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REDUCED–ORDER FRACTIONAL DESCRIPTOR OBSERVERS FOR A CLASS OF FRACTIONAL DESCRIPTOR CONTINUOUS–TIME NONLINEAR SYSTEMS

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Fractional descriptor reduced-order nonlinear observers for a class of fractional descriptor continuous-time nonlinear systems are proposed. Sufficient conditions for the existence of the observers are established. The design procedure for the observers is given and demonstrated on a numerical example.

Keywords: fractional system, descriptor system, nonlinear systems, reduced-order observer.

1. Introduction

Fractional linear systems were considered in many papers and books (Kaczorek, 2012b; 2013; 2011a; 2011b; Oldham and Spanier, 1974; Ostalczyk, 2008; Podlubny, 1999; Vinagre *et al.*, 2002). Positive linear systems consisting of n subsystems with different fractional orders were proposed by Kaczorek (2011a; 2011b). Descriptor (singular) linear systems were investigated by Dodig and Stosic (2009), Cuihong (2012), Dai (1989), Fahmy and O'Reill (1989), Gantmacher (1960), Guang-ren (2010), Kaczorek (2012a; 2012b; 1992), Kucera and Zagalak (1988), Lewis (1983), Luenberger (1978; 1977), Podlubny (1999) and Van Dooren (1979). Eigenvalue and invariant assignments by state and input feedbacks were addressed by Fahmy and O'Reill (1989), as well as Kaczorek (2004; 2015). The computation of Kronecker's canonical form of a singular pencil was analyzed by Van Dooren (1979).

A new concept of perfect observers for linear continuous-time systems was proposed by Kaczorek (2001; 2015). Observers for fractional linear systems were addressed by Kaczorek (2008), Kociszewski (2013) and Vinagre *et al.* (2002), and for descriptor linear systems by Kaczorek (2014a). Fractional descriptor full-order observers for fractional descriptor continuous-time linear systems were proposed by Kaczorek (2001), along with reduced-order observers (Kaczorek, 2014b). The stability of positive descriptor systems was investigated by Virnik (2008).

In this paper, fractional descriptor reduced-order observers for a class of fractional descriptor continuous-time nonlinear systems will be proposed and sufficient conditions for the existence of the observer will be established.

The paper is organized as follows. In Section 2, the basic definitions and theorems of fractional descriptor nonlinear continuous-time systems are recalled and their full-order fractional descriptor observers are presented. In Section 3, reduced-order fractional descriptor nonlinear observers are proposed and sufficient conditions for the existence for observers are established. A design procedure of the reduced-order observers and an illustrating example are given in Section 4. Concluding remarks are presented in Section 5.

2. Fractional descriptor systems and their full-order observers

Consider the fractional descriptor continuous-time linear system

$$E\frac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha} = Ax + Bu, \quad x_0 = x(0), \tag{1a}$$

$$y = Cx, \tag{1b}$$

where $\mathrm{d}^\alpha x/\mathrm{d}t^\alpha$ is the fractional α-order derivative defined by Caputo,

$$\begin{aligned} {}_0D_t^\alpha x(t) &= \frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{\frac{\mathrm{d}^n x}{\mathrm{d}\tau^n}}{(t-\tau)^{\alpha-n+1}}\,\mathrm{d}\tau, \\ & n-1<\alpha<n\in\mathbb{N}, \quad (2) \end{aligned}$$

and

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,\mathrm{d}t$$

is the gamma function, while $x = x(t) \in \mathbb{R}^n$, $u = u(t) \in \mathbb{R}^m$, $y = y(t) \in \mathbb{R}^p$ are the state, input and output vectors, $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$. It is assumed that $\det E = 0$ and

$$\det[E\lambda - A] \neq 0 \tag{3}$$

for some $\lambda \in \mathbb{C}$.

Let U be the set of admissible inputs $u(t) \in U \subset \mathbb{R}^m$ and $X_0 \subset \mathbb{R}^n$ be the set of consistent initial conditions $x_0 \in X_0$ for which Eqn. (1) has a solution $x(t)$ for $u(t) \in U$.

The solution of Eqn. (1a) for $x_0 \in X_0$ has been derived by Kaczorek (2014a).

Definition 1. The fractional descriptor linear system (1) is called *asymptotically stable* if $\lim_{t\to\infty} x(t) = 0$ for any finite $x_0 \in X_0$ and $u(t) = 0$.

Theorem 1. (Kaczorek, 2011b; Matignon, 1996) *The fractional descriptor linear system (1) is asymptotically stable if and only if the zeros (the eigenvalues of (E, A)) $\lambda_1, \dots, \lambda_p$ of the equation*

$$\det[E\lambda - A] = \lambda^p + a_{p-1}\lambda^{p-1} + \cdots + a_1\lambda + a_0 = 0 \tag{4}$$

satisfy the condition

$$|\arg\lambda_k| > \alpha\frac{\pi}{2} \tag{5}$$

for $k = 1, \dots, p$.

The eigenvalues satisfying the condition (5) are located in the stability region shown in Fig. 1 and denoted by S_r.

Definition 2. The fractional descriptor continuous-time linear system

$$E\frac{\mathrm{d}^\alpha \hat{x}}{\mathrm{d}t^\alpha} = F\hat{x} + Gu + Hy, \tag{6}$$

where $\hat{x} = \hat{x}(t) \in \mathbb{R}^n$ is the estimate of $x(t)$, and $u = u(t) \in \mathbb{R}^m$, $y = y(t) \in \mathbb{R}^p$ are the same input and output vectors as in (1), $E, F \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{n\times m}$, $H \in \mathbb{R}^{n\times p}$, $\det E = 0$ is called a *(full-order) state observer* for the system (1) if

$$\lim_{t\to\infty}[x(t) - \hat{x}(t)] = 0. \tag{7}$$

Theorem 2. (Guang-ren, 2010; Kaczorek, 1992) *The fractional descriptor system (1) has a full state observer (6) if and only if there exists a matrix H such that all eigenvalues of the pair $(E, A–HC)$ are located in the stable region S_r shown in Fig. 1, i.e.,*

$$\sigma(E, A - HC) \subset S_r, \tag{8}$$

where σ denotes the spectrum of the pair.

The proof is also given by Kaczorek (2014a).

From Theorem 1 it follows that the design of a stable observer (6) of the system (1) has been reduced to finding a matrix H such that the eigenvalues of the pair $(E, A–HC)$ are located in the asymptotic stability region. It is well-known (Guang-ren, 2010; Kaczorek, 1992) that there exists a matrix H such that the eigenvalues of the pair $(E, A–HC)$ are located in the asymptotic stability region if and only if the fractional descriptor system (1) is detectable (Guang-ren, 2010; Kaczorek, 1992), i.e.,

$$\operatorname{rank}\begin{bmatrix} Es_k - A \\ C \end{bmatrix} = n \tag{9}$$

for $s_k \in \sigma(E, A)$.

The problem of designing the observer (6) of the system (1) can be reduced to the procedure of designing a state-feedback of the form $v = -H^T x$ for the dual system (Guang-ren, 2010; Kaczorek, 1992)

$$E^T\frac{\mathrm{d}^\alpha x}{\mathrm{d}t^\alpha} = A^T x + C^T v. \tag{10}$$

To guarantee that the descriptor state observer is impulse-free, the matrix H must be chosen so that

$$\deg[\det(Es - A + HC)] = \operatorname{rank} E. \tag{11}$$

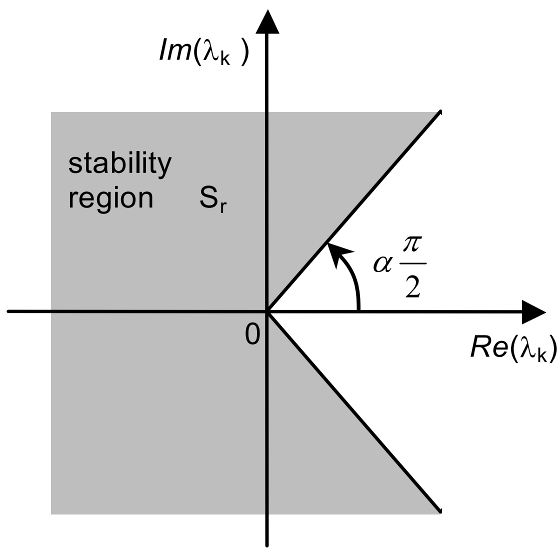

Fig. 1. Stability region.

It is well known (Cuihong, 2012; Kaczorek, 1992) that the finite observers poles (the finite eigenvalues of the pair $(E, A–HC)$ can be arbitrarily assigned if and only if the descriptor system (1) is R-observable, i.e.,

$$\operatorname{rank}\begin{bmatrix} Es - A \\ C \end{bmatrix} = n \tag{12}$$

for $s \in \mathbb{C}$.

Therefore, the following theorem has been proved.

Theorem 3. *An impulse-free fractional descriptor observer (6) with an arbitrary prescribed set of poles of the fractional descriptor system (1) satisfying (3) if and only exists if the conditions (11) and (12) are met.*

Now let us consider the fractional descriptor continuous-time nonlinear system

$$E\frac{d^\alpha x}{dt^\alpha} = Ax + f(x, u), \quad x_0 = x(0), \tag{13a}$$

$$y = Cx, \tag{13b}$$

where $x = x(t) \in \mathbb{R}^n$, $u = u(t) \in \mathbb{R}^m$, $y = y(t) \in \mathbb{R}^p$ are respectively the state, input and output vectors, $E, A \in \mathbb{R}^{n\times n}$, $f(x, u) \in \mathbb{R}^n$ is a continuous vector function of x and u.

It is assumed that $\det E = 0$ and (3) is met.

Definition 3. The fractional descriptor continuous-time nonlinear system

$$E\frac{d^\alpha \hat{x}}{dt^\alpha} = F\hat{x} + f(x, u) + Hy \tag{14}$$

is called a *full-order observer* of the nonlinear system (13) if

$$\lim_{t\to\infty}[x(t) - \hat{x}(t)] = 0, \tag{15}$$

where $\hat{x} = \hat{x}(t) \in \mathbb{R}^n$ is the estimate of $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $f(x, u)$, $y \in \mathbb{R}^p$ are the same vectors as in (13).

Definition 4. The fractional descriptor nonlinear system (14) is called a *full-order perfect observer* of the nonlinear system (13) if

$$x(t) = \hat{x}(t) \quad \text{for} \quad t > 0. \tag{16}$$

A design method of full-order perfect observers of nonlinear systems has been proposed by Kaczorek (2015).

3. Reduced-order fractional descriptor nonlinear observers

Consider the fractional descriptor nonlinear system (13) satisfying the assumption (3).

If

$$\operatorname{rank} C = p, \tag{17}$$

then there exist a permutation matrix $P \in \mathbb{R}^{n\times n}$

$$\begin{aligned} CP &= \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \quad C_1 \in \mathbb{R}^{p\times p}, \\ \det C_1 &\neq 0, \qquad\qquad C_2 \in \mathbb{R}^{p\times(n-p)} \end{aligned} \tag{18}$$

and the nonsingular matrix

$$Q_1 = \begin{bmatrix} C_1^{-1} & -C_1^{-1}C_2 \\ 0 & I_{n-p} \end{bmatrix} \in \mathbb{R}^{n\times n} \tag{19}$$

such that

$$\begin{aligned} \bar{C} &= CPQ_1 \\ &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} Q_1 \\ &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} C_1^{-1} & -C_1^{-1}C_2 \\ 0 & I_{n-p} \end{bmatrix} = \begin{bmatrix} I_p & 0 \end{bmatrix}. \end{aligned} \tag{20}$$

Substituting

$$x = PQ_1\bar{x} \tag{21}$$

into (13), we obtain

$$EPQ_1\frac{d^\alpha \bar{x}}{dt^\alpha} = APQ_1\bar{x} + f(x, u)\Big|_{x=PQ_1\bar{x}}, \tag{22a}$$

$$\begin{aligned} y &= Cx = CPQ_1\bar{x} = \begin{bmatrix} I_p & 0 \end{bmatrix}\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \bar{x}_1, \\ &\bar{x}_1 \in \mathbb{R}^p, \quad \bar{x}_2 \in \mathbb{R}^{(n-p)}. \end{aligned} \tag{22b}$$

Premultiplying (22a) by a nonsingular elementary row operations matrix $Q_2 \in \mathbb{R}^{n\times n}$, we obtain

$$\begin{aligned} Q_2EPQ_1 &= \begin{bmatrix} E_{11} & 0 \\ E_{21} & E_{22} \end{bmatrix}, \quad E_{11} \in \mathbb{R}^{p\times p}, \\ E_{21} &\in \mathbb{R}^{(n-p)\times p}, \quad E_{22} \in \mathbb{R}^{(n-p)\times(n-p)} \end{aligned} \tag{23}$$

and

$$E_{11}\frac{d^\alpha \bar{x}_1}{dt^\alpha} = A_{11}\bar{x}_1 + A_{12}\bar{x}_2 + f_1(\bar{x}_1, u), \tag{24a}$$

$$\begin{aligned} E_{21}\frac{d^\alpha \bar{x}_1}{dt^\alpha} + E_{22}\frac{d^\alpha \bar{x}_2}{dt^\alpha} &= A_{21}\bar{x}_1 + A_{22}\bar{x}_2 \\ &\quad + f_2(\bar{x}_1, \bar{x}_2, u), \end{aligned} \tag{24b}$$

where

$$Q_2APQ_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{24c}$$

$$A_{11} \in \mathbb{R}^{p\times p}, \quad A_{12} \in \mathbb{R}^{p\times(n-p)},$$

$$A_{21} \in \mathbb{R}^{(n-p)\times n}, \quad A_{22} \in \mathbb{R}^{(n-p)\times(n-p)},$$

$$\begin{bmatrix} f_1(\bar{x}_1, u) \\ f_2(\bar{x}_1, \bar{x}_2, u) \end{bmatrix} = Q_2 f(PQ\bar{x}, u),$$

$$f_1(\bar{x}_1, u) \in \mathbb{R}^p, \quad f_2(\bar{x}_1, \bar{x}_2, u) \in \mathbb{R}^{n-p}.$$

From (22b) it follows that $\bar{x}_1 = y$, and for given y the subvector $\bar{x}_1$ is known. Therefore, the reduced-order observer of the fractional descriptor system (1) should reconstruct only the subvector $\bar{x}_2 = \mathbb{R}^{(n-p)}$.

From (24) we have

$$E_{22}\frac{\mathrm{d}^\alpha \bar{x}_2}{\mathrm{d}t^\alpha} = A_{22}\bar{x}_2 + \bar{f}_2(\bar{x}_1, \bar{x}_2, u), \tag{25a}$$

$$\bar{y} = A_{12}\bar{x}_2, \tag{25b}$$

where

$$\bar{f}_2(\bar{x}_1, \bar{x}_2, u) = f_2(\bar{x}_1, \bar{x}_2, u) - E_{21}\frac{\mathrm{d}^\alpha y}{\mathrm{d}t^\alpha} + A_{21}y \tag{25c}$$

and

$$\bar{y} = E_{11}\frac{\mathrm{d}^\alpha y}{\mathrm{d}t^\alpha} - A_{11}y - f_2(\bar{x}_1, \bar{x}_2, u) \tag{25d}$$

are the new known input and output, respectively.

To find the estimate $\hat{x}_2$ of $\bar{x}_2$, the following full-order fractional descriptor nonlinear observer for the system (25) can be applied (Kaczorek, 2014a).

Definition 5. The fractional descriptor continuous-time nonlinear system

$$E_{22}\frac{\mathrm{d}^\alpha \hat{x}_2}{\mathrm{d}t^\alpha} = F\hat{x}_2 + \bar{f}_2(\bar{x}_1, \bar{x}_2, u) + H\bar{y}, \tag{26}$$

where $\hat{x}_2 \in \mathbb{R}^{n-p}$, $F \in \mathbb{R}^{(n-p)\times(n-p)}$, $H \in \mathbb{R}^{(n-p)\times p}$, is called a *reduced-order fractional descriptor observer* for the nonlinear system (13) if

$$\lim_{t\leftarrow\infty}[\bar{x}_2(t) - \hat{x}_2(t)] = 0. \tag{27}$$

Applying Theorem 1 to the fractional descriptor system (25), we obtain the following result.

Theorem 4. *For the fractional descriptor nonlinear system (13) the reduced-order observer (26) exists if the system (25) is detectable, i.e.,*

$$\operatorname{rank}\begin{bmatrix} E_{22}s - A_{22} \\ A_{12} \end{bmatrix} = n - p \tag{28}$$

for $s_k \in \sigma(E_{22}, A_{22})$.

It is well known (Guang-ren, 2010) that the eigenvalues of (E_{22}, A_{22}) (the finite poles of the observer) can be arbitrarily assigned if and only if the descriptor system (25) is R-observable, i.e.,

$$\operatorname{rank}\begin{bmatrix} E_{22}s - A_{22} \\ A_{12} \end{bmatrix} = n - p \tag{29}$$

for all $s \in \mathbb{C}$.

To guarantee that the descriptor observer (26) is impulse-free, the matrix H should be chosen so that (Guang-ren, 2010)

$$\deg\{\det[E_{22}s - A_{22} + HA_{12}]\} = \operatorname{rank} E_{22}. \tag{30}$$

Therefore, the following theorem has been proved.

Theorem 5. *The impulse-free reduced-order observer (26) for the fractional descriptor system (13) satisfying (3) exists if the conditions (24c), (29) and (30) are met.*

Remark 1. If $E_{22} = 0$ and $\det A_{22} \neq 0$, then from (25a) we have

$$\bar{x}_2 = -A_{22}^{-1}\bar{f}_2(\bar{x}_1, \bar{x}_2, u), \tag{31}$$

and we can find $\bar{x}_2$ without any observer.

Remark 2. If $\det E_{22} \neq 0$, then from (25a) we have

$$\frac{\mathrm{d}^\alpha \bar{x}_2}{\mathrm{d}t^\alpha} = E_{22}^{-1}A_{22}\bar{x}_2 + E_{22}^{-1}\bar{f}_2(\bar{x}_1, \bar{x}_2, u), \tag{32}$$

and the estimate $\hat{x}_2$ of $\bar{x}_2$ can be found using the classical (standard) fractional observer (Kociszewski, 2013; Kaczorek, 2004).

4. Procedure and examples

To design the reduced-order observer (26) for the fractional descriptor nonlinear system (13), the following procedure can be used.

Procedure 1.

Step 1. Find a permutation matrix P and a nonsingular matrix (19) transferring the matrix C to the form (20).

Step 2. Find the elementary row operations matrix Q_2, and using (23) and (24c) compute the matrices E_{11}, E_{21}, E_{22}, A_{11}, A_{12}, A_{21}, A_{22}, and $f_2(\bar{x}_1, u)$, $f_2(\bar{x}_1, \bar{x}_2, u)$.

Step 3. Check the conditions (28) and (30) for some $H \in \mathbb{R}^{(n-p)\times p}$.

Step 4. Using

$$F = A_{22} - HA_{12}, \tag{33}$$

find a matrix H such that the pair (E_{22}, F) has the desired eigenvalues located in the stability region S_r.

Step 5. Find Eqn. (26) of the desired fractional descriptor nonlinear observer.

Example 1. Consider the fractional descriptor system (13) with

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$A = \begin{bmatrix} -1 & 0 & 2 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{34}$$

$$f(x,u) = \begin{bmatrix} x_2^2 + u^2 \\ x_2x_4 + 2u \\ x_1x_2 + x_3^2 + x_2u \\ x_2x_3 + 2x_1u^2 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The system satisfies the condition (3) since

$$\det[Es - A] = \begin{bmatrix} s+1 & 0 & -2 & -1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & s \\ 0 & 0 & s & -1 \end{bmatrix} = 2(s+1)(1-s^2) \neq 0. \tag{35}$$

Using Procedure 1, we obtain what follows.

Step 1. In this case the permutation matrix is $P = I_4$ (the identity matrix)

$$CP = \begin{bmatrix} C_1 & C_2 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$
$$Q_1 = \begin{bmatrix} C_1^{-1} & -C_1^{-1}C_2 \\ 0 & I_{n-p} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{36}$$

and

$$\bar{C} = CPQ_1 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{37}$$

The new state vector is given by

$$\bar{x} = P^{-1}Q_1^{-1}x = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 + x_4 \\ x_1 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix},$$
$$\bar{x}_1 = \begin{bmatrix} x_2 + x_4 \\ x_1 \end{bmatrix}, \quad \bar{x}_2 = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}. \tag{38}$$

Step 2. The matrix of elementary operations is equal to $Q_2 = I_4$ and

$$EQ_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} E_{11} & 0 \\ 0 & E_{22} \end{bmatrix},$$

$$E_{11} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_{22} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{39a}$$

$$AQ_1 = \begin{bmatrix} -1 & 0 & 2 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 & 1 \\ -2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{39b}$$

$$A_{11} = \begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix},$$
$$A_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} f_1(\bar{x}_1, u) \\ f_2(\bar{x}_1, \bar{x}_2, u) \end{bmatrix} = f(x, u). \tag{40}$$

Step 3. The conditions (28) and (30) are satisfied since

$$\operatorname{rank} \begin{bmatrix} E_{22}s - A_{22} \\ A_{12} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} -1 & s \\ s & -1 \\ 2 & 1 \\ 0 & 2 \end{bmatrix} = 2 \tag{41}$$

for all $s \in \mathbb{C}$, and for

$$H = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$$

we have

$$\deg\{\det[E_{22}s - A_{22} + HA_{12}]\} = \deg\left\{\det \begin{bmatrix} -1 + 2h_{11} & s + h_{11} + 2h_{12} \\ s + 2h_{21} & -1 + h_{21} + 2h_{22} \end{bmatrix}\right\} = 2 = \operatorname{rank} E_2. \tag{42}$$

Step 4. Using (33) we obtain

$$F = A_{22} - HA_{12} = \begin{bmatrix} 1 - 2h_{11} & -2h_{12} - h_{11} \\ -2h_{21} & 1 - 2h_{22} - h_{21} \end{bmatrix}. \tag{43}$$

Let the desired eigenvalues of the pair (E_{22}, F) be $s_{d1} = s_{d2} = -10$. Then

$$\begin{aligned} \det[E_{22}s - F] &= \begin{vmatrix} 2h_{11} - 1 & s + h_{11} + 2h_{12} \\ s + 2h_{21} & h_{21} + 2h_{22} - 1 \end{vmatrix} \\ &= -s^2 - (2h_{21} + h_{11} + 2h_{12})s \\ &\quad + (2h_{11} - 1)(h_{21} + 2h_{22} - 1) \\ &\quad - 2h_{21}(h_{11} + 2h_{12}) \\ &= -(s+10)^2 = -(s^2 + 20s + 100) \end{aligned} \tag{44}$$

for $h_{11} = -2h_{12}$, and

$$\begin{aligned} -2h_{21} &= -20, \\ (2h_{11}-1)(h_{21}-2h_{22}-1) - 4h_{12}h_{21} &= -100. \end{aligned} \tag{45}$$

Solving (45), we obtain (for example)

$$h_{11} = 5.5, \quad h_{12} = -2.25, \quad h_{21} = 10, \quad h_{22} = -9.5. \tag{46}$$

Step 5. In this case, from (25c) we have

$$\begin{aligned} \bar{f}_2(\bar{x}_1, \bar{x}_2, u) &= f_2(\bar{x}_1, \bar{x}_2, u) - E_{21}\frac{\mathrm{d}^\alpha y}{\mathrm{d}t^\alpha} + A_{21}y \\ &= \begin{bmatrix} x_1x_2 + x_3^2 + x_2u \\ x_2x_3 + 2x_1u^2 \end{bmatrix}, \\ \bar{y} &= E_{11}\frac{d^\alpha y}{\mathrm{d}t^\alpha} + A_{11}y - f_1(\bar{x}_1, u) \\ &= \begin{bmatrix} x_2^2 + u^2 \\ x_2x_4 + 2u \end{bmatrix}. \end{aligned} \tag{47}$$

The desired reduced-order fractional observer of the system is described by the equation

$$\begin{aligned} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{\mathrm{d}^\alpha \hat{x}_2}{\mathrm{d}t^\alpha} &= \begin{bmatrix} -10 & 0 \\ 20 & 10 \end{bmatrix}\hat{x}_2 \\ &\quad - \begin{bmatrix} x_1x_2 + x_3^2 + x_2u \\ x_2x_3 + 2x_1u^2 \end{bmatrix} \\ &\quad - \begin{bmatrix} 455 & -2.75 \\ 10 & -9.5 \end{bmatrix}\bar{y} \end{aligned} \tag{48a}$$

or

$$\begin{aligned} \frac{\mathrm{d}^\alpha \hat{x}_2}{\mathrm{d}t^\alpha} &= \begin{bmatrix} -20 & 10 \\ -10 & 0 \end{bmatrix}\hat{x}_2 \\ &\quad - \begin{bmatrix} x_1x_2 + x_3^2 + x_2u \\ x_2x_3 + 2x_1u^2 \end{bmatrix} \\ &\quad - \begin{bmatrix} 10 & -9.5 \\ 5.5 & -2.75 \end{bmatrix}\bar{y}. \end{aligned} \tag{48b}$$

♦

5. Concluding remarks

Fractional descriptor reduced-order nonlinear observers for a class of fractional descriptor continuous-time nonlinear systems have been proposed. A design procedure for fractional descriptor observers has been proposed and illustrated on a numerical example.

The discussion can be easily extended to perfect fractional descriptor reduced-order observers and fractional descriptor discrete-time linear systems. An open problem is extension to fractional descriptor 2D continuous-discrete nonlinear systems.

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