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Dedicated to Professor Jan Stochel
on the occasion of his 70th birthday

CESÀRO SUMMABILITY OF TAYLOR SERIES IN HIGHER ORDER WEIGHTED DIRICHLET-TYPE SPACES

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Abstract. For a positive integer m and a finite non-negative Borel measure μ on the unit circle, we study the Hadamard multipliers of higher order weighted Dirichlet-type spaces $\mathcal{H}_{\mu,m}$. We show that if $\alpha > \frac{1}{2}$, then for any f in $\mathcal{H}_{\mu,m}$, the sequence of generalized Cesàro sums $\{\sigma_n^\alpha[f]\}$ converges to f. We further show that if $\alpha = \frac{1}{2}$ then for the Dirac delta measure supported at any point on the unit circle, the previous statement breaks down for every positive integer m.

Keywords: weighted Dirichlet-type integrals, Cesàro mean, summability, Hadamard multiplication.

Mathematics Subject Classification: 41A10, 40G05, 46E20, 41A17.

1. INTRODUCTION

The symbols $\mathbb{T}$ and $\mathbb{D}$ will denote the unit circle and the open unit disc in the complex plane $\mathbb{C}$ respectively. We use the symbols $\mathbb{Z}, \mathbb{N}$ and $\mathbb{Z}_{\geqslant 0}$ to denote the set of integers, positive integers and non-negative integers respectively. The notation $\mathcal{M}_+(\mathbb{T})$ stands for the set of all finite non-negative Borel measures on $\mathbb{T}$. Let $\mathcal{O}(\mathbb{D})$ denote the space of all complex valued holomorphic functions on $\mathbb{D}$. For a holomorphic function $f \in \mathcal{O}(\mathbb{D})$, which has a power series representation of the form $f(z) = \sum_{k=0}^\infty a_k z^k$, $z \in \mathbb{D}$, the n^{th} Taylor partial sum $s_n[f]$ and the n^{th} Cesàro sum $\sigma_n[f]$ are defined by

$$s_n[f](z) := \sum_{k=0}^n a_k z^k,$$

$$\sigma_n[f](z) := \frac{1}{n+1}\Big(\sum_{k=0}^n s_k[f](z)\Big) = \sum_{k=0}^n \Big(1 - \frac{k}{n+1}\Big) a_k z^k, \qquad n \in \mathbb{Z}_{\geqslant 0}.$$

When exploring analytic function spaces, a common concern revolves around determining the density of the polynomial set within the given function space. Once the density of polynomials is established, the next natural question to address is how to construct polynomials that can closely approximate a given function. In this context, it becomes natural to inquire whether, for a function f residing in a normed function space, the sequence of Taylor partial sum $\{s_n[f]\}$ converge to f in the norm associated with that space. It is worth noting that in several classical function spaces like Hardy space, Dirichlet space, and Bergman space defined over the unit disc $\mathbb{D}$, it is well-established that the sequence $\{s_n[f]\}$ indeed converges to f as prescribed by the associated norm. However, it is worth noting that there are instances where this convergence property does not hold. For example, it is well known that there exists a function f in the disc algebra $A(\mathbb{D})$ such that the sequence $\{s_n[f]\}$ does not converge to f, see [13, p. 57]. Another notable family of examples in the context of Hilbert function spaces is the family of weighted Dirichlet-type spaces. Richter introduced the weighted Dirichlet space $D(\mu)$ for each $\mu \in \mathcal{M}_+(\mathbb{T})$ in order to study the structure of closed M_z-invariant subspaces of the classical Dirichlet space on $\mathbb{D}$ and to obtain a model for cyclic analytic 2-isometries, see [19]. Interestingly, when $\mu = \delta_\lambda$, the Dirac delta measure at a point $\lambda \in \mathbb{T}$, it has been established that there exists a function f in $D(\delta_\lambda)$ for which the sequence $\{s_n[f]\}$ does not converge to f within the space $D(\delta_\lambda)$, see [8, Exercise 7.3 (2)]. On the contrary, a remarkable discovery surfaced when Mashreghi and Ransford recently showed that for any arbitrary, but fixed, $\mu \in \mathcal{M}_+(\mathbb{T})$, the sequence of Cesàro sums $\{\sigma_n[f]\}$ converges to f for every f in the space $D(\mu)$ (refer to [15, Theorem 1.6] and [16, Corollary 1]). Furthermore, they refined their result by establishing that the sequence of generalized Cesàro sums $\{\sigma_n^\alpha[f]\}$ also converge to f for every f in $D(\mu)$ and for each $\mu \in \mathcal{M}_+(\mathbb{T})$ whenever $\alpha > 1/2$, see [17, Theorem 1.1]. For each $n \in \mathbb{Z}_{\geqslant 0}$, the generalized n^{th}-Cesàro mean $\sigma_n^\alpha[f]$ is defined by

$$\sigma_n^\alpha[f](z) = \binom{n+\alpha}{\alpha}^{-1} \sum_{k=0}^{n} \binom{n-k+\alpha}{\alpha} a_k z^k, \quad \alpha \geqslant 0,$$

where the binomial coefficient $\binom{m+\alpha}{\alpha}$ is given the following interpretation:

$$\binom{m+\alpha}{\alpha} = \frac{\Gamma(m+\alpha+1)}{\Gamma(\alpha+1)\Gamma(m+1)},$$

with Γ denoting the usual gamma function. Note that the generalized n^{th}-Cesàro mean $\sigma_n^\alpha[f]$ corresponds to the n^{th}-partial sum $s_n[f]$ when $\alpha = 0$ and corresponds to the n^{th}-Cesàro sum $\sigma_n[f]$ when $\alpha = 1$. For $\alpha \geqslant 0$, the Taylor series of a function f is said to be (C,α)-summable to f if the sequence $\{\sigma_n^\alpha[f]\}$ converges to f in the associated normed function space. It is well known that (C,α)-summability implies (C,β)-summability if $\alpha \leqslant \beta$, see [12, Theorem 43]. It is worth mentioning that there is a de Branges-Rovnyak space $\mathcal{H}(b)$ with the property that the set of polynomials is dense in $\mathcal{H}(b)$ but there is a function f in $\mathcal{H}(b)$ such that the sequence of generalized Cesàro sum does not converge to f in $\mathcal{H}(b)$ for any $\alpha \geqslant 0$, see [18], [11, Corollary 6.14].

In this article, our primary objective is to investigate whether the sequence of generalized Cesàro means $\{\sigma_n^\alpha[f]\}$ converges to the corresponding function f that belongs to the higher order weighted Dirichlet space $\mathcal{H}_{\mu,m}$ as defined below. These spaces serve as a crucial framework for modeling a specific sub-class within the broader category of m-isometries. The motivation behind introducing these spaces is rooted in Agler's exploration of m-isometries see [2–4]. In order to find a suitable model for cyclic m-isometries, Rydhe delved into the study of higher order weighted Dirichlet-type spaces, see [20]. Following Rydhe's framework, we consider, for a measure μ in $\mathcal{M}_+(\mathbb{T})$, $f \in \mathcal{O}(\mathbb{D})$ and for a positive integer m, the concept of *weighted Dirichlet-type integral of f of order m*, $D_{\mu,m}(f)$, defined by

$$D_{\mu,m}(f) := \frac{1}{m!(m-1)!} \int\limits_{\mathbb{D}} \left|f^{(m)}(z)\right|^2 P_\mu(z)(1-|z|^2)^{m-1} dA(z).$$

Here $dA(z)$ denotes the normalized Lebesgue measure on the unit disc $\mathbb{D}$, $f^{(m)}(z)$ represents the m^{th}-order derivative of f at z, and $P_\mu(z)$ is the Poisson integral of the measure μ, that is,

$$P_\mu(z) := \int\limits_{\mathbb{T}} \frac{1-|z|^2}{|z-\zeta|^2} d\mu(\zeta), \quad z \in \mathbb{D}.$$

When dealing with Dirac delta measure δ_λ, representing a point measure at $\lambda \in \mathbb{T}$, we adopt a simpler notation $D_{\lambda,m}(\cdot)$ in place of $D_{\delta_\lambda,m}(\cdot)$. For a measure $\mu \in \mathcal{M}_+(\mathbb{T})/\{0\}$ and for each $m \in \mathbb{N}$, we consider the semi-inner product space $\mathcal{H}_{\mu,m}$ given by

$$\mathcal{H}_{\mu,m} := \big\{ f \in \mathcal{O}(\mathbb{D}) : D_{\mu,m}(f) < \infty \big\},$$

associated to the semi-norm $\sqrt{D_{\mu,m}(\cdot)}$. In case μ is δ_λ for some $\lambda \in \mathbb{T}$, we will use a simpler notation $\mathcal{H}_{\lambda,m}$ in place of $\mathcal{H}_{\delta_\lambda,m}$ and we refer it as a *local Dirichlet space of order m at λ*. If $\mu = 0$, we set $\mathcal{H}_{\mu,m} = H^2$, Hardy space on $\mathbb{D}$, for every $m \in \mathbb{N}$. If $\mu = \sigma$, the normalized Lebesgue measure on $\mathbb{T}$, by a straightforward computation, it follows that for a holomorphic function $f = \sum_{k=0}^\infty a_k z^k$ in $\mathcal{O}(\mathbb{D})$, we have

$$D_{\sigma,m}(f) = \sum_{k=m}^{\infty} \binom{k}{m} |a_k|^2, \quad m \in \mathbb{Z}_{\geqslant 0}, \tag{1.1}$$

where $\binom{k}{m} := \frac{k!}{m!(k-m)!}$ for any $k \geqslant m$. Using this, it can be easily verified that $\mathcal{H}_{\sigma,m}$ coincides with the space $\mathcal{D}_m$, studied in [21], where

$$\mathcal{D}_m = \Big\{ \sum_{k=0}^{\infty} a_k z^k : \sum_{k=0}^{\infty} |a_k|^2 (k+1)^m < \infty \Big\}.$$

The reader is referred to [9, 10, 14, 20] for several properties of the spaces $\mathcal{H}_{\mu,m}$ for an arbitrary non-negative measure μ and a positive integer m. Note that when $m = 1$, the weighted Dirichlet-type space $\mathcal{H}_{\mu,1}$ coincides with the weighted Dirichlet-type space $D(\mu)$ as introduced by Richter [19]. Theorems 1.1 and 1.3 are the main results of this article. These results in the case of $m = 1$ are well known, see [17, Theorem 1.1], see also [15, Theorem 1.6] for the special case of $m = 1$ and $\alpha = 1$.

Theorem 1.1. *Let $\mu \in \mathcal{M}_+(\mathbb{T})$ and $m \in \mathbb{N}$. If $\alpha > \frac{1}{2}$ then there exists a constant $\kappa > 0$ such that*

$$D_{\mu,m}(\sigma_n^\alpha[f]) \leqslant \kappa D_{\mu,m}(f), \quad n \in \mathbb{N},\ f \in \mathcal{H}_{\mu,m}.$$

Moreover, in this case, for every $f \in \mathcal{H}_{\mu,m}$, $D_{\mu,m}\big(\sigma_n^\alpha[f] - f\big) \to 0$ as $n \to \infty$.

It is known that the space $\mathcal{H}_{\mu,m}$ is contained in the Hardy space H^2 (see [9, Corollary 2.5]). This allows us to define a norm $\|\cdot\|_{\mu,m}$ on $\mathcal{H}_{\mu,m}$ given by

$$\|f\|_{\mu,m}^2 = \|f\|_{H^2}^2 + D_{\mu,m}(f), \quad f \in \mathcal{H}_{\mu,m},$$

where $\|f\|_{H^2}$ denotes the norm of f in H^2. As a consequence of Theorem 1.1, we obtain the following corollary.

Corollary 1.2. *Let $\mu \in \mathcal{M}_+(\mathbb{T})$ and $m \in \mathbb{N}$. If $\alpha > \frac{1}{2}$, then $\|\sigma_n^\alpha[f] - f\|_{\mu,m} \to 0$ as $n \to \infty$ for any $f \in \mathcal{H}_{\mu,m}$.*

Continuing our investigation into the convergence behavior of the sequence of generalized Cesàro means $\{\sigma_n^\alpha[f]\}$ to the respective function f when $\alpha \leqslant \frac{1}{2}$ within the higher order weighted Dirichlet-type space $\mathcal{H}_{\mu,m}$, we bring attention to the following theorem.

Theorem 1.3. *Let $m \in \mathbb{N}$ and $\lambda \in \mathbb{T}$ and $\alpha = \frac{1}{2}$. There exists a function $f \in \mathcal{H}_{\lambda,m}$ such that $D_{\lambda,m}\big(\sigma_n^\alpha[f] - f\big) \nrightarrow 0$ as $n \to \infty$.*

We will use the techniques of the Hadamard multiplication operators of $\mathcal{H}_{\mu,1}$ as developed in [15], in order to prove our results. In Section 2, first we extend the theory of Hadamard multiplication operators of $\mathcal{H}_{\mu,1}$ to higher order weighted Dirichlet-type spaces $\mathcal{H}_{\mu,m}$ and then we proceed to establish the main results.

2. HADAMARD MULTIPLICATION OPERATORS ON $\mathcal{H}_{\mu,m}$

For two formal power series $f(z) = \sum_{j=0}^\infty a_j z^j$ and $g(z) = \sum_{j=0}^\infty b_j z^j$, the Hadamard product $f * g$ of f and g is defined by a formal power series, given by the formula

$$(f * g)(z) := \sum_{j=0}^\infty a_j b_j z^j.$$

In view of the Cauchy–Hadamard formula for the radius of convergence of a power series, it is straightforward to verify that $f * g$ is in $\mathcal{O}(\mathbb{D})$ whenever f and g belong to $\mathcal{O}(\mathbb{D})$. For $\mu \in \mathcal{M}_+(\mathbb{T})$ and $m \in \mathbb{N}$, a function $c \in \mathcal{O}(\mathbb{D})$ is said to be a *Hadamard multiplier* of $\mathcal{H}_{\mu,m}$ if $(c * f) \in \mathcal{H}_{\mu,m}$ for every $f \in \mathcal{H}_{\mu,m}$. Proposition 2.2 below characterizes the Hadamard multiplier of $\mathcal{H}_{\mu,m}$. Before we state and prove Proposition 2.2, for $\mu \in \mathcal{M}_+(\mathbb{T}) \setminus \{0\}$, we introduce a linear space denoted as $\hat{\mathcal{H}}_{\mu,m}$, associated to the linear space $\mathcal{H}_{\mu,m}$, for the sake of simplicity.

$$\hat{\mathcal{H}}_{\mu,m} := \Big\{ f \in \mathcal{O}(\mathbb{D}) : f(z) = \sum_{j=m}^{\infty} a_j z^j, \ \ a_j \in \mathbb{C}, \ \ D_{\mu,m}(f) < \infty \Big\}. \tag{2.1}$$

When $\mu = 0$, we define $\hat{\mathcal{H}}_{\mu,m}$ to be the Hardy space H^2. For a non-zero μ in $\mathcal{M}_+(\mathbb{T})$, since $\sqrt{D_{\mu,m}(\cdot)}$ is a semi-norm on the linear space $\mathcal{H}_{\mu,m}$ and $D_{\mu,m}(f) = 0$ if and only if f is a polynomial of degree at most $(m-1)$, it follows that $\sqrt{D_{\mu,m}(\cdot)}$ induces a norm on $\hat{\mathcal{H}}_{\mu,m}$. It is straightforward to verify that $\hat{\mathcal{H}}_{\mu,m}$ turns out to be a Hilbert space with respect to this norm. Furthermore, as established in the following lemma, $\hat{\mathcal{H}}_{\mu,m}$ is a reproducing kernel Hilbert space on $\mathbb{D}$.

Lemma 2.1. *Let $\mu \in \mathcal{M}_+(\mathbb{T}) \setminus \{0\}$ and $m \in \mathbb{N}$. The linear space $\hat{\mathcal{H}}_{\mu,m}$, equipped with the norm $\sqrt{D_{\mu,m}(\cdot)}$, is a reproducing kernel Hilbert space on $\mathbb{D}$.*

Proof. Let $f \in \hat{\mathcal{H}}_{\mu,m}$ and $f(z) = \sum_{j=m}^{\infty} a_j z^j$, $z \in \mathbb{D}$. Then we have

$$f^{(m)}(z) = \sum_{j=m}^{\infty} \frac{\Gamma(j+1)}{\Gamma(j-m+1)} a_j z^{j-m}, \quad z \in \mathbb{D}.$$

In view of [9, Eq (2.6), p. 454], we obtain that

$$\begin{aligned}
D_{\mu,m}(f) &\geqslant \frac{\mu(\mathbb{T})}{4m!(m-1)!} \int_{\mathbb{D}} |f^{(m)}|^2(z)(1-|z|^2)^m dA(z) \\
&= \frac{\mu(\mathbb{T})}{4m!(m-1)!} \sum_{j=m}^{\infty} |a_j|^2 \frac{(\Gamma(j+1))^2}{(\Gamma(j-m+1))^2} \int_{\mathbb{D}} |z|^{2(j-m)}(1-|z|^2)^m dA(z) \\
&= \frac{\mu(\mathbb{T})}{4m!(m-1)!} \sum_{j=m}^{\infty} |a_j|^2 \frac{(\Gamma(j+1))^2}{(\Gamma(j-m+1))^2} \frac{\Gamma(m+1)\Gamma(j-m+1)}{\Gamma(j+2)} \\
&= \frac{\mu(\mathbb{T})}{4(m-1)!} \sum_{j=m}^{\infty} \frac{|a_j|^2}{(j+1)} \frac{\Gamma(j+1)}{\Gamma(j-m+1)}.
\end{aligned}$$

Using Cauchy–Schwarz inequality, we have for each $w \in \mathbb{D}$,

$$\begin{aligned}
|f(w)|^2 &= \Big| \sum_{j=m}^{\infty} a_j w^j \Big|^2 \\
&\leqslant \Big(\sum_{j=m}^{\infty} \frac{|a_j|^2}{(j+1)} \frac{\Gamma(j+1)}{\Gamma(j-m+1)} \Big) \Big(\sum_{j=m}^{\infty} \frac{(j+1)\Gamma(j-m+1)}{\Gamma(j+1)} |w|^{2j} \Big) \\
&= \frac{4(m-1)!}{\mu(\mathbb{T})} \Big(\sum_{j=m}^{\infty} \frac{(j+1)\Gamma(j-m+1)}{\Gamma(j+1)} |w|^{2j} \Big) D_{\mu,m}(f).
\end{aligned}$$

This shows that the evaluation at each point $w \in \mathbb{D}$ is a bounded linear functional on $\hat{\mathcal{H}}_{\mu,m}$ and hence $\hat{\mathcal{H}}_{\mu,m}$ is a reproducing kernel Hilbert space on $\mathbb{D}$. This completes the proof. □

Now we establish the result that gives a characterization of the Hadamard multiplier of $\mathcal{H}_{\mu,m}$.

Proposition 2.2. *Let $\mu \in \mathcal{M}_+(\mathbb{T})$ and $m \in \mathbb{N}$. A function $c \in \mathcal{O}(\mathbb{D})$ is a Hadamard multiplier of $\mathcal{H}_{\mu,m}$ if and only if there exists a constant $\kappa > 0$ such that*

$$D_{\mu,m}(c * f) \leqslant \kappa D_{\mu,m}(f), \quad \textit{for all } f \in \mathcal{H}_{\mu,m}.$$

Proof. Suppose that a function $c \in \mathcal{O}(\mathbb{D})$ is a Hadamard multiplier of $\mathcal{H}_{\mu,m}$. Thus $(c * f) \in \hat{\mathcal{H}}_{\mu,m}$ for every $f \in \hat{\mathcal{H}}_{\mu,m}$. Consider the linear transformation $H_c : \hat{\mathcal{H}}_{\mu,m} \to \hat{\mathcal{H}}_{\mu,m}$ given by $H_c(f) = c * f$, for $f \in \hat{\mathcal{H}}_{\mu,m}$. Since, by Lemma 2.1, $\hat{\mathcal{H}}_{\mu,m}$ is a reproducing kernel Hilbert space consisting of analytic functions on $\mathbb{D}$, it follows that the linear functional $L_j : f \to \frac{f^{(j)}(0)}{j!}$ on $\hat{\mathcal{H}}_{\mu,m}$ is continuous for every $j \geqslant m$ (see [6, Lemma 4.1]). This gives us that the graph of H_c is closed, and hence by closed graph theorem, H_c is bounded. That is, $D_{\mu,m}(c * f) \leqslant \|H_c\| D_{\mu,m}(f)$ for every $f \in \hat{\mathcal{H}}_{\mu,m}$. For an arbitrary $f \in \mathcal{H}_{\mu,m}$, we write

$$f = p + g,$$

where p is a polynomial of degree at most $m-1$ and $g \in \hat{\mathcal{H}}_{\mu,m}$. Hence we obtain

$$D_{\mu,m}(c * f) = D_{\mu,m}(c * g) \leqslant \|H_c\| D_{\mu,m}(g) = \|H_c\| D_{\mu,m}(f).$$

The converse part is trivial. □

For any $c(z) = \sum_{j=0}^{\infty} c_j z^j$ in $\mathcal{O}(\mathbb{D})$ and for any $m \in \mathbb{N}$, we consider the infinite matrix $T_c(m)$ defined by

$$T_c(m) := \begin{pmatrix} c_m & c_{m+1} - c_m & c_{m+2} - c_{m+1} & c_{m+3} - c_{m+2} & \cdots \\ 0 & c_{m+1} & c_{m+2} - c_{m+1} & c_{m+3} - c_{m+2} & \cdots \\ 0 & 0 & c_{m+2} & c_{m+3} - c_{m+2} & \cdots \\ 0 & 0 & 0 & c_{m+3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let W_m be the linear space in $\mathcal{O}(\mathbb{D})$ given by $W_m = \operatorname{span}\{z^j : j \geqslant m-1\}$. The matrix $T_c(m)$ induces a linear transformation $A_c(m)$ on the linear space W_m, given by

$$A_c(m)\Big(\sum_{j=m-1}^{n} b_j z^j\Big) := \sum_{j=m-1}^{n} \Big(c_{j+1} b_j + \sum_{k=j+1}^{n} (c_{k+1} - c_k) b_k\Big) z^j.$$

Let σ be the normalized Lebesgue measure on $\mathbb{T}$. Consider the Hilbert space $\hat{\mathcal{H}}_{\sigma,m-1}$ associated with the norm $\sqrt{D_{\sigma,m-1}(\cdot)}$, as defined in (2.1). Suppose $A_c(m)$ induces a bounded operator on the Hilbert space $\hat{\mathcal{H}}_{\sigma,m-1}$. As $\{z^j : j \geqslant m-1\}$ is an orthogonal basis for $\hat{\mathcal{H}}_{\sigma,m-1}$ with $D_{\sigma,m-1}(z^j) = \binom{j}{m-1}$, we obtain that

$$A_c(m)^*(z^{m-1}) = c_m z^{m-1} + \sum_{j=m}^{\infty} \frac{c_{j+1} - c_j}{\binom{j}{m-1}} z^j.$$

By (1.1), we get that

$$D_{\sigma,m-1}\Big(A_c(m)^*(z^{m-1})\Big) = |c_m|^2 + \sum_{j=m}^{\infty} |c_{j+1} - c_j|^2 \binom{j}{m-1}^{-1}. \tag{2.2}$$

Let $b(z) = \sum_{j=0}^{\infty} b_j z^j \in \mathcal{H}_{\sigma,m-1}$. By (1.1), we have

$$D_{\sigma,m-1}(b) = \sum_{j=m-1}^{\infty} |b_j|^2 \binom{j}{m-1} < \infty.$$

This together with (2.2) and an application of Cauchy–Schwarz inequality shows that the infinite series $\sum_{k=j+1}^{\infty}(c_{k+1}-c_k)b_k$ converges absolutely for every $j \geqslant m$. Moreover, using the continuity of $A_c(m)$ on $\hat{\mathcal{H}}_{\sigma,m-1}$ and the orthogonality of $\{z^j : j \geqslant m-1\}$ in $\hat{\mathcal{H}}_{\sigma,m-1}$, we get that

$$A_c(m)\Big(\sum_{j=m-1}^{\infty} b_j z^j\Big) = \sum_{j=m-1}^{\infty} \Big(c_{j+1} b_j + \sum_{k=j+1}^{\infty} (c_{k+1} - c_k) b_k\Big) z^j. \tag{2.3}$$

In this section, we shall be using the local Douglas formula for higher order local Dirichlet-type integrals [10, Theorem 1.1] repeatedly. Here we are reproducing this formula for the sake of reader's convenience.

Theorem 2.3 (A higher order local Douglas formula)**.** *Let n be a positive integer, $\lambda \in \mathbb{T}$, and $f \in \mathcal{O}(\mathbb{D})$. Then $f \in \mathcal{H}_{\lambda,n}$ if and only if $f = \alpha + (z-\lambda)g$ for some g in $\mathcal{H}_{\sigma,n-1}$ and $\alpha \in \mathbb{C}$. Moreover, in this case, the following statements hold:*

(i) $D_{\lambda,n}(f) = D_{\sigma,n-1}(g)$,

(ii) *$f(z) \to \alpha$ as $z \to \lambda$ in each oricyclic approach region $|z-\lambda| < \kappa(1-|z|^2)^{\frac{1}{2}}$, $\kappa > 0$. In particular, $f^*(\lambda)$ exists and is equal to α.*

The following theorem tells us that $A_c(m)$ being a bounded linear operator is equivalent to c being a Hadamard multiplier of $\mathcal{H}_{\mu,m}$ for every $\mu \in \mathcal{M}_+(\mathbb{T})$.

Theorem 2.4. *Let $m \in \mathbb{N}$, $c \in \mathcal{O}(\mathbb{D})$. The following statements are equivalent:*

(i) *c is a Hadamard multiplier of $\mathcal{H}_{\mu,m}$ for every $\mu \in \mathcal{M}_+(\mathbb{T})$,*

(ii) *c is a Hadamard multiplier of $\mathcal{H}_{\lambda,m}$ for some $\lambda \in \mathbb{T}$,*

(iii) *the transformation $A_c(m)$ defines a bounded operator on $\hat{\mathcal{H}}_{\sigma,m-1}$.*

Moreover, in this case, for every $\mu \in \mathcal{M}_+(\mathbb{T})$, we have

$$D_{\mu,m}(c * f) \leqslant \|A_c(m)\|^2 D_{\mu,m}(f), \qquad f \in \mathcal{H}_{\mu,m}.$$

Proof. (iii)⇒(i) Assume that the linear transformation $A_c(m)$ defines a bounded operator on the Hilbert space $\hat{\mathcal{H}}_{\sigma,m-1}$. Fix a $\lambda \in \mathbb{T}$. Let $f \in H_{\lambda,m}$ and the associated power series representation of f be given by $f(z) = \sum_{j=0}^{\infty} a_j z^j$. By the higher order local Douglas formula, $f(z) = a + (z-\lambda)L_\lambda[f](z)$, $z \in \mathbb{D}$, where $a = f^*(\lambda)$ (the boundary value of f at λ) and $L_\lambda[f] \in \mathcal{H}_{\sigma,m-1}$ with $D_{\lambda,m}(f) = D_{\sigma,m-1}(L_\lambda[f])$. Writing $L_\lambda[f](z) = \sum_{j=0}^{\infty} b_j z^j$, we obtain the relations

$$a_0 = a - b_0\lambda, \quad a_k = b_{k-1} - b_k\lambda, \quad k \geqslant 1. \tag{2.4}$$

Note that

$$D_{\lambda,m}(f) = D_{\sigma,m-1}(L_\lambda[f]) = \sum_{j=m-1}^{\infty} |b_j|^2 \binom{j}{m-1} = D_{\sigma,m-1}\Big(\sum_{j=m-1}^{\infty} b_j z^j\Big).$$

As $A_c(m)$ defines a bounded operator on $\hat{\mathcal{H}}_{\sigma,m-1}$ and $|\lambda| = 1$, we know that the series $\sum_{k=j+1}^{\infty}(c_{k+1} - c_k)b_k\lambda^k$ converges absolutely for every $j \geqslant m$ (see the discussion preceding this theorem) and it makes sense to consider the formal power series g given by

$$\begin{aligned} g(z) &= \sum_{j=0}^{\infty}\Big(c_{j+1}b_j + \sum_{k=j+1}^{\infty}(c_{k+1} - c_k)b_k\lambda^{k-j}\Big)z^j \\ &= \sum_{j=0}^{\infty}\Big(c_{j+1}b_j\lambda^j + \sum_{k=j+1}^{\infty}(c_{k+1} - c_k)b_k\lambda^k\Big)(\bar{\lambda}z)^j. \end{aligned}$$

Then by (2.4), it follows that $(c * f)(z) = A + (z-\lambda)g(z)$ for every $z \in \mathbb{D}$ for some $A \in \mathbb{C}$. Note that

$$D_{\sigma,m-1}\Big(\sum_{j=m-1}^{\infty} b_j\lambda^j z^j\Big) = D_{\sigma,m-1}\Big(\sum_{j=m-1}^{\infty} b_j z^j\Big) = D_{\lambda,m}(f),$$

and

$$\begin{aligned} D_{\sigma,m-1}(g) &= \sum_{j=m-1}^{\infty}\Big|c_{j+1}b_j\lambda^j + \sum_{k=j+1}^{\infty}(c_{k+1} - c_k)b_k\lambda^k\Big|^2\binom{j}{m-1} \\ &= D_{\sigma,m-1}\Big(A_c(m)\Big(\sum_{j=m-1}^{\infty} b_j\lambda^j z^j\Big)\Big) < \infty. \end{aligned}$$

Thus applying [10, Theorem 1.1] a second time, we obtain $c * f \in \mathcal{H}_{\lambda,m}$, and

$$D_{\lambda,m}(c * f) = D_{\sigma,m-1}(g).$$

Hence we obtain that

$$\sup\Big\{D_{\lambda,m}(c * f) : D_{\lambda,m}(f) = 1\Big\} = \|A_c(m)\|^2. \tag{2.5}$$

This gives us that $D_{\lambda,m}(c*f) \leqslant \|A_c(m)\|^2 D_{\lambda,m}(f)$ for every $f \in H_{\lambda,m}$. Since $\lambda \in \mathbb{T}$ was arbitrary, for any $f \in \mathcal{H}_{\mu,m}$, it follows that

$$\begin{aligned} D_{\mu,m}(c*f) &= \int_{\mathbb{T}} D_{\lambda,m}(c*f)d\mu(\lambda) \\ &\leqslant \|A_c(m)\|^2 \int_{\mathbb{T}} D_{\lambda,m}(f)d\mu(\lambda) \\ &= \|A_c(m)\|^2 D_{\mu,m}(f). \end{aligned}$$

Hence we obtain that c is a Hadamard multiplier of $\mathcal{H}_{\mu,m}$ for every $\mu \in \mathcal{M}_+(\mathbb{T})$.

(i)⇒(ii) is obvious.

(ii)⇒(iii) Assume that c is a Hadamard multiplier of $\mathcal{H}_{\lambda,m}$ for some $\lambda \in \mathbb{T}$. Following Proposition 2.2, we obtain that the linear transformation H_c on the Hilbert space $\hat{\mathcal{H}}_{\lambda,m}$ given by $H_c(f) = c*f$, for $f \in \hat{\mathcal{H}}_{\lambda,m}$ is bounded. Moreover, there exists a constant $\kappa = \|H_c\|^2$ such that

$$D_{\lambda,m}(c*f) \leqslant \kappa D_{\lambda,m}(f), \quad \text{for all } f \in \mathcal{H}_{\lambda,m}. \tag{2.6}$$

Note that, by the higher order local Douglas formula, every $f \in \mathcal{H}_{\lambda,m}$ can be uniquely written as

$$f(z) = a + (z-\lambda)L_\lambda[f](z),$$

where $a \in \mathbb{C}$ and $D_{\lambda,m}(f) = D_{\sigma,m-1}(L_\lambda[f])$. As c is a Hadamard multiplier of $\mathcal{H}_{\lambda,m}$, the inequality in (2.6) can be rephrased as

$$D_{\sigma,m-1}(L_\lambda[c*f]) \leqslant \kappa D_{\sigma,m-1}(L_\lambda[f]), \quad \text{for all } f \in \mathcal{H}_{\lambda,m}. \tag{2.7}$$

In view of the higher order local Douglas formula, we have that for any $g \in \mathcal{H}_{\sigma,m-1}$, the function $(z-\lambda)g \in \mathcal{H}_{\lambda,m}$ and consequently $D_{\sigma,m-1}(L_\lambda[c*(z-\lambda)g]) < \infty$. Note that for any $f \in \mathcal{H}_{\sigma,m-1}$, the function $f - s_{m-2}[f] \in \hat{\mathcal{H}}_{\sigma,m-1}$ whenever $m \geqslant 2$. Now consider the transformation $\hat{H}_c(m)$ from the Hilbert space $\hat{\mathcal{H}}_{\sigma,m-1}$ into itself defined by

$$\hat{H}_c(m)(g) := \begin{cases} L_\lambda[c*(z-\lambda)g] & \text{if } m=1, \\ L_\lambda[c*(z-\lambda)g] - s_{m-2}[L_\lambda[c*(z-\lambda)g]] & \text{if } m \geqslant 2, \end{cases} \quad g \in \hat{\mathcal{H}}_{\sigma,m-1}.$$

In view of the inequality (2.7), we obtain that

$$\begin{aligned} D_{\sigma,m-1}\Big(\hat{H}_c(m)(g)\Big) &= D_{\sigma,m-1}(L_\lambda[c*(z-\lambda)g]) \\ &\leqslant \kappa D_{\sigma,m-1}(L_\lambda[(z-\lambda)g]) \\ &= \kappa D_{\sigma,m-1}(g), \end{aligned}$$

for every $g \in \hat{\mathcal{H}}_{\sigma,m-1}$. Let's compute the matrix representation of $\hat{H}_c(m)$ with respect to the orthogonal basis $\{(\bar{\lambda}z)^j : j \geqslant m-1\}$ of $\hat{\mathcal{H}}_{\sigma,m-1}$. Note that for any $j \geqslant (m-1)$,

$$\begin{aligned} L_\lambda\Big[c * (z-\lambda)(\bar{\lambda}z)^j\Big] &= L_\lambda\Big[c_{j+1}\bar{\lambda}^j z^{j+1} - c_j\bar{\lambda}^{j-1}z^j\Big] \\ &= c_{j+1}\bar{\lambda}^j\frac{z^{j+1}-\lambda^{j+1}}{z-\lambda} - c_j\bar{\lambda}^{j-1}\frac{z^j-\lambda^j}{z-\lambda} \\ &= (c_{j+1}-c_j)\sum_{k=0}^{j-1}(\bar{\lambda}z)^k + c_{j+1}(\bar{\lambda}z)^j. \end{aligned}$$

Thus it follows that for any $j \geqslant (m-1)$, we have

$$\hat{H}_c(m)\Big((\bar{\lambda}z)^j\Big) = (c_{j+1}-c_j)\sum_{k=m-1}^{j-1}(\bar{\lambda}z)^k + c_{j+1}(\bar{\lambda}z)^j.$$

Hence the matrix representation of the operator $\hat{H}_c(m)$ with respect to the orthogonal basis $\{(\bar{\lambda}z)^j : j \geqslant m-1\}$ of $\hat{\mathcal{H}}_{\sigma,m-1}$ coincides with the matrix $T_c(m)$. Let V be the unitary operator on $\hat{\mathcal{H}}_{\sigma,m-1}$ defined by

$$V\Big(\sum_{j=m-1}^{\infty} b_j z^j\Big) = \sum_{j=m-1}^{\infty} b_j(\bar{\lambda}z)^j.$$

In view of (2.3), it is easy to verify that $V^{-1}\hat{H}_c(m)V(z^j) = A_c(m)(z^j)$ for every $j \geqslant (m-1)$. Hence $A_c(m)$ must define a bounded operator on $\hat{\mathcal{H}}_{\sigma,m-1}$ and $\|A_c(m)\| = \|\hat{H}_c(m)\|$. □

Remark 2.5. It is straightforward to verify that any function $c \in \mathcal{O}(\mathbb{D})$ with Taylor coefficients $\{c_n\}$ is a Hadamard multiplier for $\mathcal{H}_{\sigma,m}$ if and only if $\{c_n\}_{n\geqslant 0}$ is bounded. On the other hand, if we define $c_n = 1$ if n is odd and $c_n = 0$ if n is even, then the linear transformation $A_c(m)$ will be unbounded on $\hat{\mathcal{H}}_{\sigma,m-1}$, as noted in [17, p. 52] in the case $m = 1$. To see this for general m, note that for any even $n \geqslant m$,

$$\big\|A_c(m)(z^n)\big\|^2_{\sigma,m-1} = \sum_{k=m-1}^{n}\binom{k}{m-1} = \binom{n+1}{m}.$$

Therefore

$$\|A_c(m)\| \geqslant \binom{n}{m-1}^{-\frac{1}{2}}\binom{n+1}{m}^{\frac{1}{2}} \sim \sqrt{\frac{(n+1)^m}{n^{m-1}}}$$

for all even $n \geqslant m$. Hence $A_c(m)$ is unbounded on $\hat{\mathcal{H}}_{\sigma,m-1}$.

Remark 2.6. If c is a Hadamard multiplier of $\mathcal{H}_{\lambda,m}$ for some $\lambda \in \mathbb{T}$, then by Proposition 2.2, the operator $H_c : \hat{\mathcal{H}}_{\lambda,m} \to \hat{\mathcal{H}}_{\lambda,m}$, defined by $H_c(f) = c * f$, is bounded. In view of (2.5), it follows that, in this case,

$$\|H_c : \hat{\mathcal{H}}_{\lambda,m} \to \hat{\mathcal{H}}_{\lambda,m}\| = \|A_c(m)\|.$$

3. GENERALIZED CESÀRO SUMMABILITY IN HIGHER ORDER DIRICHLET SPACES

In this section, we provide proofs of Theorems 1.1 and 1.3. We shall need the following property of the Gamma function (see [1, p. 257, (6.1.46)]):

$$\lim_{k\to\infty} k^{b-a}\frac{\Gamma(k+a)}{\Gamma(k+b)} = 1, \quad a, b \in \mathbb{C}. \tag{3.1}$$

We start with the following lemma.

Lemma 3.1. *Let $\alpha \in \left(\frac{1}{2}, 1\right)$. Then there exists a positive constant C such that*

$$\sum_{j=0}^{n} \frac{\Gamma(j+\alpha)^2}{\Gamma(j+1)^2} \leqslant C(n+1)^{2\alpha-1} \quad \textit{for all } n \geqslant 0.$$

Proof. By (3.1), there exists a constant $C_1 > 0$ such that $\frac{\Gamma(k+\alpha)}{\Gamma(k+1)} \leqslant C_1(k+1)^{\alpha-1}$ for all $k \in \mathbb{Z}_{\geqslant 0}$. Thus we have

$$\begin{aligned}\sum_{j=0}^{n} \frac{\Gamma(j+\alpha)^2}{\Gamma(j+1)^2} &\leqslant C_1^2 \sum_{j=0}^{n} (j+1)^{2\alpha-2} \leqslant C_1^2 \sum_{j=0}^{n} \int_{j}^{j+1} t^{2\alpha-2}\, dt \\ &= C_1^2 \int_{j=0}^{n+1} t^{2\alpha-2}\, dt = \frac{C_1^2}{2\alpha-1}(n+1)^{2\alpha-1}.\end{aligned}$$

Choosing $C = \frac{C_1^2}{2\alpha-1}$ completes the proof of the lemma. □

Theorem 3.2. *Let $\alpha \in \left(\frac{1}{2}, 1\right)$. For each $n \in \mathbb{N}$, let*

$$h_n(z) = \binom{n+\alpha}{\alpha}^{-1} \sum_{k=0}^{n} \binom{n-k+\alpha}{\alpha} z^k.$$

Then the family of linear transformations

$$A_{h_n}(m) : \hat{\mathcal{H}}_{\sigma,m-1} \to \hat{\mathcal{H}}_{\sigma,m-1},\ n \in \mathbb{N},$$

is uniformly bounded in operator norm.

Proof. Note that $\left\{\binom{j}{m-1}^{-\frac{1}{2}} z^j\right\}_{j\geqslant m-1}$ forms an orthonormal basis of $\hat{\mathcal{H}}_{\sigma,m-1}$. Fix $n \in \mathbb{N}$. With respect to this orthonormal basis, let $((a_{i,j}))_{i,j=m-1}^{\infty}$ denote the matrix of $A_{h_n}(m)$. From (2.3), it is easy to see that

$$a_{i,j} = \begin{cases} c_{j+1}, & \text{if } i = j, \\ \sqrt{\dfrac{\binom{i}{m-1}}{\binom{j}{m-1}}}(c_{j+1} - c_j), & \text{if } i+1 \leqslant j, \\ 0, & \text{otherwise,} \end{cases}$$

where $c_j = \binom{n+\alpha}{\alpha}^{-1}\binom{n-j+\alpha}{\alpha}$ for $j \leqslant n$ and $c_j = 0$ for $j > n$. Note that all the entries of the matrix $((a_{i,j}))_{i,j=m-1}^{\infty}$ are zero except finitely many. Thus we have

$$\begin{aligned}
\|A_{h_n}(m)\| &= \|((a_{i,j}))_{i,j=m-1}^{n}\| \\
&\leqslant \left\| \begin{pmatrix} a_{m-1,m-1} & 0 & \dots & 0 \\ 0 & a_{m,m} & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & a_{n,n} \end{pmatrix} \right\| + \left\| \begin{pmatrix} 0 & a_{m-1,m} & \dots & a_{m-1,n} \\ 0 & 0 & \dots & a_{m,n} \\ \vdots & \vdots & \ddots & a_{n-1,n} \\ 0 & 0 & \dots & 0 \end{pmatrix} \right\| \\
&\leqslant \max_{m-1 \leqslant i \leqslant n} |a_{i,i}| + \sum_{j=m}^{n} \sum_{i=m-1}^{j} |a_{i,j}|^2 \\
&\leqslant 1 + \sum_{j=m}^{n} \sum_{i=m-1}^{j-1} |a_{i,j}|^2.
\end{aligned}$$

Here for the second last inequality, we have used the fact that the operator norm of a matrix is bounded by its Hilbert–Schmidt norm, see [5, Chapter 1, Proposition 1.6]. It is easily verified that for $0 \leqslant j \leqslant n$, we have

$$\begin{aligned}
c_{j+1} - c_j &= \frac{1}{\binom{n+\alpha}{\alpha}} \left(\binom{n-j-1+\alpha}{\alpha} - \binom{n-j+\alpha}{\alpha} \right) \\
&= -\frac{\alpha\, \Gamma(n-j+\alpha)}{\binom{n+\alpha}{\alpha}\Gamma(\alpha+1)\Gamma(n-j+1)}.
\end{aligned}$$

Note that

$$\begin{aligned}
\sum_{j=m}^{n} \sum_{i=m-1}^{j-1} |a_{i,j}|^2 &\leqslant \frac{\alpha^2}{(\Gamma(\alpha+1))^2\binom{n+\alpha}{\alpha}^2} \sum_{j=m}^{n} \sum_{i=m-1}^{j-1} \frac{\binom{i}{m-1}}{\binom{j}{m-1}} \frac{(\Gamma(n-j+\alpha))^2}{(\Gamma(n-j+1))^2} \\
&= \frac{\alpha^2}{(\Gamma(\alpha+1))^2\binom{n+\alpha}{\alpha}^2} \sum_{j=m}^{n} \frac{1}{\binom{j}{m-1}} \frac{(\Gamma(n-j+\alpha))^2}{(\Gamma(n-j+1))^2} \sum_{i=m-1}^{j-1} \binom{i}{m-1} \\
&= \frac{\alpha^2}{(\Gamma(\alpha+1))^2\binom{n+\alpha}{\alpha}^2} \sum_{j=m}^{n} \frac{1}{\binom{j}{m-1}} \frac{(\Gamma(n-j+\alpha))^2}{(\Gamma(n-j+1))^2} \binom{j}{m} \\
&= \frac{\alpha^2}{m(\Gamma(\alpha+1))^2\binom{n+\alpha}{\alpha}^2} \sum_{j=m}^{n} (j-m+1) \frac{(\Gamma(n-j+\alpha))^2}{(\Gamma(n-j+1))^2} \\
&\leqslant \frac{\alpha^2(n-m+1)}{m(\Gamma(\alpha+1))^2\binom{n+\alpha}{\alpha}^2} \sum_{j=m}^{n} \frac{(\Gamma(n-j+\alpha))^2}{(\Gamma(n-j+1))^2} \\
&= \frac{\alpha^2(n-m+1)}{m(\Gamma(\alpha+1))^2\binom{n+\alpha}{\alpha}^2} \sum_{j=0}^{n-m} \frac{(\Gamma(j+\alpha))^2}{(\Gamma(j+1))^2} \\
&\leqslant C \frac{\alpha^2}{m(\Gamma(\alpha+1))^2\binom{n+\alpha}{\alpha}^2} (n-m+1)^{2\alpha}.
\end{aligned}$$

Here, we have used a well-known binomial identity $\sum_{i=m-1}^{j-1}\binom{i}{m-1} = \binom{j}{m}$, (see [7, p. 46]), while the last inequality follows from Lemma 3.1. Also, by (3.1), we get $\binom{n+\alpha}{\alpha} \sim (n+1)^{\alpha}$. This completes the proof. $\square$

The following lemma might be well-known to the experts. We provide a proof for the sake of completeness.

Lemma 3.3. *Let $\{T_n\}_{n\geqslant 1}$ be a sequence of bounded linear operators on a reproducing kernel Hilbert space $\mathcal{H}$ of holomorphic functions on $\mathbb{D}$. Suppose that:*

(i) *T_n is finite-rank for each $n \in \mathbb{N}$,*
(ii) *$T_nT_m(\mathcal{H}) \subseteq T_m(\mathcal{H})$ for each $m, n \in \mathbb{N}$,*
(iii) *$T_n(f)(z) \to f(z)$ as $n \to \infty$ for all $f \in \mathcal{H}$ and for all $z \in \mathbb{D}$.*

Then $T_n(f) \to f$ in norm as $n \to \infty$ for all $f \in \mathcal{H}$ if and only if $\sup_n \|T_n\| < \infty$.

Proof. Suppose $T_n(f) \to f$ in norm as $n \to \infty$ for all $f \in \mathcal{H}$. Then, by the uniform boundedness principle, it follows that $\sup_n \|T_n\| < \infty$. For the converse, assume that $\sup_n \|T_n\| < \infty$. Let $K(z, w)$ denote the reproducing kernel of $\mathcal{H}$. Since $T_n(f)(z) \to f(z)$ as $n \to \infty$ for all $z \in \mathbb{D}$, by the reproducing property of $\mathcal{H}$, it follows that $\langle T_n(f), g\rangle \to \langle f, g\rangle$ for all g of the form $\sum_{i=1}^{\ell} a_iK(\cdot, w_i)$, where $a_i \in \mathbb{C}, w_i \in \mathbb{D}, \ell \in \mathbb{N}$. Since the set $\{\sum_{i=1}^{\ell} a_iK(\cdot, z_i) : a_i \in \mathbb{C}, z_i \in \mathbb{D}, \ell \in \mathbb{N}\}$ is dense in $\mathcal{H}$, and $\sup_n \|T_n\| < \infty$, it follows that $T_n(f) \to f$ weakly as $n \to \infty$ for all $f \in \mathcal{H}$. The proof is now complete by [11, Lemma 2.3]. $\square$

We now are ready to prove main theorem of this section.

Proof of Theorem 1.1. In view of [12, Theorem 43], it is sufficient to consider $\frac{1}{2} < \alpha < 1$. Suppose $f \in \mathcal{H}_{\mu,m}$. Note that $\sigma_n^{\alpha}[f](z) = (h_n * f)(z)$, where $h_n(z) = \binom{n+\alpha}{\alpha}^{-1}\sum_{k=0}^{n}\binom{n-k+\alpha}{\alpha}z^k$, $n \in \mathbb{N}$. By Theorem 3.2, there exists a constant $\kappa > 0$ such that $\|A_{h_n}(m)\| \leqslant \kappa$ for all $n \in \mathbb{N}$. Using this along with Theorem 2.4, we get

$$D_{\mu,m}(\sigma_n^{\alpha}[f]) = D_{\mu,m}(h_n * f) \leqslant \|A_{h_n}(m)\|^2 D_{\mu,m}(f) \leqslant \kappa^2 D_{\mu,m}(f).$$

To prove the second part, let T_n be the operator on $(\hat{\mathcal{H}}_{\mu,m}, D_{\mu,m}(\cdot))$ defined by $T_n(f) = \sigma_n^{\alpha}[f]$. It is easy to see that T_n is finite-rank and $T_nT_\ell(\hat{\mathcal{H}}_{\mu,m}) \subseteq T_\ell(\hat{\mathcal{H}}_{\mu,m})$ for each n, ℓ. Also, since for any $f \in \mathcal{O}(\mathbb{D})$, $s_n[f](z) \to f(z)$ for each $z \in \mathbb{D}$, by [12, Theorem 4.3], we obtain that $T_n(f)(z) \to f(z)$ for each $z \in \mathbb{D}$. Further, by the first part of this Theorem, $\sup_n \|T_n\| < \infty$. Hence by Lemma 3.3, it follows that $D_{\mu,m}(\sigma_n^{\alpha}[f] - f) \to 0$ as $n \to \infty$ for all $f \in \hat{\mathcal{H}}_{\mu,m}$. Now let $f = \sum_{i\geqslant 0} a_iz^i \in \mathcal{H}_{\mu,m}$. Then $f_1 := \sum_{i\geqslant m} a_iz^i \in \hat{\mathcal{H}}_{\mu,m}$ and $D_{\mu,m}(\sigma_n^{\alpha}[f] - f) = D_{\mu,m}(\sigma_n^{\alpha}[f_1] - f_1) \to 0$ as $n \to \infty$. This completes the proof. $\square$

Proof of Corollary 1.2. Let $\alpha > \frac{1}{2}$. It is easy to see that $\|s_n[f] \to f\|_{H^2} \to 0$ for all $f \in H^2$. Thus by [12, Theorem 4.3], $\|\sigma_n^{\alpha}[f] - f\|_{H^2} \to 0$ for all $f \in H^2$. This together with Theorem 1.1 completes the proof of this Corollary. $\square$

We proceed now to show that the statement of Theorem 3.2 does not hold true if $\alpha \leqslant 1/2$. It is enough to disprove the statement for $\alpha = 1/2$.

Proposition 3.4. *Suppose for each $n \in \mathbb{N}$,*

$$g_n(z) = \sum_{k=0}^{n} \Big(1 - \frac{k}{n+1}\Big)^{1/2} z^k.$$

Then for any $m \in \mathbb{N}$, the family $\{A_{g_n}(m) : n \in \mathbb{N}\}$ is not uniformly bounded.

Proof. Fix $n \in \mathbb{N}$. Let $c_k = (1 - \frac{k}{n+1})^{1/2}$, $k = 1, \ldots, n$. Consider the sub-matrix PAQ of $DT_{g_n}(m)D^{-1}$, where

$$A := \begin{pmatrix} c_{s+1} - c_s & \ldots & c_{n+1} - c_n \\ \vdots & \ddots & \vdots \\ c_{s+1} - c_s & \ldots & c_{n+1} - c_n \end{pmatrix}_{(s+1)\times(n-s+1)},$$

$$P = \operatorname{diag}\Bigg(\binom{s}{m-1}^{\frac{1}{2}}, \binom{s+1}{m-1}^{\frac{1}{2}}, \ldots, \binom{2s}{m-1}^{\frac{1}{2}}\Bigg),$$

$$Q = \operatorname{diag}\Bigg(\binom{s}{m-1}^{-\frac{1}{2}}, \binom{s+1}{m-1}^{-\frac{1}{2}}, \ldots, \binom{n}{m-1}^{-\frac{1}{2}}\Bigg),$$

and

$$D = \operatorname{diag}\Bigg(\binom{m-1}{m-1}^{\frac{1}{2}}, \binom{m}{m-1}^{\frac{1}{2}}, \ldots\Bigg).$$

Note that

$$\|PAQ\| \geqslant \alpha\beta\|A\|,$$

where α and β are minimum of the eigenvalues of P and Q respectively, and $\|\cdot\|$ denotes the operator norm. As shown in [17, Theorem 2.2], it turns out that AA^* is a square matrix of size $(s+1)$ with each of its entries equal to $(\sum_{k=s}^{n} |c_{k+1} - c_k|^2)^{1/2}$. Therefore we obtain the following:

$$\|PAQ\| \geqslant \sqrt{\frac{\binom{s}{m-1}}{\binom{n}{m-1}}}\sqrt{s+1}\Big(\sum_{k=s}^{n} |c_{k+1} - c_k|^2\Big)^{1/2}. \tag{3.2}$$

Since, for a fixed $m \in \mathbb{N}$, $\binom{s}{m-1} \sim (s+1)^{m-1}$ and $\binom{n}{m-1} \sim (n+1)^{m-1}$, it follows from (3.2) that

$$\|PAQ\| \geqslant C\sqrt{\frac{(s+1)^m}{(n+1)^{m-1}}}\Big(\sum_{k=s}^{n} |c_{k+1} - c_k|^2\Big)^{1/2},$$

for some positive constant C (independent of n and s). It follows from [17, p. 7] that

$$\|PAQ\| \geqslant C\sqrt{\frac{(s+1)^m}{(n+1)^{m-1}}}\frac{1}{2\sqrt{n+1}}\sqrt{log(n+2-s)} = \frac{C}{2}\sqrt{\frac{(s+1)^m}{(n+1)^m}}\sqrt{log(n+2-s)}.$$

Choosing $s = \lfloor \frac{n}{2} \rfloor$, we get that

$$\|PAQ\| \geqslant \frac{C}{2^{m+1}} \sqrt{log(2 + n/2)}. \tag{3.3}$$

Now as n approaches to ∞, the right hand side of (3.3) tends to ∞ as well. As $\|A_{g_n}(m)\| = \|DT_{g_n}(m)D^{-1}\|$, it follows from (3.3) that the family $\{A_{g_n}(m) : n \in \mathbb{N}\}$ is not uniformly bounded. □

Proof of Theorem 1.3. Let $\alpha = \frac{1}{2}$. Note that $\sigma_n^\alpha[f](z) = (h_n * f)(z)$, where $h_n(z) = \binom{n+\alpha}{\alpha}^{-1} \sum_{k=0}^{n} \binom{n-k+\alpha}{\alpha} z^k$, $n \in \mathbb{N}$. By [17, Theorem 3.2], for any $f \in \mathcal{H}_{\lambda,m}$, the sequence $\{h_n * f\}$ does not converge to f in $\mathcal{H}_{\lambda,m}$ if and only if the sequence $\{\phi_n * f\}$ does not converge to f in $\mathcal{H}_{\lambda,m}$, where for each $n \in \mathbb{N}$, $\phi_n(z) = \sum_{k=1}^{n} c_k z^k$ with $c_k = (1 - \frac{k}{n+1})^{1/2}$, $k = 1, \ldots, n$. By Proposition 3.4, the family $\{A_{\phi_n}(m)\}_{n\in\mathbb{N}}$ is not uniformly bounded. Thus, by Remark 2.6, it follows that the family $\{H_{\phi_n}\}_{n\in\mathbb{N}}$, where $H_{\phi_n} : \hat{\mathcal{H}}_{\lambda,m} \to \hat{\mathcal{H}}_{\lambda,m}$ is given by $H_{\phi_n}(f) = \phi_n * f$, is not uniformly bounded. An application of the uniform boundedness principle now completes the proof. □

4. CONVERGENCE OF GENERALIZED CESÀRO SUM

In this section, we provide an alternative proof of Theorem 1.1 using the method of induction. For any $f \in \mathcal{O}(\mathbb{D})$, let Lf be the function in $\mathcal{O}(\mathbb{D})$ defined by

$$Lf(z) := \frac{f(z) - f(0)}{z}, \quad z \in \mathbb{D}.$$

It is known that for any $\mu \in \mathcal{M}_+(\mathbb{T})$ and $m \in \mathbb{N}$, the inequality $D_{\mu,m}(Lf) \leqslant D_{\mu,m}(f)$ holds for every $f \in \mathcal{H}_{\mu,m}$, see [9, Lemma 2.9]. Moreover, from [9, Lemma 2.10] and [9, Corollary 3.3], we have the following result which will be crucial for the proof of Theorem 1.1 presented in this section.

Lemma 4.1. *Let $m \geqslant 1$ and $\mu \in \mathcal{M}_+(\mathbb{T})$. Then for any function f in $\mathcal{H}_{\mu,m}$, we have*

$$\sum_{k=1}^{\infty} D_{\mu,j}(L^k f) = D_{\mu,j+1}(f), \quad 0 \leqslant j \leqslant m-1.$$

Now we start with the following proposition which describes a relationship between $L^j(\sigma_n^\alpha[f])$ and $\sigma_{n-j}^\alpha[L^j f]$ for any function $f \in \mathcal{O}(\mathbb{D})$.

Proposition 4.2. *For every $f \in \mathcal{O}(\mathbb{D})$, $n \in \mathbb{Z}_{\geqslant 0}$, $\alpha \geqslant 0$, and $j \in \mathbb{N}$, we have*

$$L^j(\sigma_n^\alpha[f]) = \begin{cases} \frac{\Gamma(n+1)\Gamma(n+\alpha-j+1)}{\Gamma(n-j+1)\Gamma(n+\alpha+1)} \sigma_{n-j}^\alpha[L^j f], & j \leqslant n, \\ 0, & j > n. \end{cases}$$

Proof. Since for any $f \in \mathcal{O}(\mathbb{D})$ and $n \in \mathbb{Z}_{\geqslant 0}$, $\sigma_n^\alpha[f]$ is a polynomial of degree at most n, it follows that $L^j(\sigma_n^\alpha[f]) = 0$ whenever $j > n$. For the remaining case, let $f \in \mathcal{O}(\mathbb{D})$. Note that

$$\sigma_n^\alpha[f](z) = \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \sum_{k=0}^{n} \frac{\Gamma(n-k+\alpha+1)}{\Gamma(n-k+1)} a_k z^k, \quad n \in \mathbb{Z}_{\geqslant 0},\ \alpha \geqslant 0.$$

Operating L on both sides, we get

$$L(\sigma_n^\alpha[f])(z) = \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \sum_{k=0}^{n-1} \frac{\Gamma(n-k+\alpha)}{\Gamma(n-k)} a_{k+1} z^k, \quad n \in \mathbb{N},\ \alpha \geqslant 0.$$

Hence it follows that

$$L(\sigma_n^\alpha[f]) = \begin{cases} \frac{\Gamma(n+1)\Gamma(n+\alpha)}{\Gamma(n+\alpha+1)\Gamma(n)} \sigma_{n-1}^\alpha[Lf], & n \geqslant 1, \\ 0, & n = 0. \end{cases}$$

This completes the proof of the proposition for the case $j = 1$. Fix $k \in \mathbb{N}$ with $k < n$. Assume that the statement of the proposition holds for $j = k$. Then applying the induction hypothesis for the function Lf we obtain that for any $m \in \mathbb{Z}_{\geqslant 0}$ satisfying $m \geqslant k$,

$$L^k(\sigma_m^\alpha[Lf]) = \frac{\Gamma(m+1)\Gamma(m+\alpha-k+1)}{\Gamma(m-k+1)\Gamma(m+\alpha+1)}\, \sigma_{m-k}^\alpha[L^{k+1}f].$$

Since $L(\sigma_{m+1}^\alpha[f]) = \frac{\Gamma(m+2)\Gamma(m+\alpha+1)}{\Gamma(m+\alpha+2)\Gamma(m+1)}\, \sigma_m^\alpha[Lf]$ for every $m \in \mathbb{Z}_{\geqslant 0}$, we obtain that

$$L^{k+1}(\sigma_{m+1}^\alpha[f]) = \frac{\Gamma(m+2)\Gamma(m+\alpha-k+1)}{\Gamma(m+\alpha+2)\Gamma(m-k+1)}\, \sigma_{m-k}^\alpha[L^{k+1}f],\ m \geqslant k.$$

This gives us that for any $n \in \mathbb{Z}_{\geqslant 0}$,

$$L^{k+1}(\sigma_n^\alpha[f]) = \begin{cases} \frac{\Gamma(n+1)\Gamma(n+\alpha-k)}{\Gamma(n+\alpha+1)\Gamma(n-k)}\, \sigma_{n-k-1}^\alpha[L^{k+1}f], & k+1 \leqslant n, \\ 0, & k+1 > n. \end{cases}$$

This finishes the induction step for $j = k+1$ and completes the proof of the proposition. □

Alternative proof of Theorem 1.1. Let $\mu \in \mathcal{M}_+(\mathbb{T})$, $n \in \mathbb{N}$, and $\alpha > \frac{1}{2}$. We will prove this theorem by induction on m. From [17, Theorem 1.1], it follows that there exists a constant M_α, independent of n, such that $D_{\mu,1}(\sigma_n^\alpha[f]) \leqslant M_\alpha D_{\mu,1}(f)$ for every $f \in \mathcal{H}_{\mu,1}$. Let us assume that $D_{\mu,m}(\sigma_n^\alpha[f]) \leqslant M_\alpha D_{\mu,m}(f)$ for every $f \in \mathcal{H}_{\mu,m}$, and for $m = 1, \ldots, k$. Now take $f \in \mathcal{H}_{\mu,k+1}$. Since $\sigma_n^\alpha[f]$ is a polynomial of degree at most n, from Lemma 4.1, it follows that

$$D_{\mu,k+1}(\sigma_n^\alpha[f]) = \sum_{j=1}^{n} D_{\mu,k}(L^j(\sigma_n^\alpha[f])).$$

Now applying Proposition 4.2 and the induction hypothesis, we obtain that

$$\begin{aligned} D_{\mu,k+1}(\sigma_n^\alpha[f]) &= \sum_{j=1}^{n} \left(\frac{\Gamma(n+1)\Gamma(n+\alpha-j+1)}{\Gamma(n-j+1)\Gamma(n+\alpha+1)} \right)^2 D_{\mu,k}(\sigma_{n-j}^\alpha[L^j f]) \\ &\leqslant \sum_{j=1}^{n} D_{\mu,k}(\sigma_{n-j}^\alpha[L^j f]) \leqslant M_\alpha \sum_{j=1}^{n} D_{\mu,k}(L^j f) \\ &\leqslant M_\alpha \sum_{j=1}^{\infty} D_{\mu,k}(L^j f) = M_\alpha D_{\mu,k+1}(f). \end{aligned}$$

This completes the induction step for $m = k + 1$ and the proof of the theorem. □

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