& A_{jirs}e_{ij}e_{rs} + 2B_{ijpqr}e_{ij}\kappa_{pqr} + C_{ijkpqr}\kappa_{ijk}\kappa_{pqr} + 2D_{ijk}\kappa_{ijk}\varphi\\
&+ 2E_{ijk}\kappa_{ijk}\psi + 2F_{ijkm}\kappa_{ijk}\varphi_{,m} + 2G_{ijkm}\kappa_{ijk}\psi_{,m} + 2d_{ij}e_{ij}\varphi\\
&+ 2f_{ij}e_{ij}\psi + 2g_{ijk}e_{ij}\varphi_{,k} + 2h_{ijk}e_{ij}\psi_{,k} + a_{ij}\varphi_{,i}\varphi_{,j} + 2b_{ij}\varphi_{,i}\psi_{,j}\\
&+ c_{ij}\psi_{,i}\psi_{,j} + a_1\varphi^2 + a_2\psi^2 + 2a_3\varphi\psi + 2d_i\varphi_{,i}\varphi + 2e_i\psi_{,i}\psi\\
&+ 2p_i\varphi\psi_{,i} + 2q_i\psi\varphi_{,i},
\end{aligned} \tag{3.3}
$$

where the constitutive coefficients have the following symmetries

$$
\begin{aligned}
&A_{ijrs} = A_{jirs} = A_{rsij}, \qquad B_{ijpqr} = B_{jipqr} = B_{ijqpr},\\
&C_{ijkpqr} = C_{jikpqr} = C_{pqrijk}, \qquad D_{ijk} = D_{jik},\\
&E_{ijk} = E_{jik}, \qquad F_{ijkm} = F_{jikm}, \qquad G_{ijkm} = G_{jikm},\\
&d_{ij} = d_{ji}, \qquad c_{ij} = c_{ji}.
\end{aligned} \tag{3.4}
$$

We introduce the notation

$$
\begin{aligned}
&\tau_{ij} = \delta_{iK} S_{Kj}, \qquad \mu_{rsi} = \delta_{rK}\delta_{sL} M_{KLi},\\
&\sigma_i = \delta_{iK}\sigma_K, \qquad \tau_i = \delta_{iK}\tau_K, t_{ij} = \delta_{iK} T_{Kj},
\end{aligned}
\tag{3.5}
$$

where δ_{iK} is the Kronecker delta. In view of (2.16) and (3.5) we get

$$
\begin{aligned}
&\tau_{ij} = \frac{\partial W}{\partial e_{ij}}, \qquad \mu_{ijk} = \frac{\partial W}{\partial \kappa_{ijk}}, \qquad \sigma_i = \frac{\partial W}{\partial \varphi_{,i}}, \qquad \tau_i = \frac{\partial W}{\partial \psi_{,i}},\\
&\xi = -\frac{\partial W}{\partial \varphi}, \qquad \zeta = -\frac{\partial W}{\partial \psi}, \qquad t_{ij} = \tau_{ij} - \mu_{kij,k}.
\end{aligned}
\tag{3.6}
$$

From (2.5), (2.9), (2.16) and (3.6) we obtain the equations

$$
\begin{aligned}
&\tau_{ji,j} - \mu_{kij,kj} + \rho f_i = \rho \ddot{u}_i,\\
&\sigma_{i,i} + \xi + \rho g = \rho\kappa_1 \ddot{\varphi}, \tau_{i,i} + \zeta + \rho l = \rho \kappa_2 \ddot{\psi}.
\end{aligned}
\tag{3.7}
$$

We note that in the context of linear theory, the relation (2.12) becomes $\tau_{ij} = \tau_{ji}$.

By (3.3) and (3.6) we obtain the following constitutive equations

$$
\begin{aligned}
&\tau_{ij} = A_{ijrs} e_{rs} + B_{ijpqr}\kappa_{pqr} + d_{ij}\varphi + f_{ij}\psi + g_{ijk}\varphi_{,k} + h_{ijk}\psi_{,k},\\
&\mu_{ijk} = B_{rsijk} e_{rs} + C_{ijkpqr}\kappa_{pqr} + D_{ijk}\varphi + E_{ijk}\psi + F_{ijkm}\varphi_{,m} + G_{ijkm}\psi_{,m},\\
&\sigma_i = g_{rsi} e_{rs} + F_{pqri}\kappa_{pqr} + a_{ij}\varphi_{,j} + b_{ij}\psi_{,j} + d_i\varphi + q_i\psi,\\
&\tau_i = h_{rsi} e_{rs} + G_{pqri}\kappa_{pqr} + b_{ji}\varphi_{,j} + c_{ij}\psi_{,j} + p_i\varphi + e_i\psi,\\
&\xi = -d_{ij} e_{ij} - D_{ijk}\kappa_{ijk} - a_1\varphi - a_3\psi - d_i\varphi_{,i} - p_i\psi_{,i},\\
&\zeta = -f_{ij} e_{ij} - E_{ijk}\kappa_{ijk} - a_3\varphi - a_2\psi - q_i\varphi_{,i} - e_i\psi_{,i}.
\end{aligned}
\tag{3.8}
$$

In the linear theory the basic equations are (3.2), (3.7) and (3.8). The relations (2.19) reduce to

$$
\begin{aligned}
&P_i = (\tau_{ji} - \mu_{kji,k}) n_j - D_j(\mu_{kji} n_k) + (D_j n_j)\mu_{pqi} n_p n_q,\\
&R_i = \mu_{rsi} n_r n_s, \qquad \Pi_i = \langle \mu_{rsi} n_r \gamma_s \rangle, \qquad \gamma_j = \varepsilon_{jpq} s_p n_q,
\end{aligned}
\tag{3.9}
$$

where $n_i = \delta_{iK} n_K$, ε_{ijk} is the alternating symbol, and D_j are the components of the surface gradient. The boundary conditions (2.20) become

$$
\begin{aligned}
u_i &= \widetilde{u}_i \ \text{ on } \overline{S}_1 \times I, & Du_i &= \widetilde{d}_i \ \text{ on } \overline{S}_3 \times I,\\
\varphi &= \widetilde{\varphi} \ \text{ on } \overline{S}_5 \times I, & \psi &= \widetilde{\psi} \ \text{ on } \overline{S}_7 \times I,\\
P_i &= \widetilde{P}_i \ \text{ on } \overline{S}_2 \times I, & R_i &= \widetilde{R}_i \ \text{ on } \overline{S}_4 \times I,\\
\sigma &= \widetilde{\sigma} \ \text{ on } \overline{S}_6 \times I, & \tau &= \widetilde{\tau} \ \text{ on } \overline{S}_8 \times I,\\
\Pi_i &= \widetilde{\Pi}_i \ \text{ on } C \times I,
\end{aligned}
\tag{3.10}
$$

where $\widetilde{u}_i$, $\widetilde{d}_i$, $\widetilde{\varphi}$, $\widetilde{\psi}$, $\widetilde{P}_i$, $\widetilde{R}_i$, $\widetilde{\sigma}$, $\widetilde{\tau}$ and $\widetilde{\Pi}_i$ are given. We add the following initial conditions

$$
\begin{aligned}
u_i(X_j,0) &= u_i^0(X_j), & \dot{u}_i(X_j,0) &= v_i^0(X_j),\\
\varphi(X_j,0) &= \varphi^0(X_j), & \dot{\varphi}(X_j,0) &= \zeta_1^0(X_j),\\
\psi(X_j,0) &= \psi^0(X_j), & \dot{\psi}(X_j,0) &= \zeta_2^0(X_j),
\end{aligned}
\tag{3.11}
$$

where the functions u_i^0, v_i^0, φ^0, ψ^0 and ζ_α^0 are prescribed. In the case of isotropic chiral materials the constituive equations (3.8) have the form [3, 14]

$$
\begin{aligned}
\tau_{ij} &= \lambda e_{rr}\delta_{ij} + 2\mu e_{ij} + b_1\delta_{ij}\varphi + b_2\delta_{ij}\psi + f(\varepsilon_{ikm}\kappa_{jkm} + \varepsilon_{jkm}\kappa_{ikm}),\\
\mu_{ijk} &= \frac{1}{2}\alpha_1(\kappa_{rri}\delta_{jk} + 2\kappa_{krr}\delta_{ij} + \kappa_{rrj}\delta_{ik}) + \alpha_2(\kappa_{irr}\delta_{jk} + \kappa_{jrr}\delta_{ik})\\
&\quad + 2\alpha_3\kappa_{rrk}\delta_{ij} + 2\alpha_4\kappa_{ijk} + \alpha_5(\kappa_{kji} + \kappa_{kij}) + \beta_1\delta_{ij}\varphi_{,k}\\
&\quad + \beta_2(\delta_{ik}\varphi_{,j} + \delta_{jk}\varphi_{,i}) + \eta_1\delta_{ij}\psi_{,k} + \eta_2(\delta_{ik}\psi_{,j} + \delta_{jk}\psi_{,i})\\
&\quad + f(\varepsilon_{iks}e_{js} + \varepsilon_{jks}e_{is}),\\
\sigma_i &= \beta_1\kappa_{rri} + 2\beta_2\kappa_{irr} + a\varphi_{,i} + b\psi_{,i},\\
\tau_i &= \eta_1\kappa_{rri} + 2\eta_2\kappa_{irr} + b\varphi_{,i} + c\psi_{,i},\\
\xi &= -b_1 e_{rr} - a_1\varphi - a_3\psi,\\
\zeta &= -b_2 e_{rr} - a_3\varphi - a_2\psi,
\end{aligned}
\tag{3.12}
$$

where λ, μ, α_k $(k = 1, 2, \dots, 5)$, β_α, η_α, b_α $(\alpha = 1, 2)$, a, b, c, a_j and f are constituitve coefficients. For centrosymmetric materials we have $f = 0$. By using (3.2) and (3.12), we can express Eq. (3.7) in terms of the functions u_i, φ and ψ,

$$
\begin{aligned}
&(\mu - p_1\Delta)\Delta u_i + (\lambda + \mu - p_2\Delta)u_{j,ji} + b_1\varphi_{,i} + b_2\psi_{,i}\\
&\qquad\qquad\qquad\qquad + 2f\varepsilon_{ikm}\Delta u_{m,k} + \rho f_i = \rho\ddot{u}_i,\\
&a\Delta\varphi + b\Delta\psi - a_1\varphi - a_3\psi + (\beta_1 + 2\beta_2)\Delta u_{j,j} - b_1 u_{i,i} + \rho g = \rho\kappa_1\ddot{\varphi},\\
&b\Delta\varphi + c\Delta\psi - a_3\varphi - a_2\psi + (\eta_1 + 2\eta_2)\Delta u_{j,j} - b_2 u_{i,i} + \rho l = \rho\kappa_2\ddot{\psi},
\end{aligned}
\tag{3.13}
$$

where

$$
p_1 = 2(\alpha_3 + \alpha_4), \qquad p_2 = 2(\alpha_1 + \alpha_2 + \alpha_3).
$$

4. Deformation of a chiral circular cylinder

In this section we consider the equilibrium theory of isotropic and homogeneous elastic solids characterized by the constitutive equations (3.12). We suppose that the region B from here on refers to a right cylinder of the length h

with the cross section Σ. We assume that the lateral boundary Π is a smooth surface. We denote by Γ the boundary of Σ and assume that the rectangular Cartesian coordinate system is chosen such that X_3-axis is parallel to the axis of cylinder, and X_1X_2-plane contains one of the ends. The cross-sections located at $X_3 = 0$ and $X_3 = h$ are denoted by Σ_1 and Σ_2, respectively. We suppose that the surface Π is free from loading so that we have the conditions

$$P_i = 0, \quad R_i = 0, \quad \sigma_\alpha n_\alpha = 0, \quad \tau_\alpha n_\alpha = 0 \quad \text{on } \Pi, \tag{4.1}$$

where $(n_1, n_2, 0)$ are the direction cosines of the exterior normal to Π. In what follows we assume that the body loads are absent and that the load on the cylinder is distributed over its ends so that the conditions of equilibrium are satisfied. In this case the equations (3.7) reduce to

$$\tau_{ji,j} - \mu_{kji,kj} = 0, \quad \sigma_{j,j} + \xi = 0, \quad \tau_{i,i} + \zeta = 0. \tag{4.2}$$

We study the extension and torsion of porous elastic beams, and assume that the loading applied on Σ_1 is statically equivalent to the force $\mathbf{F} = (0, 0, F_3)$ and the moment $\mathbf{M} = (0, 0, M_3)$. Thus, on Σ_1 we have the conditions

$$\int_{\Sigma_1} P_\alpha \, da + \int_{\Gamma_1} Q_\alpha \, ds = 0, \tag{4.3}$$

$$\int_{\Sigma_1} P_3 \, da + \int_{\Gamma_1} Q_3 \, ds = F_3, \tag{4.4}$$

$$\int_{\Sigma_1} (X_\alpha P_3 + R_\alpha) \, da + \int_{\Gamma_1} X_\alpha Q_3 \, ds = 0, \tag{4.5}$$

$$\int_{\Sigma_1} \varepsilon_{\alpha\beta 3} X_\alpha P_\beta \, da + \int_{\Gamma_1} \varepsilon_{\alpha\beta 3} X_\alpha Q_\beta \, ds = M_3, \tag{4.6}$$

where Γ_1 is the boundary of Σ_1. From (3.9) we get

$$\begin{aligned} &P_i = -\tau_{3i} + 2\mu_{\alpha 3i,\alpha} + \mu_{33i,3}, \quad R_i = \mu_{33i} \quad \text{on } \Sigma_1, \\ &Q_i = -2\mu_{\alpha 3i} n_\alpha \quad \text{on } \Gamma_1. \end{aligned} \tag{4.7}$$

Here and in what follows Greek subscripts are confined to the range $(1, 2)$. Let us consider the case of a circular cylinder of the radius a with the cross-section Σ_1 defined by $\Sigma_1 = \{X : X_1^2 + X_2^2 < a^2, X_3 = 0\}$. Deformation of porous circular cylinders in the theory of simple materials with voids has been studied in various papers (see, e.g., [21, 23] and references therein). We try to find the solution assuming that

$$
\begin{aligned}
u_\alpha &= \varepsilon_{3\beta\alpha}c_1X_\beta X_3 + c_2u_\alpha^{(1)} + c_2u_\alpha^{(2)},\\
u_3 &= c_2X_3 + c_1u_3^{(1)} + c_2u_3^{(2)},\\
\varphi &= c_1\varphi^{(1)} + c_2\varphi^{(2)},\\
\psi &= c_1\psi^{(1)} + c_2\psi^{(2)},
\end{aligned}
\tag{4.8}
$$

where c_1 and c_2 are unknown constants, and $u_j^{(\rho)}$, $\varphi^{(\rho)}$ and $\psi^{(\rho)}$, $(\rho = 1, 2)$, are unknown functions which are independent of X_3. The terms which depend on X_3 coincide with those from the classical elasticity.We introduce the notations

$$
2e_{\alpha\beta}^{(\rho)} = u_{\alpha,\beta}^{(\rho)} + u_{\beta,\alpha}^{(\rho)}, \quad 2e_{\alpha3}^{(\rho)} = u_{3,\alpha}^{(\rho)}, \quad \kappa_{\alpha\beta j}^{(\rho)} = u_{j,\alpha\beta}^{(\rho)}.
\tag{4.9}
$$

The stresses in the plane strain associated to the functions $u_j^{(\rho)}, \varphi^{(\rho)}$ and $\psi^{(\rho)}$ are given by

$$
\begin{aligned}
\tau_{\alpha\beta}^{(\rho)} &= \lambda e_{\eta\eta}^{(\rho)}\delta_{\alpha\beta} + 2\mu e_{\alpha\beta}^{(\rho)} + b_1\varphi^{(\rho)}\delta_{\alpha\beta} + b_2\psi^{(\rho)}\delta_{\alpha\beta} + f(\varepsilon_{\alpha\eta3}\kappa_{\beta\eta3}^{(\rho)}\\
&\quad + \varepsilon_{\beta\eta3}\kappa_{\alpha\eta3}^{(\rho)}),\\
\tau_{\alpha3}^{(\rho)} &= 2\mu e_{\alpha3}^{(\rho)} + f\varepsilon_{\eta\beta3}\kappa_{\alpha\eta\beta}^{(\rho)},\\
\mu_{\alpha\beta\gamma}^{(\rho)} &= \frac{1}{2}\alpha_1(\kappa_{\eta\eta\alpha}^{(\rho)}\delta_{\beta\gamma} + 2\kappa_{\gamma\eta\eta}^{(\rho)}\delta_{\alpha\beta} + \kappa_{\eta\eta\beta}^{(\rho)}\delta_{\alpha\gamma}) + \alpha_2(\kappa_{\alpha\eta\eta}^{(\rho)}\delta_{\beta\gamma} + \kappa_{\beta\eta\eta}^{(\rho)}\delta_{\alpha\gamma})\\
&\quad + 2\alpha_3\kappa_{\eta\eta\gamma}^{(\rho)}\delta_{\alpha\beta} + 2\alpha_4\kappa_{\alpha\beta\gamma}^{(\rho)} + \alpha_5(\kappa_{\gamma\beta\alpha}^{(\rho)} + \kappa_{\gamma\alpha\beta}^{(\rho)}) + \beta_1\delta_{\alpha\beta}\varphi_{,\gamma}^{(\rho)}\\
&\quad + \beta_2(\delta_{\alpha\gamma}\varphi_{,\beta}^{(\rho)} + \delta_{\beta\gamma}\varphi_{,\alpha}^{(\rho)}) + \eta_1\delta_{\alpha\beta}\psi_{,\gamma}^{(\rho)}\\
&\quad + \eta_2(\delta_{\alpha\gamma}\psi_{,\beta}^{(\rho)} + \delta_{\beta\gamma}\psi_{,\alpha}^{(\rho)}) + f(\varepsilon_{\alpha\gamma3}e_{\beta3}^{(\rho)} + \varepsilon_{\beta\gamma3}e_{\alpha3}^{(\rho)}),\\
\mu_{\alpha\beta3}^{(\rho)} &= 2\alpha_3\kappa_{\eta\eta3}^{(\rho)}\delta_{\alpha\beta} + 2\alpha_4\kappa_{\alpha\beta3}^{(\rho)} + f(\varepsilon_{\eta\alpha3}e_{\beta\eta}^{(\rho)} + \varepsilon_{\eta\beta3}e_{\alpha\eta}^{(\rho)}),\\
\sigma_\alpha^{(\rho)} &= \beta_1\kappa_{\eta\eta\alpha}^{(\rho)} + 2\beta_2\kappa_{\alpha\eta\eta}^{(\rho)} + a\varphi_{,\alpha}^{(\rho)} + b\psi_{,\alpha}^{(\rho)},\\
\tau_\alpha^{(\rho)} &= \eta_1\kappa_{\eta\eta\alpha}^{(\rho)} + 2\eta_2\kappa_{\alpha\eta\eta}^{(\rho)} + b\varphi_{,\alpha}^{(\rho)} + c\psi_{,\alpha}^{(\rho)},\\
\xi^{(\rho)} &= -b_1e_{\alpha\alpha}^{(\rho)} - a_1\varphi^{(\rho)} - a_3\psi^{(\rho)},\\
\zeta^{(\rho)} &= -b_2e_{\alpha\alpha}^{(\rho)} - a_3\varphi^{(\rho)} - a_2\psi^{(\rho)}.
\end{aligned}
\tag{4.10}
$$

In view of (4.8)–(4.10) and the constitutive equations (3.12) we obtain

$$
\begin{aligned}
\tau_{\alpha\beta} &= (\lambda c_2 - 2fc_1)\delta_{\alpha\beta} + \sum_{\rho=1}^{2} c_\rho \tau_{\alpha\beta}^{(\rho)},\\
\tau_{\alpha 3} &= \mu c_1 \varepsilon_{3\beta\alpha} X_\beta + \sum_{\rho=1}^{2} c_\rho \tau_{\alpha 3}^{(\rho)},\\
\tau_{33} &= (\lambda + 2\mu)c_2 + 4fc_1 + \sum_{\rho=1}^{2} c_\rho (\lambda e_{\eta\eta}^{(\rho)} + b_1\varphi^{(\rho)} + b_2\psi^{(\rho)}),\\
\mu_{111} &= \sum_{\rho=1}^{2} c_\rho \mu_{111}^{(\rho)}, \quad \mu_{222} = \sum_{\rho=1}^{2} c_\rho \mu_{222}^{(\rho)},\\
\mu_{221} &= -fc_1 X_1 + \sum_{\rho=1}^{2} c_\rho \mu_{221}^{(\rho)}, \quad \mu_{112} = -fc_1 X_2 + \sum_{\rho=1}^{2} c_\rho \mu_{112}^{(\rho)},\\
\mu_{121} &= \frac{1}{2} fc_1 X_2 + \sum_{\rho=1}^{2} c_\rho \mu_{121}^{(\rho)}, \quad \mu_{122} = \frac{1}{2} fc_1 X_1 + \sum_{\rho=1}^{2} c_\rho \mu_{122}^{(\rho)},\\
\mu_{\alpha 33} &= -\frac{1}{2} fc_1 X_\alpha\\
&\quad + \sum_{\rho=1}^{2} c_\rho \left(\alpha_2 \kappa_{a\lambda\lambda}^{(\rho)} + \frac{1}{2}\alpha_1 \kappa_{\lambda\lambda\alpha}^{(\rho)} + f\varepsilon_{\lambda\alpha 3} e_{3\lambda}^{(\rho)} + \beta_2 \varphi_{,\alpha}^{(\rho)} + \eta_2 \psi_{,\alpha}^{(\rho)} \right),\\
\mu_{33\alpha} &= fc_1 X_\alpha + \sum_{\rho=1}^{2} c_\rho \left(\alpha_1 \kappa_{\alpha\lambda\lambda}^{(\rho)} + 2\alpha_3 \kappa_{\lambda\lambda\alpha}^{(\rho)} + f\varepsilon_{3\alpha\lambda} u_{3,\lambda}^{(\rho)} \right),\\
\mu_{\alpha 3\beta} &= \varepsilon_{3\alpha\beta} fc_2 + \varepsilon_{3\alpha\beta}(2\alpha_4 - \alpha_5)c_1\\
&\quad + \sum_{\rho=1}^{2} c_\rho \left(\frac{1}{2}\alpha_1 \kappa_{\lambda\lambda 3}^{(\rho)} \delta_{\alpha\beta} + \alpha_5 \kappa_{\beta\alpha 3}^{(\rho)} + fc_{3\beta\lambda} e_{\alpha\lambda}^{(\rho)} \right),\\
\mu_{\alpha\beta 3} &= \sum_{\rho=1}^{2} c_\rho \mu_{\alpha\beta 3}^{(\rho)}, \quad \mu_{333} = (\alpha_1 + 2\alpha_3) \sum_{\rho=1}^{2} c_\rho \kappa_{\lambda\lambda 3}^{(\rho)},\\
\sigma_\alpha &= \sum_{\rho=1}^{2} c_\rho \sigma_\alpha^{(\rho)}, \quad \sigma_3 = \beta_1 \sum_{\rho=1}^{2} c_\rho \kappa_{\lambda\lambda 3}^{(\rho)},\\
\tau_\alpha &= \sum_{\rho=1}^{2} c_\rho \tau_\alpha^{(\rho)}, \quad \tau_3 = \eta_1 \sum_{\rho=1}^{2} c_\rho \kappa_{\eta\eta 3}^{(\rho)},\\
\xi &= -b_1 c_2 + \sum_{\rho=1}^{2} c_\rho \xi^{(\rho)}, \quad \zeta = -b_2 c_2 + \sum_{\rho=1}^{2} c_\rho \zeta^{(\rho)}.
\end{aligned}
\tag{4.11}
$$

We require that the equilibrium equations (4.2) and the boundary conditions (4.1) be satisfied for any c_1 and c_2. From the equilibrium equations we find that the functions $u_j^{(\rho)}, \varphi^{(\rho)}$ and $\psi^{(\rho)}$ satisfy the equations

$$\begin{aligned}
&\tau^{(\rho)}_{\beta j,\beta} - \mu^{(\rho)}_{\eta\nu j,\eta\nu} = 0, \quad \sigma^{(1)}_{\alpha,\alpha} + \xi^{(1)} = 0,\\
&\sigma^{(2)}_{\alpha,\alpha} + \xi^{(2)} - b_1 = 0, \qquad\qquad (4.12)\\
&\tau^{(1)}_{\alpha,\alpha} + \zeta^{(1)} = 0, \quad \tau^{(2)}_{\alpha,\alpha} + \zeta^{(2)} - b_2 = 0 \quad \text{on } \Sigma.
\end{aligned}$$

On the lateral surface we have $n_\alpha = X_\alpha/a$ and $n_3 = 0$. The boundary conditions (3.9) imply the following conditions,

$$\begin{aligned}
&P^{(1)}_\alpha = 2fn_\alpha, \quad P^{(1)}_3 = 0, R^{(1)}_i = 0, \quad \sigma^{(1)}_\alpha n_\alpha = 0,\\
&\tau^{(1)}_\alpha n_\alpha = 0, \quad P^{(2)}_\alpha = -\lambda n_\alpha, \quad P^{(2)}_3 = 0, \qquad\qquad (4.13)\\
&R^{(2)}_i = 0, \quad \sigma^{(2)}_\alpha n_\alpha = 0, \quad \tau^{(2)}_\alpha n_\alpha = 0,
\end{aligned}$$

on Γ_1. Here we have denoted by $P_j^{(\rho)}$ and $R_j^{(\rho)}$ the functions P_i and R_i from (3.9), associated to the stresses $\tau_{ij}^{(\rho)}$ and $\mu_{ijk}^{(\rho)}$. We seek the functions $u_j^{(1)}$, $\varphi^{(1)}$ and $\psi^{(1)}$ in the form

$$u^{(1)}_\alpha = A_1 X_\alpha, \quad u^{(1)}_3 = 0, \quad \varphi^{(1)} = A_2, \quad \psi^{(1)} = A_3, \qquad (4.14)$$

where A_j are unknown constants.

It follows from (3.9), (4.8) and (4.10) that

$$\begin{aligned}
&P^{(1)}_\alpha = [2(\lambda+\mu)A_1 + b_1A_2 + b_2A_3]n_\alpha,\\
&P^{(1)}_3 = 0, \quad R^{(1)}_i = 0, \qquad\qquad (4.15)\\
&\sigma^{(1)}_\alpha = 0, \quad \tau^{(1)}_\alpha = 0.
\end{aligned}$$

The equations of equilibrium and the conditions (4.13), for the functions $u^{(1)}_\alpha, \varphi^{(1)}$ and $\psi^{(1)}$, reduce to the following equations for the constants A_j,

$$\begin{aligned}
&2(\lambda+\mu)A_1 + b_1A_2 + b_2A_3 = 2f,\\
&2A_1b_1 + a_1A_2 + a_3A_3 = 0,\\
&2A_1b_2 + a_3A_2 + a_2A_3 = 0.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&A_1 = f(a_1a_2 - a_3^2)/d, \quad A_2 = -2f(b_1a_2 - b_2a_3)/d,\\
&A_3 = 2f(b_1a_3 - a_1b_2)/d, \qquad\qquad (4.16)\\
&\quad d = (\lambda+\mu)(a_1a_2 - a_3^2) - b_1(b_1a_2 - b_2a_3) + b_2(b_1a_3 - a_1b_2).
\end{aligned}$$

The positive definiteness of the elastic potential implies that d is different from zero. To prove this assertion we consider the elastic potential W in the form

$$2W = \tau_{ij}e_{ij} + \mu_{ijk}\kappa_{ijk} + \sigma_i\varphi_{,i} + \tau_i\psi_{,i} - \xi\varphi - \zeta\psi.$$

In the case of the deformation described by the functions $u_j^{(1)}, \varphi^{(1)}$ and $\psi^{(1)}$ we find

$$\begin{aligned}
&e^{(1)}_{\alpha\beta} = A_1\delta_{\alpha\beta}, \quad e^{(1)}_{j3} = 0, \quad \kappa^{(1)}_{ijk} = 0,\\
&\tau^{(1)}_{\alpha\beta} = [2(\lambda+\mu)A_1 + b_1A_2 + b_2A_3]\delta_{\alpha\beta}, \quad \tau^{(1)}_{j3} = \mu^{(1)}_{ijk} = 0,\\
&\sigma^{(1)}_j = \tau^{(1)}_j = 0, \quad \xi^{(1)} = -2b_1A_1 - a_1A_2 - a_3A_3,\\
&\zeta^{(1)} = -2b_2A_1 - a_3A_2 - a_2A_3,
\end{aligned}$$

and

$$2W = 4(\lambda+\mu)A_1^2 + a_1^2A_2^2 + a_2^2A_3^2 + 4b_1A_1A_2 + 4b_2A_1A_3 + 2a_3A_2A_3.$$

Let us consider the matrix

$$\begin{pmatrix} 4(\lambda+\mu) & 2b_1 & 2b_2\\ 2b_1 & a_1 & a_3\\ 2b_2 & a_3 & a_2 \end{pmatrix}.$$

Since W is a positive-definite quadratic form, the determinant of this matrix is nonzero. It is easy to see that $detA = 4d$. We conclude that $d \neq 0$.

We seek the functions $u_j^{(2)}$, $\varphi^{(2)}$ and $\psi^{(2)}$ in the form

$$u^{(2)}_\alpha = B_1X_\alpha, \;\; u^{(2)}_3 = 0, \;\; \varphi^{(2)} = B_2, \quad \psi^{(2)} = B_3, \tag{4.17}$$

where B_j are unknown constants. The equations and the boundary conditions associated to these functions are satisfied if

$$\begin{aligned}
&B_1 = [b_1(b_1a_2 - b_2a_3) - \lambda(a_1a_2 - a_3^2) + b_2(a_1b_2 - b_1a_3)]/2d,\\
&B_2 = \mu(b_2a_3 - b_1a_2)/d, \quad B_3 = \mu(b_1a_3 - a_1b_2)/d.
\end{aligned} \tag{4.18}$$

In view of (4.7), (4.11), (4.14) and (4.8) we get

$$\begin{aligned}
&\int\limits_{\Sigma_1} P_j\,da + \int\limits_{\Gamma_1} Q_j\,ds = -\int\limits_{\Sigma_1}\tau_{3j}\,da,\\
&\int\limits_{\Sigma_1}(X_\alpha P_3 + R_\alpha)\,da + \int\limits_{\Gamma_1} X_\alpha Q_3\,ds = -\int\limits_{\Sigma_1}(X_\alpha\tau_{33} + 2\mu_{\alpha33})\,da,\\
&\int\limits_{\Sigma_1}\varepsilon_{\alpha\beta3}X_\alpha P_\beta\,da + \int\limits_{\Gamma_1}\varepsilon_{\alpha\beta3}X_\alpha Q_\beta\,ds = -\int\limits_{\Sigma_1}\varepsilon_{\alpha\beta3}(X_\alpha\tau_{3\beta} + 2\mu_{\alpha3\beta})\,da.
\end{aligned} \tag{4.19}$$

It follows from (4.11), (4.14) and (4.18) that

$$
\begin{aligned}
\tau_{3\alpha} &= \mu c_1 \varepsilon_{3\beta\alpha} X_\beta, \\
\tau_{33} &= c_1(4f + 2\lambda A_1 + b_1 A_2 + b_2 A_3) \\
&\quad + c_2(\lambda + 2\mu + 2\lambda B_1 + b_1 B_2 + b_2 B_3), \\
\mu_{\alpha 33} &= -\frac{1}{2} f c_1 X_\alpha, \\
\mu_{\alpha 3\beta} &= \varepsilon_{3\alpha\beta} f c_2 + \varepsilon_{3\alpha\beta} c_1 (2\alpha_4 - \alpha_5) + \varepsilon_{3\beta\alpha} f (c_1 A_1 + c_2 B_1).
\end{aligned}
\tag{4.20}
$$

The conditions (4.3) and (4.5) are satisfied on the basis of (4.19) and (4.20). The conditions (4.4) and (4.6) reduce to the following conditions for the constants c_1 and c_2,

$$
D_{11} c_1 + D_{12} c_2 = -M_3, D_{21} c_1 + D_{22} c_2 = -F_3, \tag{4.21}
$$

where we have used the notation

$$
\begin{aligned}
D_{11} &= \frac{1}{2}\pi a^4 + 4\pi(2\alpha_4 - \alpha_5 - f A_1) a^2, \\
D_{12} &= 4f(1 - B_1), \\
D_{21} &= \pi(4f + 2\lambda A_1 + b_1 A_2 + b_2 A_3) a^2, \\
D_{22} &= \pi(\lambda + 2\mu + 2\lambda B_1 + b_1 B_2 + b_2 B_3) a^2.
\end{aligned}
$$

As in classical elasticity [24] we can prove that the system (4.21) determines the constants c_1 and c_2. In the case of a centrosymmetric material ($f = 0$) the coefficients A_j, D_{12} and D_{21} are equal to zero and we obtain

$$
c_1 = -M_3/D_{11}, \qquad c_2 = -F_3/D_{22},
$$

where D_{11} is the torsional rigidity. In contrast with the classical theory the torsion and extension of a chiral cylinder cannot be treated independently of each other. It follows from (4.8), (4.14) and (4.17) that the solution of the problem is

$$
\begin{aligned}
u_\alpha &= \varepsilon_{3\beta\alpha} c_1 X_\beta X_3 + (c_1 A_1 + c_2 B_1) X_\alpha, \\
u_3 &= c_2 X_3, \\
\varphi &= c_1 A_2 + c_2 B_2, \psi = c_1 A_3 + c_2 B_3,
\end{aligned}
$$

where the constants A_j and B_j are given by (4.16) and (4.18), and the constants c_α are determined by the system (4.21). We can see that the torsion of the cylinder produces a variation of porosities.

5. Conclusions

We study a strain gradient theory of elastic materials which have a double porosity structure. The original results established in this paper can be summarized as follows:

(a) We establish the basic equations of the nonlinear theory of porous elastic solids and formulate the boundary-initial-value problems.

(b) We derive the field equations of the linear theory and present the constitutive equations for chiral elastic solids.

(c) We investigate the deformation of a chiral elastic cylinder subjected to extension and torsion.

(d) We show that the torsion of the cylinder produces a variation of porosities and an extension.

Declaration of competing interest

There is no any conflict of interest.

Acknowledgements

I express my gratitude to the referees for their helpful suggestions.

References

1. R.A. Toupin, *Elastic materials with couple stresses*, Archive for Rational Mechanics and Analysis, **11**, 385–414, 1962.
2. R.A. Toupin, *Theories of elasticity with couple-stress*, Archive for Rational Mechanics and Analysis, **17**, 85–112, 1964.
3. R.D. Mindlin, *Microstructure in linear elasticity*, Archive for Rational Mechanics and Analysis, **16**, 51–77, 1964.
4. R.D. Mindlin, N.N. Eshel, *On first strain gradient theories in linear elasticity*, International Journal for Solids and Structures, **4**, 109–124, 1968.
5. C. Liebold, W.H. Müller, *Applications of strain gradient theories to the size effect in submicro-structures incl. experimental analysis of elastic material parameters*, Bulletin of TICMI, **19**, 45–55, 2015.
6. S.C.Cowin, *Bone poroelasticity*, Journal of Biomechanics, **32**, 217–238, 1999.
7. J.G. Berryman, H.F. Wang, *Elastic wave propagation and attenuation in a double-porosity dual-permeability medium*, International Journal of Rock Mechanics and Mining Sciences, **37**, 63–78, 2000.
8. N. Khalili, A.P.S. Selvadurai, *A fully coupled constitutive model for thermohydromechanical analysis in elastic media with double porosity*, Geophysical Research Letters, **30**, 2268, 2003.

9. B. Straughan, *Stability and uniqueness in double porosity elasticity*, International Journal of Engineering Science, **65**, 1–8, 2013.

10. B.Straughan, *Waves and uniqueness in multi-porosity elasticity*, Journal of Thermal Stresses, **39**, 704–721, 2016.

11. M. Svanadze, *Potential method in the theory of elasticity for triple porosity materials*, Journal of Elasticity, **130**, 1–24, 2018.

12. R.Lakes, *Elastic and viscoelastic behaviour of chiral materials*, International Journal of Mechanical Sciences, **43**, 1579–1589, 2001.

13. C.S. Ha, M.E. Plesha, R.S. Lakes, *Chiral three dimensional lattices with tunable Poisson's ratio*, Smart Materials Structures, **25**, 054005, 2016.

14. S.A. Papanicolopulos, *Chirality in isotropic linear gradient elasticity*, International Journal for Solids and Structures, **48**, 745–752, 2011.

15. N. Auffray, H. Le Quang, Q.C. Hea, *Matrix representations for 3D strain-gradient elasticity*, Journal of the Mechanics and Physics of Solids, **57**, 1202–1223, 2013.

16. M. Poncelet, A.Somera, C. Morel, C. Jailin, N. Auffray, *An experimental evidence of the failure of Cauchy elasticity for the overall modeling of a non-centro-symmetric lattice under static loading*, International Journal for Solids and Structures, **69-70**, 195–206, 2015.

17. D. Iesan, *On the grade consistent theories of micromorphic solids*, American Institute of Physics, Conference Proceedings, **1329**, 130–149, 2011.

18. J.W. Nunziato, S.C. Cowin, *A nonlinear theory of elastic materials with voids*, Archive for Rational Mechanics and Analysis, **72**, 175–201, 1979.

19. D. Iesan, R. Quintanilla, *On a theory of thermoelastic materials with a double porosity structure*, Journal of Thermal Stresses, **37**, 1017–1036, 2014.

20. A.E. Green, R.S. Rivlin, *Multipolar continuum mechanics*, Archive for Rational Mechanics and Analysis, **17**, 113–147, 1964.

21. S. De Cicco, D. Iesan, *Thermal effects in anisotropic porous elastic rods*, Journal of Thermal Stresses, **36**, 364–377, 2013.

22. S.De Cicco , F. De Angelis, *A plane strain problem in the theory of elastic materials with voids*, Mathematics and Mechanics of Solids, **25**, 46–59, 2020.

23. S. De Cicco, D. Iesan, *On the theory of thermoelastic materials with a double porosity structure*, Journal of Thermal Stresses, **44** (12), 1514–1533, 2021.

24. D. Iesan, *Classical and Generalized Models of Elastic Rods*, Chapman & Hall/CRC Press, New York, 2009.

Received February 18, 2022; revised version May 19, 2022.
Published online June 07, 2022.