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SWITCHING TIME ESTIMATION AND ACTIVE MODE RECOGNITION USING A DATA PROJECTION METHOD

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This paper proposes a data projection method (DPM) to detect a mode switching and recognize the current mode in a switching system. The main feature of this method is that the precise knowledge of the system model, i.e., the parameter values, is not needed. One direct application of this technique is fault detection and identification (FDI) when a fault produces a change in the system dynamics. Mode detection and recognition correspond to fault detection and identification, and switching time estimation to fault occurrence time estimation. The general principle of the DPM is to generate mode indicators, namely, residuals, using matrix projection techniques, where matrices are composed of input and output measured data. The DPM is presented in detail, and properties of switching detectability (fault detectability) and discernability between modes (fault identifiability) are characterized and discussed. The great advantage of this method, compared with other techniques in the literature, is that it does not need the model parameter values and thus can be applied to systems of the same type without identifying their parameters. This is particularly interesting in the design of generic embedded fault diagnosis algorithms.

Keywords: switching systems, mode recognition, fault detection and isolation, data-driven method, mode discernability, switching detectability, fault identifiability.

1. Introduction

Switching systems are characterized by the interaction between a finite state automaton and a finite number of dynamical subsystems called operating modes (Liberzon, 2005). These operating modes may be described by differential or difference equations (Lin and Antsaklis, 2009). The switching between modes could be governed by a logical switching rule called the switching law. It determines which mode is active at each time instant and is governed by (Lin and Antsaklis, 2009)

- internal features: system input, output, state variables or system parameters changing, etc.;
- external actions: human operators actions, environment conditions changing, etc.

Recently, switching systems have been the subject of intensive investigations. A motivation comes from the fact that they represent a large class of physical systems, such as mechanical and chemical processes (Engell *et al.*, 2000), communication networks, aircraft and air traffic control systems (Livadas *et al.*, 2000), automotive systems (Antsaklis, 2000), robotics (Petroff, 2007), embedded systems (Zhang *et al.*, 2007), DC/DC converters (Ma *et al.*, 2004), oscillators (Torikai and Saito, 1998) or chaos generators (Mitsubori and Saito, 1997).

Another motivation for studying switching systems comes from the simplicity of representing complex non-linear systems using a set of simple structure models (linear time invariant subsystems, for example), where each operating zone is described by a mode (Goebel *et al.*, 2012).

Various problems for switching systems have been investigated, such as modeling (Heemels *et al.*, 2001), stability studies (Williams and Hoft, 1991), observability and controllability analysis (Daizhan, 2007), or fault detection and identification (FDI) (Akhenak *et al.*, 2008; Domlan *et al.*, 2007b), etc.

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All these studies show that, at each time instant, it is very important to know the exact active mode. Because it is not always possible to implement specialized sensors that indicate the active mode, estimation techniques have to be designed. The aims of such algorithms are to detect any switching and to accurately recognize the current mode. One direct use of such techniques is FDI.

The data projection method (DPM) is different from other methods proposed in the literature for switching detection, mode recognition and discernability characterization (Narasimhan and Biswas, 2007; Anderson *et al.*, 2001; Akhenak *et al.*, 2008; Domlan *et al.*, 2007a). The DPM is a data-driven method, which is guided by the structure of the model which has to be known. However, differently from the cited literature, the parameter values are not needed to apply the DPM. This makes this method evidently intrinsically robust to parameters values and easier to apply. In this paper, the discernability conditions between modes and the conditions of mode switching detectability are revisited. It is shown that these conditions are not equivalent: a transition between non-discernable modes can be detected in certain situations.

This original result was not obtained by Narasimhan and Biswas (2007), Anderson *et al.* (2001), Akhenak *et al.* (2008) or Domlan *et al.* (2007a). Early fault detection and identification (FDI) is crucial for human and system safety. Indeed, if a fault occurs in the system and is not detected, it may produce a severe damage in the system and in its environment. Even if the fault does not cause a severe damage in the system, it can weaken its dependability and performance. If the fault is detected, the control law can be adjusted in order to maintain the system performance (Yang *et al.*, 2010) or maintenance actions can be performed. It is thus of paramount importance to detect accurately and as fast as possible a fault, to localize precisely the faulty component and to characterize (identify) the fault.

Internal component faults modify the system dynamics (Cocquempot *et al.*, 2004). A way to deal with these faults is to consider faulty modes. Fault detection is thus equivalent to detect a faulty mode switching, and fault identification is equivalent to recognize the mode after switching.

Several approaches for switching detection and mode recognition have been studied in the literature (Narasimhan and Biswas, 2007; Anderson *et al.*, 2001), including model-based methods and data-driven ones. One difficulty in model-based methods (Narasimhan and Biswas, 2007; Anderson *et al.*, 2001) is accurate estimation of system parameters in each mode.

Moreover, even for the same kind of physical systems, the parameter values are not exactly the same and may slightly change in the system's life, which leads to model uncertainties. To cope with this problem, robust methods are proposed in the literature. These methods are based on observers (Belkhiat, 2011), Kalman filters (Akhenak *et al.*, 2008), and analytical redundancy relations (ARRs) (Bayoudh and Travé Massuyès, 2014; El Mezyani, 2005). However, these methods are limited to given classes of uncertainties.

The DPM, which is considered in this paper, uses the collected data and the knowledge of the mode class. However, it does not need the values of the model parameters. Previous publications have introduced the DPM for linear systems (Pekpe *et al.*, 2006) to detect and to isolate sensor faults. A residual is generated by projecting the system output matrix onto the kernel of an input Hankel matrix. The proposed residual is calculated using only on-line input-output data.

The DPM is extended in this paper for switching detection and mode recognition. The condition for mode recognition is the discernability between modes. Indeed, if two modes are not discernable, it is not possible to determine which one is active. Finding the conditions for discernability between modes has been the subject of intensive studies, and several results have been reported in the literature (see, e.g., Cocquempot *et al.*, 2004; Bayoudh and Travé Massuyès, 2014).

Discernability and switching detectability will be characterized using the DPM. It is shown that these two conditions are not equivalent; in other words, a switching between two non discernable modes could be the following detected.

The main contributions of this paper are

- a method, called the DPM, to estimate the switching time by using only on-line collected measured data. This method will be extended to recognize the active mode by using on-line collected data and a database of inputs and outputs collected off-line;

- a characterization of several properties, such as discernability between modes and switching detectability.

The rest of the paper is organized as follows. In Section 2, the switching system with linear modes is described. In Section 3, the data projection method (DPM) is detailed and used for switching time estimation. In Section 4, the DPM is extended for active mode recognition. In Section 5, DPM tuning is explained. Finally, two illustrative examples are presented to show the efficiency of the proposed method.

2. Problem setting

Consider the dynamic switching system with linear discrete-time modes described by

$$\begin{cases} x_{k+1} = A_{\sigma_k} x_k + B_{\sigma_k} u_k, \\ y_k = C_{\sigma_k} x_k + D_{\sigma_k} u_k + w_k, \\ (x_0,\ \sigma_0) \in \text{Init}, \end{cases} \tag{1}$$

where $A_{\sigma_k} \in \mathbb{R}^{n\times n}$, $B_{\sigma_k} \in \mathbb{R}^{n\times m}$, $C_{\sigma_k} \in \mathbb{R}^{\ell\times n}$, $D_{\sigma_k} \in \mathbb{R}^{\ell\times m}$ are constant matrices, and vectors $u_k \in \mathbb{R}^m$, $x_k \in \mathbb{R}^n$ and $y_k \in \mathbb{R}^\ell$ are respectively input, state and output signals at time-instant kT_e, with T_e being the sampling period. The system outputs are affected by a centered Gaussian noise $w_k \in \mathbb{R}^\ell$, where $\text{var}(w_k^s)$ represents the variance of w_k^s, $w_k = \begin{pmatrix} w_k^1 & \dots & w_k^\ell \end{pmatrix}^T$. $\sigma_k \in \{1, 2, \cdots, Q\}$ is the mode index and 'Init' is the set of initial states $(x_0,\ \sigma_0)$, Q being the number of modes.

2.1. Definition, objectives and hypotheses. Given the switching system described by Eqn. (1), the objectives of this paper are to estimate the switching time and to recognize the current mode under the following hypotheses:

1. Matrices A_{σ_k}, B_{σ_k}, C_{σ_k}, D_{σ_k} are all *unknown*.
2. State matrices A_{σ_k} are stable for all σ_k.
3. u_k and y_k are known for all values of k.
4. The time period between two successive switchings is long enough to allow mode identification, i.e., the system has a dwell-time (Hespanha and Morse, 1999). This condition will be precisely characterized later when the method is detailed.

Definition 1. Two modes $(m_1,\ m_2)$ are *discernible* for all inputs in a time interval $[0,\ T]$ (T is a positive integer), if for all initial states and for the same input applied in modes m_1 and m_2, the outputs in the two modes m_1 and m_2 are different.

2.2. Data projection method for sensor FDI. For simplicity, let us consider first the dynamic linear system described by

$$\begin{cases} x_{k+1} = Ax_k + Bu_k, \\ y_k = Cx_k + Du_k + w_k + f_k, \end{cases} \tag{2}$$

where f_k represents the vector of sensor faults at time k.

The DPM framework is described here for one mode (see the work of Pekpe *et al.* (2006) for more details):

1. By stacking Eqn. (2) on a time window of size L ($L \in \mathbb{N}$), we obtain

$$\begin{aligned} Y_{k-L+1:k} &= CA^i X_{k-L-i+1:k-i} + H_i U_{k-L+1:k} \\ &\quad + F_{k-L+1:k} + W_{k-L+1:k}, \end{aligned} \tag{3}$$

with

$$\begin{aligned} H_i &= \begin{bmatrix} CA^{i-1}B| & \cdots| & CB| & D \end{bmatrix}, \\ U_{k-L+1:k} &= \begin{bmatrix} \overline{u}_{k-L+1,\ i} & \overline{u}_{k-L+2,\ i} \\ \cdots & \overline{u}_{k,\ i} \end{bmatrix}, \end{aligned} \tag{4}$$

where

$$\begin{aligned} \overline{u}_{k,\ i} &= \begin{pmatrix} u_{k-i}^{\mathrm{T}} & u_{k-i+1}^{\mathrm{T}} & \dots & u_k^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}} \\ &\in \mathbb{R}^{m(i+1)\times 1} \end{aligned} \tag{5}$$

and

$$\begin{aligned} Y_{k-L+1:k} &= \begin{bmatrix} y_{k-L+1} & \cdots & y_{k-1} & y_k \end{bmatrix} \\ &\in \mathbb{R}^{\ell\times L}. \end{aligned} \tag{6}$$

Matrices $X_{k-L-i+1:k-i}$ and $W_{k-L+1:k}$ are defined as $Y_{k-L+1:k}$ (using x_{k-i} (resp. w_k) instead of y_k). Matrix $F_{k-L+1:k}$ is also defined as $Y_{k-L+1:k}$ using f_k instead of y_k, where f_k represents a sensor fault at time instant k. Moreover,

- the term $CA^i X_{k-L-i+1:k-i}$ depends on model parameters and a set of states in a time window of size L;
- the term $H_i U_{k-L+1:k}$ depends on the inputs $U_{k-L+1:k}$ and H_i on model parameters;
- the term $F_{k-L+1:k}$ depends on additive sensor faults;
- the term $W_{k-L+1:k}$ depends on measurement noise.

2. A judicious choice of the time window (integers i and L):

- owing to the stability of matrix A, the term $CA^i X_{k-L-i+1:k-i}$ becomes very small if i is large enough;
- there exist projection matrices Π, such that $U_{k-L+1:k}\Pi = 0$; these matrices project onto the right orthogonal space of the matrix $U_{k-L+1:k}$ and one of them is $\Pi_{U_{k-L+1:k}}$:

$$\begin{aligned} &\Pi_{U_{k-L+1:k}} \\ &\quad = I_L - U_{k-L+1:k}{}^{\mathrm{T}} \\ &\quad\quad \times (U_{k-L+1:k} U_{k-L+1:k}{}^{\mathrm{T}})^{+} \overline{u}_{k,i} \end{aligned} \tag{7}$$

where I_L is an identity matrix of size L and M^+ represents the Moore–Penrose pseudo-inverse of matrix M.

3. Post-multiplying (3) by $\Pi_{U_{k-L+1:k}}$ gives the residual

$$\begin{aligned} \epsilon_k &= Y_{k-L+1:k} \Pi_{U_{k-L+1:k}} \\ &= \Delta_i + F_{k-L+1:k} \Pi_{U_{k-L+1:k}} \\ &\quad + W_{k-L+1:k} \Pi_{U_{k-L+1:k}}. \end{aligned} \tag{8}$$

Since the state matrix A is stable, the term

$$\Delta_i = CA^i X_{k-L-i+1:k-i}$$

can be neglected if i is large enough (see Pekpe *et al.*, 2006).

4. Finally, a statistical test on ϵ_k is used for fault detection.

3. Switching time estimation

3.1. Residual generation. In this section, the DPM is proposed for switching time estimation. It is proved that a switching can be detected under a detectability condition which will be given in Section 3.2).

We assume that integers i and L satisfy the condition $L > m(i+1)$ and L is an even integer such that $\dim(\ker(U_{k-L+1:k})) > 1$, $\forall u_k (k \in N)$, where $\ker(U_{k-L+1:k})$ represents the kernel of matrix $U_{k-L+1:k}$.

Theorem 1. *Let i and L be two integers and suppose all the inputs are not null and no change occurs in time window $[k-L-i+1, k]$. Then there exist $i_s \in \mathbb{N}$ such that $\forall i \geq i_s$ and $L > m(i+1)$, and the vector ϵ_k defined by*

$$\epsilon_k = Y_{k-L+1:k} \Pi_{U_{k-L+1:k}} \in \mathbb{R}^\ell \tag{9}$$

is a centered Gaussian noise with variance R_ϵ ($\epsilon_k \sim N(0, \mathrm{R}_\epsilon)$), where

$$R_\epsilon = \mathrm{E}[W_{k-L+1:k} \Pi_{U_{k-L+1:k}} \Pi^T_{U_{k-L+1:k}} W^T_{k-L+1:k}]. \tag{10}$$

Proof. Equation (8) applied to (1) gives

$$\epsilon_k = \Delta_i + W_{k-L+1:k} \Pi_{U_{k-L+1:k}}. \tag{11}$$

Since $W_{k-L+1:k}$ is a zero mean Gaussian noise and the inputs are deterministic, $W_{k-L+1:k}\Pi_{U_{k-L+1:k}}$ is also a zero mean Gaussian noise.

Consider a multiplicative norm $\|\Delta_i\|$ of Δ_i. We have

$$\begin{aligned} \|\Delta_i\| &= \left\|CA^i X_{k-L-i+1:k-i}\right\| \\ &\leq \|C\| \, \|A\|^i \, \|X_{k-L-i+1:k-i}\| . \end{aligned} \tag{12}$$

Since all the modes are stable, the state norm is bounded. Let $\|X_m\|$ be the maximum of $\|X_{k-L-i+1:k-i}\|$, $k \in \mathbb{N}$. Then, the following inequality holds:

$$\|\Delta_i\| \leq \|C\| \, \|A\|^i \, \|X_m\| . \tag{13}$$

Therefore, if we choose i_s such that

$$i_s \geq \frac{\log(V_m) - \log(N \, \|C\| \, \|X_m\|)}{\log(\|A\|)}, \tag{14}$$

where V_m is the minimum of $\mathrm{var}(w_k^s)$, $s \in 1, 2, \ldots, \ell$, and N is an integer which is supposed to be sufficiently large. If the inequality (14) holds, then $\forall i > i_s$,

$$\|\Delta_i\| \leq \left\| \frac{V_m}{N} \right\|. \tag{15}$$

If N is a large integer, then the influence of Δ_i is negligible before the noise and ϵ_k is Gaussian zero mean.

Finally, the variance of ϵ_k is

$$R_\epsilon = \mathrm{E}[W_{k-L+1:k} \Pi_{U_{k-L+1:k}} \Pi^T_{U_{k-L+1:k}} W^T_{k-L+1:k}]. \tag{16}$$

■

3.2. Residual analysis. Introduce matrices

$$\begin{aligned} \Omega = \big[& \mathcal{H}_{(\sigma_\tau,\, \sigma_{\tau+1}),\, i,\, 1} - H_{(\sigma_\tau),\, i} | \\ & \mathcal{H}_{(\sigma_\tau,\, \sigma_{\tau+1}),\, i,\, 2} - H_{(\sigma_\tau),\, i} | \cdots | \\ & \mathcal{H}_{(\sigma_\tau,\, \sigma_{\tau+1}),\, i,\, i-1} - H_{(\sigma_\tau),\, i} | \\ & \mathcal{H}_{(\sigma_\tau,\, \sigma_{\tau+1}),\, i,\, i} - H_{(\sigma_\tau),\, i} \big] \big[0_{mi(i+1) \times \tau + L - k - 1} | \\ & \breve{U} | 0_{mi(i+1) \times k - \tau - i + 1} \big] + (H_{(\sigma_{\tau+1}),\, i} \\ & - H_{(\sigma_\tau),\, i}) \left[0_{m(i+1) \times L - k + \tau + i - 1} | U_{\tau+i:k} \right] . \end{aligned} \tag{17}$$

$$\begin{aligned} H_{(\sigma_k),\, i} = \big[& C_{\sigma_k} A^{i-1}_{\sigma_k} B_{\sigma_k} | C_{\sigma_k} A^{i-2}_{\sigma_k} B_{\sigma_k} | \cdots | \\ & C_{\sigma_k} B_{\sigma_k} | D_{\sigma_k} \big], \end{aligned} \tag{18}$$

where

$$\begin{aligned} & \mathcal{H}_{(\sigma_\tau,\, \sigma_{\tau+1}),\, i,\, t-\tau+1} \\ & \quad = [H_{(\sigma_\tau,\, \sigma_{\tau+1}),\, i,\, t-\tau+1} | H_{(\sigma_{\tau+1}),\, t-\tau+1}] \\ & \quad \in \mathbb{R}^{\ell \times m(i+1)} \end{aligned}$$

and $H_{(\sigma_\tau,\, \sigma_{\tau+1}),\, i,\, t-\tau+1}$ is constructed as follows:

$$\begin{aligned} & H_{(\sigma_\tau,\, \sigma_{\tau+1}),\, i,\, t-\tau+1} \\ & \quad = C_{\sigma_{\tau+1}} A^{t-\tau+1}_{\sigma_{\tau+1}} \big[A^{i-t+\tau-2}_{\sigma_\tau} B_{\sigma_\tau} | \\ & \qquad A^{i-t+\tau-3}_{\sigma_\tau} B_{\sigma_\tau} | \cdots | A_{\sigma_\tau} B_{\sigma_\tau} | B_{\sigma_\tau} \big] \\ & \quad \in \mathbb{R}^{\ell \times m(i-t+\tau-1)}. \end{aligned} \tag{19}$$

A condition of switching detectability (internal fault detectability) is given by the following theorem and proposition.

Theorem 2. *If a switching occurs on time interval $[k-L-i+1, k]$ and the inputs are not identically zero, then for all $i \in \mathbb{N}$ and $i > i_s$ (i_s given by Theorem 1), vector ϵ_k is not a centered Gaussian noise with variance R_ϵ if and only if*

$$\mathrm{span}(\Omega) \not\subset \mathrm{span}(U_{k-L+1:k}), \tag{20}$$

where $\mathrm{span}(\mathrm{M})$ *denotes the row space of matrix M and the condition (20) implies*

1. $H_{\sigma_\tau,\, i} \neq H_{\sigma_{\tau+1},\, i}$ *or*
2. $H_{\sigma_\tau,i} = H_{\sigma_{\tau+1},i}$ *and* $\exists r \in \{1, 2, \ldots, i\}$: $\mathcal{H}_{(\sigma_\tau,\sigma_{\tau+1}),i,r} \neq H_{(\sigma_\tau),i}$.

The proof of Theorem 2 is provided in Appendix.

We specify now two particular cases which may respect the previous detectability condition (20):

1. Case of $H_{\sigma_\tau,\, i} \neq H_{\sigma_{\tau+1},\, i}$. This condition implies that modes σ_τ and $\sigma_{\tau+1}$ have different Markov parameters (their minimal realizations are different); in other words, these two modes are *discernible* as defined by Cocquempot *et al.* (2004), Hofbaur *et al.* (2010) or Bayoudh and Travé Massuyès (2014):

   The first condition is sufficient for switching detectability but not necessary. This means that the proposed residual can detect a switching between discernable modes. In addition to that, it can detect switching between non-discernible modes under the second condition (so discernbility between modes is not necessary for switching detectability).

2. The second condition consists of two sub-conditions:

   - non-discernible modes: if $H_{\sigma_\tau,i} = H_{\sigma_{\tau+1},i}$, then there exists a non-singular matrix $\Psi \in \mathbb{R}^{n\times n}$ satisfying
   $$\begin{cases} A_{\sigma_{\tau+1}} = \Psi^{-1} A_{\sigma_\tau} \Psi, \\ B_{\sigma_{\tau+1}} = \Psi^{-1} B_{\sigma_\tau}, \\ C_{\sigma_{\tau+1}} = C_{\sigma_\tau} \Psi, \\ D_{\sigma_{\tau+1}} = D_{\sigma_\tau}, \end{cases}$$
   - and there is an index $r \in \{1, 2, \ldots, i\}$ such that if $\mathcal{H}_{(\sigma_\tau,\sigma_{\tau+1}),i,r} \neq H_{(\sigma_\tau),i}$, then Ψ is not an identity matrix.

   In other words, this condition can be derived by substituting $A_{\sigma_{\tau+1}}$, $B_{\sigma_{\tau+1}}$, $C_{\sigma_{\tau+1}}$ and $D_{\sigma_{\tau+1}}$ in Eqn. (19).

   The *second condition* expresses that the transition between two non-discernible modes is transiently detectable (there exists $r \in \{1,\ 2,\ \ldots,\ i\}$ such that $\mathcal{H}_{(\sigma_\tau,\, \sigma_{\tau+1}),\, i,\, r} \neq H_{(\sigma_\tau),\, i}$). This is due to the modification of the state by matrix Ψ after a switching.

The following proposition provides a statistical tool for switching detection.

Proposition 1. *If a switching occurs on time interval* $[k - L - i + 1, k]$, *the inputs are not identically zero and for all* $i \in \mathbb{N}$ *and* $i > i_s$ *(*i_s *given by Theorem 1) such that the condition (20) is satisfied. Then*

$$\varpi \geq \chi^2_{L,\alpha}, \tag{21}$$

where $\chi_{L,\alpha}$ *is the critical value with significance level* α *and*

$$\varpi = \epsilon_k^{\mathrm{T}} R_\epsilon^{-1} \epsilon_k, \tag{22}$$

otherwise

$$\varpi < \chi^2_{L,\alpha}. \tag{23}$$

4. Current mode recognition

In this section, the DPM is extended to recognize the current mode and the *discernibility condition* is derived.

4.1. Residual generation using on-line and off-line data.

Let us consider two input matrices

$$U_{k-\frac{L}{2}+1:k} \in \mathbb{R}^{m(i+1)\times\frac{L}{2}}$$

and

$$U^*_{(\gamma),\, 1:\frac{L}{2}} \in \mathbb{R}^{m(i+1)\times\frac{L}{2}}$$

which are constructed using respectively inputs collected on-line in the current mode (to be identified) and collected off-line in mode γ.

It should be noted that the inputs collected off-line are persistently exciting (see Van Overschee and De Moor, 1996), and vary to excite all modes in each operating mode, which implies that the matrix $U^*_{(\gamma),\, 1:\frac{L}{2}}$ is of full row rank. Unlike for inputs collected on-line, the persistence condition is no longer indispensable.

Let us consider the two output matrices $Y_{k-\frac{L}{2}+1:k}$ and $Y^*_{(\gamma),\, 1:\frac{L}{2}}$ constructed using respectively outputs collected on-line in the current mode (to be identified) and collected off-line in mode γ.

Let us construct the input and output matrices as follows:

$$\begin{aligned} U_{(\gamma,\, \sigma_k),\, k-\frac{L}{2}+1:k} &= \left[\ U^*_{(\gamma),\, 1:\frac{L}{2}} | U_{k-\frac{L}{2}+1:k}\ \right], \\ Y_{(\gamma,\, \sigma_k),\, k-\frac{L}{2}+1:k} &= \left[\ Y^*_{(\gamma),\, 1:\frac{L}{2}} | Y_{k-\frac{L}{2}+1:k}\ \right]. \end{aligned} \tag{24}$$

The following theorem presents the residual generator for the current mode recognition.

Theorem 3. *If no change occurs in time interval* $[k - L/2 - i + 1, k]$ *and the inputs are not identically zero, then for all* $i \in \mathbb{N}$ *and* $i > i_s$ *(*i_s *given by Theorem 1) the residual* $\overline{\epsilon}_{(\gamma),\, k}$ *defined by*

$$\overline{\epsilon}_{(\gamma),\, k} = Y_{(\gamma,\, \sigma_k),\, k-\frac{L}{2}+1:k} \Pi_{U_{(\gamma,\, \sigma_k),\, k-\frac{L}{2}+1:k}} \tag{25}$$

has the following evaluation form:

$$\begin{aligned} \overline{\epsilon}_{(\gamma),\, k} = \delta_k^i &+ (H_{(\sigma_k),\, i} - H_{(\gamma),\, i}) \Big[0_{m(i+1)\times\frac{L}{2}} | \\ & U_{k-\frac{L}{2}+1:k} \Big] \Pi_{U_{(\gamma,\, \sigma_k),\, k-\frac{L}{2}+1:k}} \\ & + W_k \Pi_{U_{(\gamma,\, \sigma_k),\, k-\frac{L}{2}+1:k}}, \end{aligned} \tag{26}$$

where

$$\tilde{\delta}^i_{(\sigma_k),\ k-L+1-i:k-i} = C_{\sigma_k} A^i_{\sigma_k} X_k$$

and

$$\delta^i_k = \left[\tilde{\delta}^i_{(\gamma),\ 1:\frac{L}{2}} | \tilde{\delta}^i_{(\sigma_k),\ k-\frac{L}{2}+1-i:k-i}\right] \Pi_{U_{(\gamma,\ \sigma_k),\ k-\frac{L}{2}+1:k}}$$

If $\gamma = \sigma$ (σ_k is denoted by σ since it does not change on $[k - L/2 - i + 1, k]$), then

$$\overline{\epsilon}_{(\gamma),\ k} = \delta^i_k + W_k \Pi_{U_{(\gamma,\ \sigma_k),\ k-\frac{L}{2}+1:k}} \tag{27}$$

and $\forall \eta > 0$, $\exists i_0 \in \mathbb{N}$ such that $\forall i > i_0 (i \in \mathbb{N})$: $\left|\delta^i_k\right| < \eta$.

The proof of Theorem 3 is provided in Appendix. Using Theorem 3, one can recognize the active mode using Proposition 1.

4.2. Residual analysis. A condition for mode discernibility (internal faults identifiability) is given in the following theorem.

Theorem 4. *If no change occurs in time interval $[k - L/2 - i + 1, k]$ and the inputs are not identically null, then for all $i \in \mathbb{N}$ and $i > i_s$ (i_s given by Theorem 1), a necessary and sufficient condition for mode discernibility is*

$$\mathrm{span}\Big((H_{(\sigma_k),\ i} - H_{(\gamma),\ i}) \left[0_{m(i+1)\times \frac{L}{2}} | U_{k-\frac{L}{2}+1:k}\right]\Big) \not\subset \mathrm{span}(U_{(\gamma,\ \sigma_k),\ k-\frac{L}{2}+1:k}). \tag{28}$$

The proof of Theorem 4 is provided in Appendix.

5. DPM tuning

Consider the linear dynamic system described by

$$\begin{cases} x_{k+1} = Ax_k + Bu^*_k, \\ y^*_k = Cx_k + Du^*_k + w_k. \end{cases} \tag{29}$$

The time window size i in the DPM allows neglecting the past state influence. In order to find a good trade-off between low sensitivity to neglected terms and a reasonable complexity of on-line computation, this integer is determined using a criterion $J(p)$ which minimizes the approximation error between the reference model of the system and the implicit model used in the projection. The integer i has to be determined in a preliminary phase executed off-line in the healthy case using a persistently excited input.

Let us define

$$\overline{u}^*_{k,\ p} = \begin{pmatrix} u^{*,\,\mathrm{T}}_{k-p} & u^{*,\,\mathrm{T}}_{k-p+1} & \cdots & u^{*,\,\mathrm{T}}_{k} \end{pmatrix}^{\mathrm{T}} \in \mathbb{R}^{m(p+1)\times 1},$$

the input matrix

$$U^*_k = \begin{bmatrix} \overline{u}^*_{k-L+1,\ p} & \overline{u}^*_{k-L+2,\ p} & \cdots & \overline{u}^*_{k,\ p} \end{bmatrix} \in \mathbb{R}^{m(p+1)\times L},$$

the output matrix

$$Y^*_k = \begin{bmatrix} y^*_{k-L+1} & \cdots & y^*_{k-1} & y^*_k \end{bmatrix} \in \mathbb{R}^{\ell \times L}$$

and the state matrix

$$X_{k-L-i+1:k-i} = \begin{bmatrix} x_{k-i-L+1} & \cdots & x_{k-i-1} & x_{k-i} \end{bmatrix} \in \mathbb{R}^{n\times L}.$$

Theorem 5. *Let r, p and L be three integers such that $L > m(p + 1)$ and $J(p)$ is defined by (with $\|\cdot\|_2$, the 2-norm)*

$$J(p) = \frac{1}{r}\sum_{k=p+1}^{r} \left\|Y^*_k \Pi_{U^*_k}\right\|^2_2. \tag{30}$$

Let $\mathcal{X}_1$ be a positive real. There exists p_0 such that $\forall p > p_0$, and the criterion $J(p)$ defined by Eqn. (30) satisfies the following inequality:

$$J(p) \le \mathrm{var}(W_k) + \mathcal{X}_1. \tag{31}$$

Proof. From the residual expressions (9), the criterion $J(p)$ given by (30) can be written as

$$J(p) = \frac{1}{r}\sum_{k=p+1}^{r} \left\|W_k \Pi_{U^*_k} + CA^p X_{k-L-i+1:k-i} \Pi_{U^*_k}\right\|^2_2. \tag{32}$$

From the 2-norm properties and from a certain rank p_0, we have

$$J(p) \le \frac{1}{r}\sum_{k=p+1}^{r} \left\|W_k \Pi_{U^*_k}\right\|^2_2 + \frac{1}{r}\sum_{k=p+1}^{r} \left\|CA^p X_{k-L-p+1:k-p} \Pi_{U^*_k}\right\|^2_2, \tag{33}$$

where

$$\left\|W_k \Pi_{U^*_k}\right\|_2 \le \|W_k\|_2 \underbrace{\left\|\Pi_{U^*_k}\right\|_2}_{=1} \Rightarrow \left\|W_k \Pi_{U^*_k}\right\|_2 \le \|W_k\|_2. \tag{34}$$

The inequality (33) becomes

$$J(p) \le \frac{1}{r}\sum_{k=p+1}^{r} \|W_k\|^2_2 + \mathcal{X}, \tag{35}$$

where

$$\mathcal{X} = \frac{1}{r} \sum_{k=p+1}^{r} \|CA^p X_{k-L-i+1:k-i}\|_2^2.$$

Since

$$\mathrm{var}(W_k) = \lim_{r\to\infty} \frac{1}{r} \sum_{k=p+1}^{r} \|W_k\|_2^2,$$

under the stability hypothesis and using the same method as in the proof of Theorem 1, one can have (31).

Remark 1. The integer i should be chosen in the interval $[p_0,\ p_x[$ $(p_x \in \mathbb{N})$, where p_x is the maximum value with acceptable computational complexity and p_0 the minimum integer which makes the criterion acceptable. ■

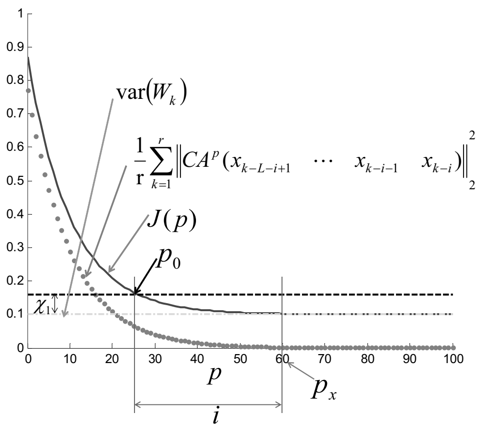

Fig. 1. Illustration of the criterion.

6. Illustrative examples

Consider a system with three operating modes ($\sigma_k \in \{1,\ 2,\ 3\}$), where Mode 1 is a normal operating mode and the other two modes represent faulty modes resulting from two internal faults (Fig. 2). Output measurements are affected by Gaussian white noise with zero mean and variance $\mathrm{var}(w_k) = 0.15$.

The switching sequence is given by Table 1.

Table 1. Simulated switching sequence.

$k \in$	$[0,\ 1500[$	$[1500,\ 2500[$	$[2500,\ 4000[$
mode number	1	3	2

The 3 modes are stable. Numerical values of the parameters in these modes are given below. These parameters are used to simulate the system output, but they are not used to compute the residuals.

Two examples are described below. In the first one, all modes are discernible, while in the second, Modes 2 and 3 are not discernible.

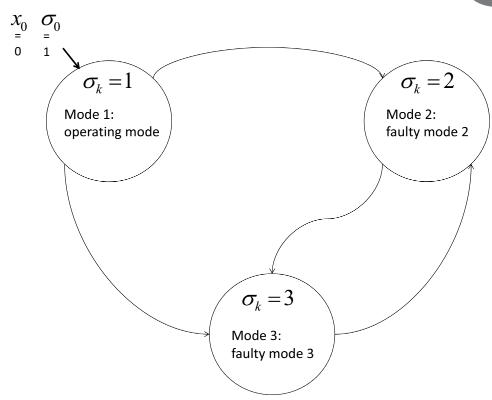

Fig. 2. Switching system.

6.1. Example 1 with all discernible modes.

The parameters of Mode 1 are given by

$$A_1 = \begin{bmatrix} -0.7 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \\ 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix}.$$

The parameters of Mode 2 are given by

$$A_2 = \begin{bmatrix} -0.5 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0.3 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0.1 & 0.1 \\ 1.2 & 1.2 \\ 0.5 & 0.5 \\ 0.8 & 0.8 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0.3 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0.3 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0.3 & 0.3 \\ 0.3 & 0.3 \\ 0.3 & 0.3 \\ 0.3 & 0.3 \end{bmatrix}.$$

The parameters of Mode 3 are given by

$$A_3 = \begin{bmatrix} -0.4 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0.6 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 0.3 & 0.3 \\ 0.1 & 0.1 \\ 0.7 & 0.7 \\ 0.9 & 0.9 \end{bmatrix},$$

$$C_3 = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix},$$

$$D_3 = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \\ 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}.$$

The three modes are discernible since they do not have the same Markov parameters.

DPM tuning. The first step of DMP is to tune parameters i and L.

The integers $i = 15$ and $L = 68$ are chosen to calculate the proposed residual. This choice is based on the proposed criterion detailed in Section 5.

Input and output generation. Figures 3 and 4 represent the system inputs and outputs, which are the only data used for residual computation to estimate the switching time and to recognize the active mode on-line.

Switching time estimation. As shown in Fig. 5, the switching times $\tau = 1500$ and $\tau = 2500$ are well estimated by the proposed residuals. All residual components are sensitive to all switching times.

Active mode recognition. The residual $\overline{\epsilon}_{(1),\,k}$ is calculated using input-output data collected on-line and off-line in Mode 1. The residual components allow Mode 1 recognition during interval $[0,\ 1500]$, as is shown in Fig. 6.

The residual $\overline{\epsilon}_{(2),\,k}$ is calculated using input-output data collected on-line and off-line in Mode 2. The residual components allow Mode 2 recognition during interval $[2501,\ 4000]$, as shown in Fig. 7.

The residual $\overline{\epsilon}_{(3),\,k}$ is calculated using input-output data collected on-line and off-line in Mode 3. The residual components allow Mode 3 recognition during interval $[1501,\ 2500]$, as shown in Fig. 8.

6.2. Example 2 with non-discernible modes. The parameters of Mode 1 are

$$A_1 = \begin{bmatrix} 1.0792 & 1.9072 & 0.9395 & 0.5389 \\ -0.1542 & -0.3322 & -0.2895 & -0.2139 \\ -0.3538 & -0.7776 & -0.3280 & -0.2391 \\ -0.6090 & -1.4554 & -0.8149 & -0.1190 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -0.1152 & -0.1152 \\ 0.1152 & 0.1152 \\ 0.0419 & 0.0419 \\ 0.1859 & 0.1859 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.1 & 0.3 & 0.2 & 0.1 \\ 0.5 & 0.7 & 0.2 & 0.1 \\ 0.3 & 0 & -0.2 & 0.5 \\ -0.2 & 0.4 & -0.9 & 0.1 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0.7 & 0.7 \\ 1.5 & 1.5 \\ 0.6 & 0.6 \\ -0.6 & -0.6 \end{bmatrix}.$$

The parameters of Mode 2 are

$$A_2 = \begin{bmatrix} 0.5669 & 0.9342 & 0.6730 & 0.3711 \\ -0.1919 & -0.2092 & -0.3230 & -0.1961 \\ -0.1630 & -0.5192 & -0.0584 & -0.1918 \\ -0.1653 & -0.7682 & -0.5872 & 0.1007 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0.1054 & 0.1054 \\ 0.0321 & 0.0321 \\ -0.0577 & -0.0577 \\ -0.0363 & -0.0363 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0.3 & 0.9 & 0.6 & 0.3 \\ 1.5 & 2.1 & 0.6 & 0.3 \\ 0.9 & 0 & -0.6 & 1.5 \\ -0.6 & 1.2 & -2.7 & 0.3 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0.7 & 0.7 \\ 1.5 & 1.5 \\ 0.6 & 0.6 \\ -0.6 & -0.6 \end{bmatrix}.$$

The parameters of Mode 3 are

$$A_3 = \begin{bmatrix} -3.1682 & -5.0342 & 1.2697 & -2.7353 \\ 1.0859 & 1.8972 & -0.4467 & 0.8729 \\ 2.0977 & 2.9268 & -0.3946 & 1.6441 \\ 2.2298 & 3.2131 & -0.8241 & 2.0657 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} -0.0418 & -0.0418 \\ 0.0234 & 0.0234 \\ 0.0264 & 0.0264 \\ 0.0241 & 0.0241 \end{bmatrix},$$

$$C_3 = \begin{bmatrix} 6 & 8.4 & -1.5 & 4.5 \\ 13.2 & 20.4 & 3.3 & 6.9 \\ -3.9 & 8.7 & -10.5 & -0.6 \\ -3.3 & 7.8 & 3.9 & -12.6 \end{bmatrix},$$

$$D_3 = \begin{bmatrix} 0.7 & 0.7 \\ 1.5 & 1.5 \\ 0.6 & 0.6 \\ -0.6 & -0.6 \end{bmatrix}.$$

The matrix Ψ such that

$$\begin{cases} A_3 = \Psi^{-1} A_2 \Psi, \\ B_3 = \Psi^{-1} B_2, \\ C_3 = C_2 \Psi, \\ D_3 = D_2 \end{cases}$$

is

$$\Psi = \begin{bmatrix} 1 & 3 & 2 & 1 \\ 5 & 7 & 2 & 1 \\ 3 & 0 & -2 & 5 \\ -2 & 4 & -9 & 1 \end{bmatrix}.$$

Modes 2 and 3 are not discernible (i.e., they have the same Markov parameters $H_2 = H_3$) and $\exists r \in \{1, 2, \ldots, i\} : \mathcal{H}_{(2,3),i,r} \neq H_{(2),i}$.

DPM tuning. The two integers $i = 20$ and $L = 88$ are chosen for the three modes by calculating the criterion given by (30).

Input and output generation. Figures 9 and 10 represent the system inputs and outputs, which are the only data used for residual computation to estimate switching times and to recognize the active mode on-line.

Estimation of switching times.

- Switching between discernible modes: A switching at time instant $\tau = 1500$ between Modes 1 and 3 is well detected as shown in Fig. 11.

- Switching between non discernible modes: A switching at time instant $\tau = 2500$ between Modes 3 and 2 is also well detected despite the fact that these modes are not discernible, $H_2 = H_3$ this means that $\exists r \in \{1, 2, \ldots, i\} : \mathcal{H}_{(2,3),i,r} \neq H_{(2),i}$.

The switching times $\tau = 1500$ and $\tau = 2500$ have been well estimated by the proposed residual, as shown in Fig. 11. All residuals are sensitive to these switchings.

Active mode recognition. The bank of residuals $\overline{\epsilon}_{(1),k}$, $\overline{\epsilon}_{(2),k}$ and $\overline{\epsilon}_{(3),k}$ (cf. Figs. 12–14) shows the following:

- From Fig. 12: Mode 1 is active during the time interval $[0, 1500]$.

- From Figs. 13 and 14: Modes 2 and 3 are not discernible and one of them is active. The exact active mode cannot be determined, but one can just conclude that Mode 2 or 3 is active during the time interval $[1501, 2500]$. This non-discernibility is due to the fact that the condition of discernibility of Theorem 4 is not satisfied.

- Since the switching occurs during these two modes' activity at time instant $\tau = 2500$: if Mode 2 (resp. 3) is active during the time interval $[1501, 2500]$, then Mode 3 (resp. 2) is active during the time interval $[2501, 4000]$.

The switching at time instant $\tau = 2500$ occurs between Modes 2 and 3, which are not discernible. Despite the fact that these modes are not discernable the switching at time instant $\tau = 2500$ is well detected in Fig. 11 and also in Figs. 13 and 14. Indeed, the detectability condition given by the third equation of Theorem 2 is satisfied.

7. Conclusion

A data projection method (DPM) was proposed in this paper to estimate the switching time and recognize the active mode in a switching system. This method can be used to detect and identify internal faults. The diagnosis problem may be viewed as that of estimating the switching time and recognizing the faulty mode. Two conditions, namely, those of discernibility and detectability, are established. Under the discernibility condition, the active mode can be well recognized, and under the detectability condition, the switching time can be well estimated.

The main advantage of this method, compared with others described in the literature (e.g., Akhenak *et al.*, 2008; Domlan *et al.*, 2007a; Cocquempot *et al.*, 2004; Bayoudh and Travé Massuyès, 2014), is that the DPM does not need the parameter values of the model. The residuals are generated by projecting the input-output data in a way that depends on the model structure, which is supposed to linear in the paper. As a consequence, the DPM can be directly implemented on systems of the same type (the same model structure) without identifying, for each system, the parameters. This is of great interest in practice. The main drawback of the method is what can happen if the eigenvalue of the state matrix for one mode is close to 1. One can have in this case a large size of input, output and projection matrices, and the time complexity of the algorithm will increase. This problem will be considered in our future works.

References

Akhenak, A., Bako, L., Duviella, E., Pekpe, K.M. and Lecoeuche, S. (2008). Fault diagnosis for switching system

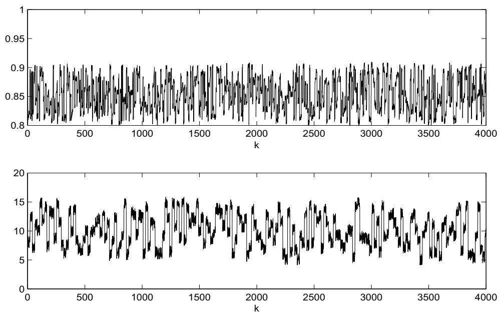

Fig. 3. System inputs u_k of Example 1.

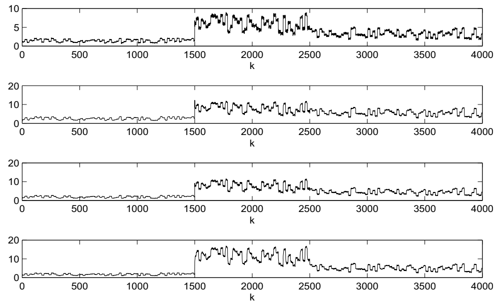

Fig. 4. System outputs y_k of Example 1.

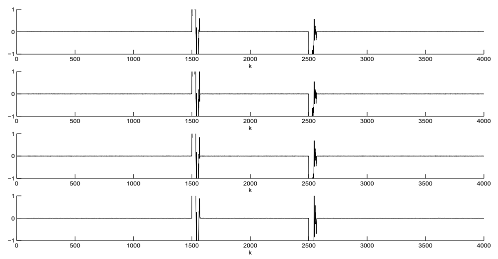

Fig. 5. Switching times estimation ϵ_k of Example 1.

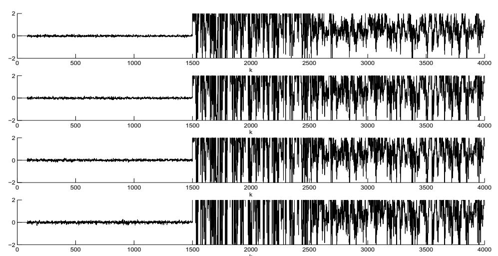

Fig. 6. First mode recognition $\overline{\epsilon}_{(1),\ k}$ of Example 1.

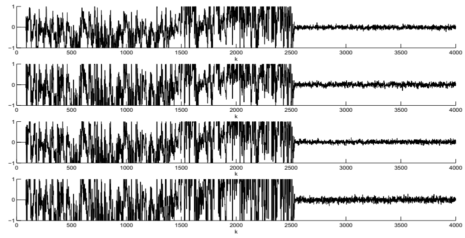

Fig. 7. Second mode recognition $\overline{\epsilon}_{(2),\ k}$ of Example 1.

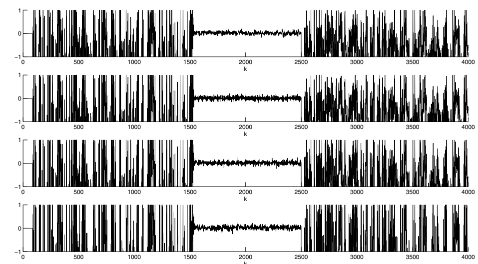

Fig. 8. Third mode recognition $\overline{\epsilon}_{(3),\ k}$ of Example 1.

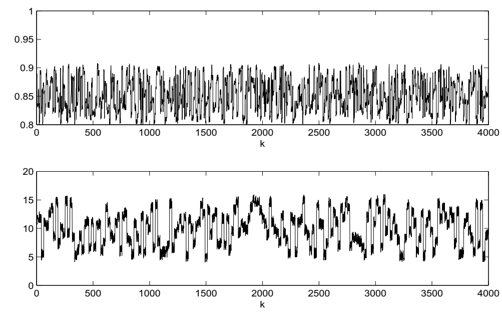

Fig. 9. System inputs u_k of Example 2.

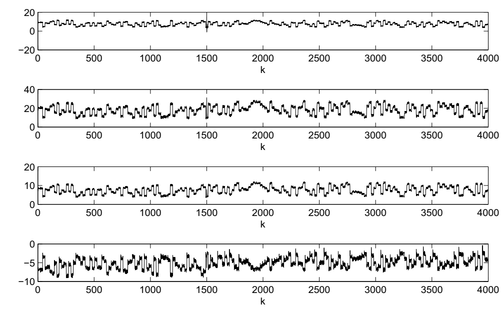

Fig. 10. System outputs y_k of Example 2.

using observer Kalman filter identification, *Proceedings of the 17th IFAC World Congress, Seoul, Korea*, pp. 10142–10147.

Anderson, B.D.O., Brinsmead, T., Liberzon, D. and Morse, A.S. (2001). Multiple model adaptive control with safe switching, *International Journal of Adaptive Control and Signal Processing* **15**(5): 445–470, DOI:10.1002/acs.684.

Antsaklis, P.J. (2000). A brief introduction to the theory and applications of hybrid systems, *Proceedings of the IEEE* **88**(7): 887–897.

Bayoudh, M. and Travé Massuyès, L. (2014). Diagnosability analysis of hybrid systems cast in a discrete-event framework, *Discrete Event Dynamic Systems* **24**(3): 309–338.

Belkhiat, D.E.C. (2011). *Diagnosis of a Class of Switching Linear Systems: Robust Observer Based Approach*, Ph.D. thesis, University of Reims Champagne Ardenne, Reims.

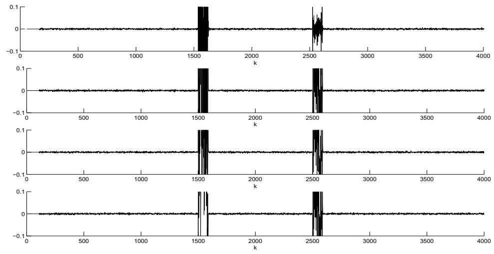

Fig. 11. Switching time estimation ϵ_k of Example 2.

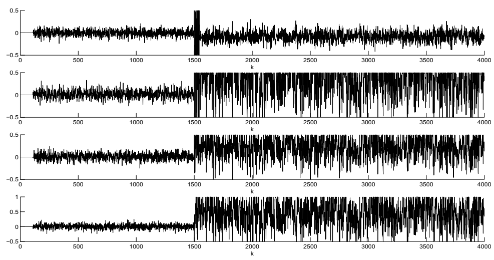

Fig. 12. First mode recognition $\overline{\epsilon}_{(1),\ k}$ of Example 2.

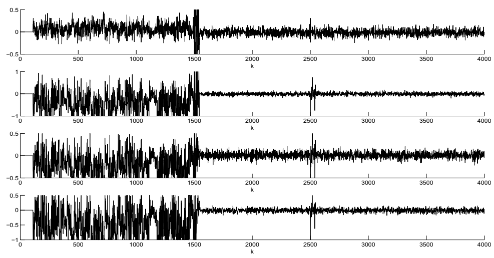

Fig. 13. Second mode recognition $\overline{\epsilon}_{(2),\ k}$ of Example 2.

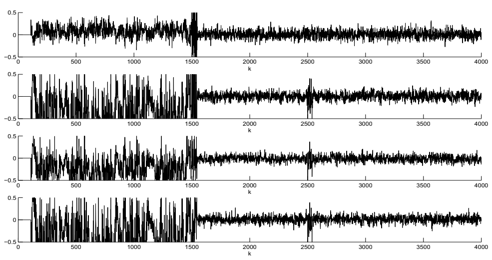

Fig. 14. Third mode recognition $\overline{\epsilon}_{(3),\ k}$ of Example 2.

Cocquempot, V., El Mezyani, T. and Staroswiecki, M. (2004). Fault detection and isolation for hybrid systems using structured parity residuals, *IEEE/IFAC-ASCC, 5th Asian Control Conference, Melbourne, Victoria, Australia*, pp. 1204–1212, DOI:10.1109/ASCC.2004.18502.

Daizhan, C. (2007). Controlability of switched systems, *IFAC Proceedings Volumes* **40**(12): 194–201.

Domlan, E.A., Ragot, J. and Maquin, D. (2007a). Active mode estimation for switching systems, *American Control Conference, New York City, NY, USA*, pp. 1143–1148.

Domlan, E.A., Ragot, J. and Maquin, D. (2007b). Switching systems: Active mode recognition, identification of the switching law, *Journal of Control Science and Engineering* **2007**: 1–11, Article ID: 50796, DOI:10.1155/2007/50796.

El Mezyani, T. (2005). *Methodology for Fault Detection and Isolation in Hybrid Dynamic Systems*, Ph.D. thesis, Université Lille1, Lille, France.

Engell, S., Kowalewski, S., Schulz, C. and Strusberg, O. (2000). Continuous discrete interactions in chemical processing plants, *Proceedings of the IEEE* **88**(7): 1050–1068.

Goebel, R., Ricardo Sanfelice, G. and Andrew Teel, R. (2012). *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*, Princeton University Press, Princeton, NJ.

Heemels, W.P.M.H., De Schutter, B. and Bemporad, A. (2001). Equivalence of hybrid dynamical models, *Automatica* **37**(7), 1085–1091, DOI: 10.1016/S0005-1098(01)00059-0.

Hespanha, J.P. and Morse, A.S. (1999). Stability of switched systems with average dwell-time, *38th IEEE Conference on Decision and Control, Phoenix, AZ, USA,* Vol. 3, pp. 2655–2660.

Hofbaur, M., Travé-Massuyès, L., Rienmüller, T. and Bayoudh, M. (2010). Overcoming non-discernibility through mode-sequence analytic redundancy relations in hybrid diagnosis and estimation, *21st International Workshop on Principles of Diagnosis DX-10, Portland, OR, USA*, pp. 1–7.

Kailath, T. (1980). *Linear Systems*, Englewood Cliffs, NJ.

Liberzon, D. (2005). *Switched Systems*, Birkhauser, Boston, MA.

Lin, H. and Antsaklis, J.P. (2009). Stability and stabilizability of switched linear systems: A survey of recent results, *IEEE Transactions on Automatic Control* **54**(2): 308–322.

Livadas, C., Lygeros, J. and Lynch, N.A. (2000). High-level modeling and analysis of the traffic alert and collision avoidance system (TCAS), *Proceedings of the IEEE*, **88**(7): 926–948.

Ma, Y., Kawakami, H. and Tse, C.K. (2004). Bifurcation analysis of switched dynamical systems with periodically moving borders, *IEEE Transactions on Circuits and Systems* **51**(6): 1184–1193.

Mitsubori, K. and Saito, T. (1997). Dependent switched capacitor chaos generator and its synchronization, *IEEE Transactions on Circuits and Systems* **44**(12): 1122–1128.

Narasimhan, S. and Biswas, G. (2007). Model-based diagnosis of hybrid systems, *IEEE Transaction on Systems, Man, and Cybernetics A: Systems and Humans* **37**(3): 348–361, DOI:10.1109/TSMCA.2007.893487.

Pekpe, K.M., Mourot, G. and Ragot, J. (2006). Subspace method for sensor fault detection and isolation–application to grinding circuit monitoring, *11th IFAC Symposium on automation in Mining, Mineral and Metal Processing, Nancy, France*, pp. 47–52.

Petroff, B.N. (2007). *Biomimetic Sensing for Robotic Manipulation*, Ph.D. thesis, Graduate School of the University of Notre Dame, Notre Dame, IN.

Torikai, H. and Saito, T. (1998), Synchronization of chaos and its itinerancy from a network by occasional linear connection, *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications* **45**(4): 464–472.

Van Overschee, P. and De Moor, B. (1996), *Subspace Identification for Linear Systems Theory: Implementation and Applications*, Kluwer Academic Publishers, Boston, MA.

Williams, S.M. and Hoft, R.G. (1991), Adaptive frequency domain control of ppm switched power line conditioner, *IEEE Transactions on Power Electronics* **6**(4): 665–670.

Yang, H., Jiang, B. and Cocquempot, V. (2010), Fault tolerant control and hybrid systems, *in* H. Yang *et al.* (Eds.), *Fault Tolerant Control Design for Hybrid Systems*, Springer Verlag, Berlin/Heidelberg.

Zhang, W., Hu, J. and Lu, Y.H. (2007), Optimal power modes scheduling using hybrid systems, *Proceedings of the American Control Conference, New York City, NY, USA*, pp. 2781–2786.

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Appendix

A1. Theorems proofs

A1.1. Proof of Theorem 2. The first part of the proof determines the evaluation form of the proposed residual when a switching occurs in the time window considered ($\tau \in [k-L-i+1,\ k]$).

The first step is to determine the output expression in the time window $[k-L-i+1,\ k]$ that contains the switching time τ. The output y_t is expressed differently, depending on whether $t < \tau$, $\tau \le t \le \tau+i-1$ and $t > \tau+i-1$, as detailed in Fig. A1.

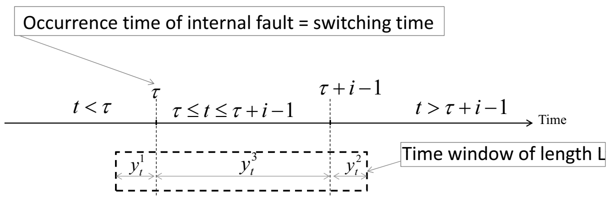

Fig. A1. Computation time window decomposition in three parts.

1. Output expression $y_t = y_t^1$ for $t < \tau$:

$$\begin{aligned} y_t^1 &= C_{\sigma_\tau} A_{\sigma_\tau}^i x_{t-i} \\ &\quad + \sum_{j=0}^{i-1} C_{\sigma_\tau} A_{\sigma_\tau}^j B_{\sigma_\tau} u_{t-1-j} \\ &\quad + D_{\sigma_\tau} u_t + w_t \\ &= C_{\sigma_\tau} A_{\sigma_\tau}^i x_{t-i} + H_{(\sigma_\tau),\, i} \overline{u}_{t,\, i} + w_t. \end{aligned} \tag{A1}$$

2. Output expression $y_t = y_t^2$ for $t > \tau+i-1$:

$$\begin{aligned} y_t^2 &= C_{\sigma_{\tau+1}} A_{\sigma_{\tau+1}}^i x_{t-i} \\ &\quad + \sum_{j=0}^{i-1} C_{\sigma_{\tau+1}} A_{\sigma_{\tau+1}}^j B_{\sigma_{\tau+1}} u_{t-1-j} \\ &\quad + D_{\sigma_{\tau+1}} u_t + w_t \\ &= C_{\sigma_{\tau+1}} A_{\sigma_{\tau+1}}^i x_{t-i} + H_{(\sigma_{\tau+1}),\, i} \overline{u}_{t,\, i} + w_t. \end{aligned} \tag{A2}$$

3. Output expression $y_t = y_t^3$ for $\tau \le t \le \tau+i-1$:

$$\begin{aligned} y_t^3 &= C_{\sigma_{\tau+1}} A_{\sigma_{\tau+1}}^{t-\tau+1} A_{\sigma_\tau}^{i-t+\tau-1} x_{t-i} \\ &\quad + \sum_{j=0}^{i-t+\tau-2} C_{\sigma_{\tau+1}} A_{\sigma_{\tau+1}}^{t-\tau+1} A_{\sigma_\tau}^j B_{\sigma_\tau} u_{\tau-2-j} \\ &\quad + \sum_{j=0}^{t-\tau} C_{\sigma_{\tau+1}} A_{\sigma_{\tau+1}}^j B_{\sigma_{\tau+1}} u_{t-j-1} \\ &\quad + D_{\sigma_{\tau+1}} u_t + w_t \\ &= C_{\sigma_{\tau+1}} A_{\sigma_{\tau+1}}^{t-\tau+1} A_{\sigma_\tau}^{i-t+\tau-1} x_{t-i} \\ &\quad + \mathcal{H}_{(\sigma_\tau,\, \sigma_{\tau+1}),\, i,\, t-\tau+1} \overline{u}_{t,\, i} + w_t. \end{aligned} \tag{A3}$$

Let us construct now the matrix $Y_k = Y_{k-L+1:k}$. For the three expressions for y_t, we have

$$Y_k = \begin{bmatrix} y^1_{k-L+1} & \cdots & y^1_{\tau-2} & y^1_{\tau-1} & y^3_\tau & \cdots \\ y^3_{\tau+i-2} & y^3_{\tau+i-1} & y^2_{\tau+i} & \cdots & y^2_{k-1} & y^2_k \end{bmatrix},$$

which gives

$$\begin{aligned} &Y_{k-L+1:k} \\ &= \Big[\tilde{\delta}^i_{(\sigma_\tau),\, k-L+1-i:\tau-1-i} | \\ &\quad \tilde{\delta}^i_{(\sigma_\tau,\, \sigma_{\tau+1}),\, \tau-i:\tau-1} | \tilde{\delta}^i_{(\sigma_{\tau+1}),\, \tau:k-i}\Big] \\ &\quad + H_{(\sigma_\tau),\, i}\left[U_{k-L+1:\tau-1} | 0_{m(i+1)\times k-\tau+1}\right] \\ &\quad + \big[\mathcal{H}_{(\sigma_\tau,\, \sigma_{\tau+1}),\, i,\, 1} | \mathcal{H}_{(\sigma_\tau,\, \sigma_{\tau+1}),\, i,\, 2} | \cdots | \\ &\quad \mathcal{H}_{(\sigma_\tau,\, \sigma_{\tau+1}),\, i,\, i-1} | \\ &\quad \mathcal{H}_{(\sigma_\tau,\, \sigma_{\tau+1}),\, i,\, i}\big] \left[0_{mi(i+1)\times \tau+L-k-1} | \right. \\ &\quad \left. \breve{U} | 0_{mi(i+1)\times k-\tau-i+1}\right] \\ &\quad + H_{(\sigma_{\tau+1}),\, i}\left[0_{m(i+1)\times L-k+\tau+i-1} | U_{\tau+i:k}\right] + W_k \end{aligned} \tag{A4}$$

where

$$\begin{aligned} &\tilde{\delta}^i_{(\sigma_\tau,\, \sigma_{\tau+1}),\, \tau-i:\tau-1} \\ &\quad = \big[C_{\sigma_{\tau+1}} A_{\sigma_{\tau+1}} A^{i-1}_{\sigma_\tau} x_{\tau-i} | \\ &\qquad C_{\sigma_{\tau+1}} A^2_{\sigma_{\tau+1}} A^{i-2}_{\sigma_\tau} x_{\tau-i+1} | \\ &\qquad \cdots | C_{\sigma_{\tau+1}} A^{i-1}_{\sigma_{\tau+1}} A_{\sigma_\tau} x_{\tau-2} |\ C_{\sigma_{\tau+1}} A^i_{\sigma_{\tau+1}} x_{\tau-1}\big] \end{aligned}$$

and $\breve{U} \in \mathbb{R}^{mi(i+1)\times i}$ is defined as follows:

$$\breve{U} = \begin{bmatrix} \overline{u}_{\tau,\, i} & \mathbb{O} & \cdots & \cdots & \mathbb{O} \\ \mathbb{O} & \overline{u}_{\tau+1,\, i} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \mathbb{O} \\ \mathbb{O} & \cdots & \cdots & \mathbb{O} & \overline{u}_{\tau+i-1,\, i} \end{bmatrix},$$

with $\mathbb{O} = 0_{m(i+1)\times 1} \in \mathbb{R}^{m(i+1)\times 1}$ being a zero column vector.

In order to make the matrix $U_{k-L+1:k}$ appear, we add and we subtract the term

$$H_{(\sigma_\tau),\, i}\left[\ 0_{m(i+1)\times \tau-k+L-1} | U_{\tau:k}\ \right]$$

from Eqn. (A4). Thereafter, the output matrix $Y_{k-L+1:k}$ is given by

$$\begin{aligned} &Y_{k-L+1:k} \\ &= \Big[\tilde{\delta}^i_{(\sigma_\tau),\, k-L+1-i:\tau-1-i} | \tilde{\delta}^i_{(\sigma_\tau,\, \sigma_{\tau+1}),\, \tau-i:\tau-1} | \\ &\quad \tilde{\delta}^i_{(\sigma_{\tau+1}),\, \tau:k-i}\Big] + H_{(\sigma_\tau),\, i} U_{k-L+1:k} \\ &\quad + \Omega + W_k, \end{aligned} \tag{A5}$$

Post-multiplying Eqn. (A5) by $\Pi_{U_{k-L+1:k}}$, we get the residual evaluation form

$$\epsilon_k = \Omega \Pi_{U_{k-L+1:k}} + W_k \Pi_{U_{k-L+1:k}} + \delta^i_k, \tag{A6}$$

where

$$\begin{aligned} \delta^i_k = &\Big[\tilde{\delta}^i_{(\sigma_\tau),\, k-L+1-i:\tau-1-i} | \tilde{\delta}^i_{(\sigma_\tau,\, \sigma_{\tau+1}),\, \tau-i:\tau-1} | \\ &\tilde{\delta}^i_{(\sigma_{\tau+1}),\, \tau:k-i}\Big] \Pi_{U_{k-L+1:k}} \in \mathbb{R}^{\ell\times 1}. \end{aligned}$$

The initial state contribution δ^i_k can be neglected for the following reasons (Kailath, 1980):

1. For $t < \tau$ and $t > \tau + i - 1$:
Under stability hypothesis of A_{σ_τ} and $A_{\sigma_{\tau+1}}$, the initial state contribution can be neglected for i sufficiently large, i.e., $\lim_{i\to\infty} C_{\sigma_\tau} A^i_{\sigma_\tau} x_{t-i} = 0$ and $\lim_{i\to\infty} C_{\sigma_{\tau+1}} A^i_{\sigma_{\tau+1}} x_{t-i} = 0$.
2. For $\tau \le t \le \tau + i - 1$):
The state is multiplied by a term of a general form $A^{t-\tau+1}_{\sigma_{\tau+1}} A^{i-t+\tau-1}_{\sigma_\tau}$ as shown in Eqn. (A3), where the sum of powers is always equal to i. Let $\|\cdot\|$ represent a multiplicative norm. Then we have
$$\left\| A^{t-\tau+1}_{\sigma_{\tau+1}} A^{i-t+\tau-1}_{\sigma_\tau} \right\| < \left\| A^{t-\tau+1}_{\sigma_{\tau+1}} \right\| \left\| A^{i-t+\tau-1}_{\sigma_\tau} \right\|,$$
and we also have
$$\left\| A^{t-\tau+1}_{\sigma_{\tau+1}} A^{i-t+\tau-1}_{\sigma_\tau} \right\| < (\max(\left\| A_{\sigma_{\tau+1}} \right\|,\ \|A_{\sigma_\tau}\|))^i.$$
For $i \to \infty$, the term $(\max(\left\| A_{\sigma_{\tau+1}} \right\|,\ \|A_{\sigma_\tau}\|))^i$ is negligible. Consequently,
$$\lim_{i\to\infty} C_{\sigma_{\tau+1}} A^{t-\tau+1}_{\sigma_{\tau+1}} A^{i-t+\tau-1}_{\sigma_\tau} x_{t-i} = 0.$$

Therefore, $\forall \eta > 0$, $\exists i_0 \in \mathbb{N}$ such that $\forall i > i_0 (i \in \mathbb{N})$ and $\left\|\delta^i_k\right\| < \eta$.

- *Sufficient condition:*
If $\mathrm{span}(\Omega) \not\subset \mathrm{span}(U_{k-L+1:k})$, then $\forall \mathcal{X}$ and we have $\Omega \neq \mathcal{X} U_{k-L+1:k}$. Consequently, the residual mean and variance change.
- *Necessary condition:*
If the residual mean and variance change, then $\Omega \Pi_{U_{k-L+1:k}} \neq 0$, which implies that $\mathrm{span}(\Omega) \not\subset \mathrm{span}(U_{k-L+1:k})$.

A1.2. Proof of Theorem 3. If a mode σ_k is active in a time window $[k-L-i+1,\ k]$, then the output matrix $Y_{k-L+1:k}$ is given by

$$Y_{k-L+1:k} = \tilde{\delta}^i_{(\sigma_k),\ k-L+1-i:k-i} + H_{(\sigma_k),\ i} U_{k-L+1:k} + W_k. \tag{A7}$$

By replacing the first $L/2$ columns of the matrix $U_{k-L+1:k}$ (resp. $Y_{k-L+1:k}$) by the input matrix $U^*_{(\gamma),\ 1:\frac{L}{2}}$ (resp. the output matrix $Y^*_{(\gamma),\ 1:\frac{L}{2}}$) constructed with input-output data collected off-line from mode γ ($\gamma \in \{1,\ 2,\ \ldots,\ d\}$), the resulting input and output matrices are given (24). From Eqn. (A7), the general expression of $Y_{(\gamma,\ \sigma_k),\ k-\frac{L}{2}+1:k}$ becomes

$$\begin{aligned} Y_{(\gamma,\ \sigma_k),\ k-\frac{L}{2}+1:k} &= \left[\tilde{\delta}^i_{(\gamma),\ 1:\frac{L}{2}} | \tilde{\delta}^i_{(\sigma_k),\ k-\frac{L}{2}+1-i:k-i}\right] \\ &+ H_{(\gamma),\ i} \left[\ U^*_{(\gamma),\ 1:\frac{L}{2}} | 0_{m(i+1)\times \frac{L}{2}}\ \right] \\ &+ H_{(\sigma_k),\ i} \left[\ 0_{m(i+1)\times\frac{L}{2}} | U_{k-\frac{L}{2}+1:k}\ \right] + W_k, \end{aligned} \tag{A8}$$

In order to make the matrix $U_{(\gamma,\ \sigma_k),\ k-\frac{L}{2}+1:k}$ appear, we add and subtract the term $H_{(\gamma),\ i} \left[\ 0_{m(i+1)\times\frac{L}{2}} | U_{k-\frac{L}{2}+1:k}\ \right]$ from Eqn. (A8):

$$\begin{aligned} Y_{(\gamma,\ \sigma_k),\ k-\frac{L}{2}+1:k} &= \left[\tilde{\delta}^i_{(\gamma),\ 1:\frac{L}{2}} | \tilde{\delta}^i_{(\sigma_k),\ k-\frac{L}{2}+1-i:k-i}\right] \\ &+ H_{(\gamma),\ i} U_{(\gamma,\ \sigma_k),\ k-\frac{L}{2}+1:k} \\ &+ (H_{(\sigma_k),\ i} - H_{(\gamma),\ i}) \left[\ 0_{m(i+1)\times\frac{L}{2}} | U_{k-\frac{L}{2}+1:k}\ \right] \\ &+ W_k. \end{aligned} \tag{A9}$$

Post-multiplying both the sides of Eqn. (A9) by $\Pi_{U_{(\gamma,\ \sigma_k),\ k-\frac{L}{2}+1:k}}$, the evaluation form of the proposed residual yields

$$\overline{\epsilon}_{(\gamma),\ k} = (H_{(\sigma_k),\ i} - H_{(\gamma),\ i}) \left[0_{m(i+1)\times\frac{L}{2}} | U_{k-\frac{L}{2}+1:k}\right] \Pi_{U_{(\gamma,\ \sigma_k),\ k-\frac{L}{2}+1:k}} + \delta^i_k \tag{A10}$$

with

$$\delta^i_k = \left[\tilde{\delta}^i_{(\gamma),\ 1:\frac{L}{2}} | \tilde{\delta}^i_{(\sigma_k),\ k-\frac{L}{2}+1-i:k-i}\right] \Pi_{U_{(\gamma,\ \sigma_k),\ k-\frac{L}{2}+1:k}} \in \mathbb{R}^{\ell\times 1}.$$

on the assumption that the modes are stable, which is equivalent to stating that eigenvalues of matrices A_γ and A_{σ_k} are inside the unit circle. Then we have (Kailath, 1980)

$$\lim_{i\to\infty} A^i_\gamma = 0, \quad \lim_{i\to\infty} A^i_{\sigma_k} = 0. \tag{A11}$$

As a consequence, δ^i_k in Eqn. (A10) becomes negligible for i sufficiently large.

By neglecting the initial state contribution, where the approximation term is δ^i_k and $\forall\eta > 0$, $\exists i_0 \in \mathbb{N}$ such that $\forall i > i_0 (i \in \mathbb{N})$: $\left|\delta^i_k\right| < \eta$.

The evaluation form (A10) of the residual can be approximated by

$$\overline{\epsilon}_{(\gamma),\ k} = (H_{(\sigma_k),\ i} - H_{(\gamma),\ i}) \left[0_{m(i+1)\times\frac{L}{2}} | U_{k-\frac{L}{2}+1:k}\right] \Pi_{U_{(\gamma,\ \sigma_k),\ k-\frac{L}{2}+1:k}}. \tag{A12}$$

If the active mode is $\sigma_k = \gamma$, then $H_{(\sigma_k),\ i} = H_{(\gamma),\ i}$ and we have (27).

A1.3. Proof of Theorem 4.

- *Sufficient condition:*
  If

$$\mathrm{span}((H_{(\sigma_k),\ i} - H_{(\gamma),\ i}) \left[0_{m(i+1)\times\frac{L}{2}} | U_{k-\frac{L}{2}+1:k}\right]) \not\subset \mathrm{span}(U_{(\gamma,\ \sigma_k),\ k-\frac{L}{2}+1:k}),$$

  then for any $\mathcal{X}$ we have

$$(H_{(\sigma_k),\ i} - H_{(\gamma),\ i}) \left[\ 0_{m(i+1)\times\frac{L}{2}} | U_{k-\frac{L}{2}+1:k}\ \right] \neq \mathcal{X} U_{(\gamma,\ \sigma_k),\ k-\frac{L}{2}+1:k}$$

  and, according to Theorem 3, the residual is not zero mean Gaussian noise while the mode is discernible.

- *Necessary condition:*
  If the residual is not zero mean Gaussian noise, then, according to Theorem 3,

$$(H_{(\sigma_k),\ i} - H_{(\gamma),\ i}) \left[0_{m(i+1)\times\frac{L}{2}} | U_{k-\frac{L}{2}+1:k}\right] \Pi_{U_{(\gamma,\ \sigma_k),\ k-\frac{L}{2}+1:k}} \neq 0,$$

  which implies that

$$\mathrm{span}((H_{(\sigma_k),\ i} - H_{(\gamma),\ i}) \left[0_{m(i+1)\times\frac{L}{2}} | U_{k-\frac{L}{2}+1:k}\right]) \not\subset \mathrm{span}(U_{(\gamma,\ \sigma_k),\ k-\frac{L}{2}+1:k}).$$

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