Opuscula Math. 40, no. 2 (2020), 271–290
https://doi.org/10.7494/OpMath.2020.40.2.271 Opuscula Mathematica

Dedicated to Mr. Akbar Javadi,
a teacher of mathematics and humanity

SUBDIVISION OF HYPERGRAPHS AND THEIR COLORINGS

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Communicated by Gyula O.H. Katona

Abstract. In this paper we introduce the subdivision of hypergraphs, study their properties and parameters and investigate their weak and strong chromatic numbers in various cases.

Keywords: hypergraph, uniform hypergraph, subdivision of hypergraph.

Mathematics Subject Classification: 05C15.

1. INTRODUCTION

Hypergraphs are not as common as graphs, but they do arise in many application areas. For example hypergraphs are used in VLSI design for circuit visualization [10, 18] and also appear in computational biology [14, 17] and social networks [6].

Throughout this paper, unless stated otherwise, we use the terminology of [4, 20] for hypergraphs. We also assume that the basic definitions from graph theory are familiar to the reader but for necessary definitions and notations we refer the reader to textbook [5]. To shorten notation, we often write $[k]$ instead of $\{1, 2, \dots, k\}$.

Let $X = \{x_1, x_2, \dots, x_n\}$ be a finite set, and $\mathcal{D} = \{D_1, D_2, \dots, D_m\}$ be a family of subsets of X. The pair $\mathcal{H} = (X, \mathcal{D})$ is called a *hypergraph* with *vertex set* X also denoted by $\mathcal{V}(\mathcal{H})$ and with *edge set* $\mathcal{D}$ also denoted by $\mathcal{D}(\mathcal{H})$. $|X| = n$ is called the *order* of the hypergraph and $|\mathcal{D}| = m$ is called the *size* of the hypergraph. The elements $x_1, x_2, \dots, x_n$ are called the *vertices* and the sets $D_1, D_2, \dots, D_m$ are called the *edges*. The *rank* of $\mathcal{H}$, denoted by $r(\mathcal{H})$, is the maximum size of any of the edges of $\mathcal{H}$. If a hypergraph $\mathcal{H}$ has no multiple edges and all its edges are of size r, then $\mathcal{H}$ is called *r-uniform hypergraph*. From this point of view, a simple graph is a 2-uniform hypergraph. If, in a hypergraph $\mathcal{H}$, the degree of each of its vertices is equal to k, then $\mathcal{H}$ is called *k-regular hypergraph*. For $1 \leq r \leq n$, we define the *complete r-uniform* *hypergraph* to be the hypergraph $\mathcal{K}_n^r = (X, \mathcal{D})$ such that $|X| = n$ and $\mathcal{D}(\mathcal{K}_n^r) = \binom{X}{r}$ which denotes the set of all r-subsets of X. Thus a complete graph on n vertices is a complete 2-uniform hypergraph $\mathcal{K}_n^2$ also denoted by K_n. Also in special case, $\mathcal{K}_n^n$ is a hypergraph with one edge of size n.

In a hypergraph $\mathcal{H} = (X, \mathcal{D})$, an alternating sequence $x_0 D_1 x_1 D_2 x_2 \ldots x_{t-1} D_t x_t$ of distinct vertices x_0, x_1, x_2, …, x_t and distinct edges D_1, D_2, D_3, …, D_t satisfying $x_{i-1}, x_i \in D_i$, $i = 1, 2, \ldots, t$, is called a *path* of length t connecting the vertices x_0 and x_t, or, equivalently, (x_0, x_t)-path. In addition, an alternating sequence $x_1 D_1 x_2 D_2 x_3 \ldots x_t D_t x_1$ of distinct vertices x_1, x_2, …, x_t and distinct edges D_1, D_2, D_3, …, D_t satisfying $\{x_i, x_{i+1}\} \subseteq D_i$, where $1 \leq i < t$ and $\{x_t, x_1\} \subseteq D_t$, is called a *cycle* of length t.

We refer to a cycle with k edges as a k-cycle, and denote the family of all k-cycles by C_k. For example, a 2-cycle consists of a pair of vertices and a pair of edges such that the pair of vertices is a subset of each edge. The girth of a hypergraph $\mathcal{H}$, containing a cycle, is the minimum length of a cycle in $\mathcal{H}$. In addition, the minimum length of a cycle in $\mathcal{H}$ that contains vertex x is called x-girth of $\mathcal{H}$ and is denoted by $g_x(\mathcal{H})$.

The coloring of hypergraphs started in 1966 when P. Erdős and A. Hajnal [9] introduced the notions of coloring and of the chromatic number of a hypergraph and obtained the first important result about the minimum number of edges in uniform hypergraphs that are not 2-colorable. For further information see [1, 3, 12, 19].

There are two basic colorings of hypergraphs. Let $[\lambda]$ be the set of colors. A *proper λ-coloring* of a hypergraph $\mathcal{H} = (X, \mathcal{D})$ is a labeling of its vertices X with the colors $[\lambda]$ in such a way that every edge D of size at least two has at least two vertices colored differently. This proper λ-coloring is called a *weak coloring* of a hypergraph. The minimum λ for which there exists a proper λ-coloring is called the (*weak*) *chromatic number* of $\mathcal{H}$ and is denoted by $\chi(\mathcal{H})$.

A *strong λ-coloring* of $\mathcal{H}$ is a partition of X into λ strong stable sets S_i, $i = 1, 2, \ldots, \lambda$, such that $|D \cap S_i| \leq 1$ holds for every $D \in \mathcal{D}(\mathcal{H})$ and for every i. The *strong chromatic number* $\gamma(\mathcal{H})$ is the smallest λ for which there exists a strong λ-coloring of $\mathcal{H}$. Evidently the strong and weak coloring coincide when $\mathcal{H}$ is a graph.

There are several operations that produce new graphs from old ones such as products of graphs and complement of a graph. To produce a new graph with the same topological properties, we use the operation of subdivision of a graph. This is an important notion in graph theory, for example, the celebrated theorem of Kuratowski uses it to characterize planar graphs. In [11], the author introduced fractional power of a graph by use of k-subdivision. For any $k \in \mathbb{N}$, the *k-subdivision* of graph G, denoted by $G^{\frac{1}{k}}$, is constructed by replacing each edge xy of G with a path of length k. The aim of this paper is to introduce the subdivision of hypergraphs and study some of their properties.

1.1. ORGANISATION OF THE PAPER

In the following section, Section 2, we describe the idea of the subdivision of a hypergraph and give its definition in Subsection 2.1. Then in Subsection 2.2, we derive exact formulas for the order, size, the degrees of the vertices and the distances between the vertices of a subdivision of a hypergraph in terms of the parameters of the original hypergraph. We also present upper and lower bounds on the diameter of a subdivision of a hypergraph and calculate x-girth and the girth of these graphs in terms of the parameters of the original hypergraph. In Section 3, we investigate the weak and strong colorings of a subdivision of a hypergraph and prove tight lower and upper bound on the weak and strong chromatic numbers of these hypergraphs. In addition we suggest some new research problems.

2. SUBDIVISION OF A HYPERGRAPH

In topology, there are various kinds of simplex subdivisions such as *edgewise subdivision* [7], *jet subdivision* [21], *chromatic subdivision* [13], *interval subdivision* [16], and *barycentric subdivision* [22]. Consider the $(n-1)$-dimensional simplex defined by

$$\Delta = \Big\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0, \sum_{i=1}^{n} x_i = 1\Big\}.$$

If we represent an edge of size n with the set of lattice points of Δ, then any topological subdivision of Δ, give us a subdivision of that edge.

In [8] and [15], the authors introduced a hypergraph related to a simplex as follows: Let $m \geq 1$ be an integer and consider the $(n-1)$-dimensional simplex defined by

$$\Delta_{n,m} = \Big\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0, \sum_{i=1}^{n} x_i = m\Big\}.$$

Consider $V_{n,m} = \mathbb{Z}^n \cap \Delta_{n,m}$, the set of all the points in $\Delta_{n,m}$ with integer coordinates. The *simplex-lattice hypergraph* is a n-uniform hypergraph $\mathcal{H}_{n,m} = (V_{n,m}, E_{n,m})$ whose edges are indexed by the elements of $V_{n,m-1}$: we have $E_{n,m} = \{u + V_{n,1} : u \in V_{n,m-1}\}$, where $u + V := \{u + v : v \in V\}$. For example, $\mathcal{H}_{n,1} = \mathcal{K}_n^n$ and $\mathcal{H}_{2,m}$ is a path of size m or $K_2^{\frac{1}{m}}$. We use this n-uniform hypergraph to defining the sudivision of hypergraph.

2.1. DEFINITIONS

Let $n \in \mathbb{N}$ and G be a graph. We know that the n-subdivision of graph G is a simple graph $G^{\frac{1}{n}}$, which is constructed by replacing each edge of G with a path of length n. As with many definitions and problems in graph theory, it seems natural attempt a generalisation to hypergraphs. To generalize the concept of subdivision of graph, we need to define the concept of subdivision of an edge of a hypergraph.

As you see, we call $V_{n,m}$ the set of nonnegative integer solutions of the equation $\sum_{i=1}^{n} x_i = m$ which can be represented by a set of points of $\mathbb{R}^n$. We know that this equation has $\binom{m+n-1}{n-1}$ nonnegative integer solutions (see Theorem 1.11 in [2]). For example, $V_{n,0} = \{(0, \ldots, 0, 0)\}$ and $V_{n,1}$ is the set of unite standard points $e_1 = (1, 0, 0, \ldots, 0)$, $e_2 = (0, 1, 0 \ldots, 0)$, $\ldots$, and $e_n = (0, 0, \ldots, 0, 1)$.

Definition 2.1. Let S be a set of points of $\mathbb{R}^n$. S is called a d-equidistant set of $\mathbb{R}^n$ if the distance between any two points of S is equal to d.

Note that the Euclidean distance of any two points of $V_{n,1}$ is $\sqrt{2}$ and so $V_{n,1}$ is a $\sqrt{2}$-equidistant set of $\mathbb{R}^n$. In Theorem 2.2 we characterize all maximal $\sqrt{2}$-equidistant sets of $V_{n,m}$. Let $u + \alpha V := \{u + \alpha v : \ v \in V\}$.

Theorem 2.2. *Let $n, m \in \mathbb{N}$, $n \geq 2$ and S be a $\sqrt{2}$-equidistant subset of $V_{n,m}$. Then*

(i) $|S| \leq n$,

(ii) *if $|S| = n$ then $S = u + V_{n,1}$ or $S = u' - V_{n,1}$, where $u = (u_1, u_2, \ldots, u_n) \in V_{n,m-1}$ and $u' = (u'_1, u'_2, \ldots, u'_n)$ is a positive integer solution of $\sum_{i=1}^{n} x_i = m + 1$,*

(iii) *there are exactly $\binom{m+n-2}{n-1}$ n-sets S of the form $S = u + V_{n,1}$ and $\binom{m}{n-1}$ n-sets S of the form $S = u' - V_{n,1}$.*

Proof. (i) Suppose that $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ are two nonnegative integer solutions of the equation $\sum_{i=1}^{n} z_i = m$. We have

$$d(X, Y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} = \sqrt{2}.$$

Therefore, there are exactly two indices $p, q \in [n]$ such that $Y = X + e_p - e_q$, where e_p and e_q were introduced before. Now suppose that $S = \{X^1, X^2, \ldots, X^k\}$ and so for every $i \in \{2, 3, \ldots, k\}$ we have $X^i = X^1 + e_{p_i} - e_{q_i}$, where $p_i, q_i \in [n]$. Because

$$d(X^i, X^j) = \|e_{p_i} - e_{q_i} - e_{p_j} + e_{q_j}\| = \sqrt{2},$$

it follows that $p_i = p_j$, $q_i \neq q_j$ or $p_i \neq p_j$, $q_i = q_j$. Now one can easily conclude that there are only two cases for the elements of S: either $p_2 = p_3 = \ldots = p_k$ and $q_i \neq q_j$ for each two indices $i, j \in \{2, 3, \ldots, k\}$ or $q_2 = q_3 = \ldots = q_k$ and $p_i \neq p_j$ for each two indices $i, j \in \{2, 3, \ldots, k\}$. Therefore, $|\{p_2, p_3, \ldots, p_k, q_2, q_3, \ldots, q_k\}| \leq n$ and so $k \leq n$.

(ii) In the first case ($p = p_2 = p_3 = \ldots = p_n$ and $q_i \neq q_j$), let $u' = X^1 + e_p$ and in the second case ($q = q_2 = q_3 = \ldots = q_n$ and $p_i \neq p_j$), let $u = X^1 - e_q$. One can easily check that in the first case $S = u' - V_{n,1}$ and in the second case $S = u + V_{n,1}$. In addition,

$$u' = X^1 + e_p = X^2 + e_{q_2} = X^3 + e_{q_3} = \ldots = X^n + e_{q_n}$$

which shows that the components of u' are positive integers and

$$n \sum_{i=1}^{n} u'_i = \sum_{i=1}^{n} \Big(\sum_{j=1}^{n} X^i_j \Big) + n = nm + n.$$

So $\sum_{i=1}^{n} u'_i = m + 1$ and u' is a positive integer solution of the equation $x_1 + x_2 + \ldots + x_n = m + 1$. Similarly

$$u = X^1 - e_q = X^2 - e_{p_2} = X^3 - e_{p_3} = \ldots = X^k - e_{p_k}$$

which shows that the components of u are nonnegative integers and

$$n\sum_{i=1}^{n} u_i = \sum_{i=1}^{n}\Big(\sum_{j=1}^{n} X_j^i\Big) - n = nm - n.$$

So $\sum_{i=1}^{n} u_i = m-1$ and u is a nonnegative integer solution of the equation $x_1 + x_2 + \ldots + x_n = m-1$.

(iii) In fact, $\binom{m+n-2}{n-1}$ is the number of nonnegative integer solutions of the equation $\sum_{i=1}^{n} x_i = m-1$ and $\binom{m}{n-1}$ is the number of positive integer solutions of the equation $\sum_{i=1}^{n} x_i = m+1$. $\square$

Remark 2.3.

(i) When $n=2$, the set of $\sqrt{2}$-equidistant subsets of the form $u + V_{n,1}$ is equal to the set of $\sqrt{2}$-equidistant subsets of the form $u' - V_{n,1}$.

(ii) Geometrically any maximal $\sqrt{2}$-equidistant subset of the form $u + V_{n,1}$ is a transferred copy of $V_{n,1}$ without rotation.

We can use these maximal $\sqrt{2}$-equidistant sets of the form $u + V_{n,1}$ to define m-subdivision of an edge of size n.

Definition 2.4. Let $\mathcal{H}$ be a hypergraph, $n, m \in \mathbb{N}$ and D be an edge of $\mathcal{H}$ of size n. If we represent the vertices of D with the points of $V_{n,1}$ in $\mathbb{R}^n$, then in m-subdivision of D, we replace D with a copy of $\mathcal{H}_{n,m}$ such that each vertex e_i of D is replaced by the vertex which assigned by $m.e_i$ in $V_{n,m}$. Precisely, If $D = \{v_1, v_2, \ldots, v_n\}$ then the vertices of $D^{\frac{1}{m}}$ are denoted by v_x, where $x = (x_1, x_2, \ldots, x_n) \in V_{n,m}$ such that $v_i = v_{m\cdot e_i}$ for any $i \in [n]$ and

$$\mathcal{D}(D^{\frac{1}{m}}) = \{D_u : u \in V_{n,m-1}\},$$

where

$$D_u = \{v_x : \; x \in u + V_{n,1}\}.$$

Example 2.5. Suppose that D is an edge of size 3 from the hypergraph $\mathcal{H}$. To define $D^{\frac{1}{2}}$, we represent the vertices of D with $\{v_{(1,0,0)}, v_{(0,1,0)}, v_{(0,0,1)}\}$. Now the set of vertices of $D^{\frac{1}{2}}$ is

$$\mathcal{V}(D^{\frac{1}{2}}) = \{v_{(2,0,0)}, v_{(0,2,0)}, v_{(0,0,2)}, v_{(1,1,0)}, v_{(1,0,1)}, v_{(0,1,1)}\}$$

and we replace D with $C(3,2) = 3$ edges as follows (see Figure 1):

$$\begin{aligned}
D_{(1,0,0)} &= \{v_x \;:\; x \in (1,0,0) + V_{n,1}\} = \{v_{(2,0,0)}, v_{(1,1,0)}, v_{(1,0,1)}\},\\
D_{(0,1,0)} &= \{v_x \;:\; x \in (0,1,0) + V_{n,1}\} = \{v_{(1,1,0)}, v_{(0,2,0)}, v_{(0,1,1)}\},\\
D_{(0,0,1)} &= \{v_x \;: x \in (0,0,1) + V_{n,1}\} = \{v_{(1,0,1)}, v_{(0,1,1)}, v_{(0,0,2)}\}.
\end{aligned}$$

Note that in $D^{\frac{1}{2}}$, we replace the vertices $v_{(1,0,0)}$, $v_{(0,1,0)}$ and $v_{(0,0,1)}$ of D consequently with the vertices $v_{(2,0,0)}$, $v_{(0,2,0)}$ and $v_{(0,0,2)}$ (we call them the vertices of D in $D^{\frac{1}{2}}$) and then we add some new vertices between them such that the distance between any two vertices of D in $D^{\frac{1}{2}}$ is two.

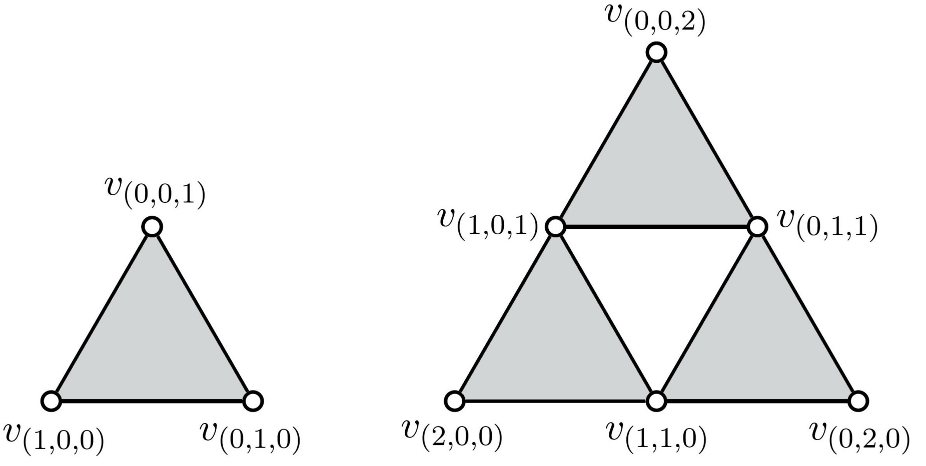

Fig. 1. D and $D^{\frac{1}{2}}$

Definition 2.6. Let $\mathcal{H}$ be a hypergraph. For any mapping $f : \mathcal{D}(\mathcal{H}) \to \mathbb{N}$, the *$f$-subdivision of* $\mathcal{H}$, denoted by $\mathcal{H}^{\frac{1}{f}}$, is constructed by replacing each edge D of $\mathcal{H}$ with its $f(D)$-subdivision. In special case, when $f(D) = k$ for each edge D, we use $\mathcal{H}^{\frac{1}{k}}$ instead of $\mathcal{H}^{\frac{1}{f}}$. Additionally, $\mathcal{H}^{\frac{1}{f}}$ is called *fully subdivided* if $f(D) \geq 2$ for each edge D of $\mathcal{H}$.

Each vertex of $\mathcal{H}$ in $\mathcal{H}^{\frac{1}{f}}$ is called a *terminal* vertex, and each of the remained vertices of $\mathcal{H}^{\frac{1}{f}}$ is called an *internal* vertex. In addition, for each $D \in \mathcal{D}(\mathcal{H})$, the sets of terminal vertices and internal vertices of $D^{\frac{1}{n}}$ and $\mathcal{H}^{\frac{1}{f}}$ is denoted by $\mathcal{V}_t(D^{\frac{1}{n}})$, $\mathcal{V}_i(D^{\frac{1}{n}})$, $\mathcal{V}_t(\mathcal{H}^{\frac{1}{n}})$ and $\mathcal{V}_i(\mathcal{H}^{\frac{1}{n}})$ consequently and D is called the *underlying edge* of the vertex x, when $x \in \mathcal{V}_i(D^{\frac{1}{n}})$. Similarly, each edge of $\mathcal{H}^{\frac{1}{f}}$ is called an *internal* edge when all of its vertices are internal, and each of the remained edges of $\mathcal{H}^{\frac{1}{f}}$ is called a *terminal* edge. D is called the *underlying edge* of the edge $D' \in \mathcal{D}(\mathcal{H}^{\frac{1}{f}})$, when $f(D) = k$ and $D' \subseteq \mathcal{V}(D^{\frac{1}{k}})$. Easily one can show that

$$|\mathcal{V}_t(D^{\frac{1}{n}})| = |D| \quad \text{and} \quad |\mathcal{V}_i(D^{\frac{1}{n}})| = \binom{n+|D|-1}{|D|-1} - |D|.$$

Example 2.7. Suppose that $\mathcal{H}$ is a hypergraph with vertex set $\mathcal{V}(\mathcal{H}) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and edge set $\mathcal{D}(\mathcal{H}) = \{D_1, D_2, D_3\}$ such that $D_1 = \{v_1, v_2, v_3\}$, $D_2 = \{v_2, v_4\}$, $D_3 = \{v_3, v_4, v_5, v_6\}$. Also assume that $f : D(H) \to \mathbb{N}$ is a mapping with the following outputs:

$$f(D_1) = 3, \quad f(D_2) = 4, \quad f(D_3) = 2.$$

Then $\mathcal{H}^{\frac{1}{f}}$ has four edges of size 2, six edges of size 3 and four edges of size 4 as follows (see Figure 2):

$$
\begin{aligned}
D_{(2,0,0)} &= \{v_{(3,0,0)}, v_{(2,1,0)}, v_{(2,0,1)}\}, \quad D_{(0,2,0)} = \{v_{(0,3,0)}, v_{(1,2,0)}, v_{(0,2,1)}\},\\
D_{(0,0,2)} &= \{v_{(0,0,3)}, v_{(1,0,2)}, v_{(0,1,2)}\}, \quad D_{(1,1,0)} = \{v_{(2,1,0)}, v_{(1,2,0)}, v_{(1,1,1)}\},\\
D_{(1,0,1)} &= \{v_{(2,0,1)}, v_{(1,1,1)}, v_{(1,0,2)}\}, \quad D_{(0,1,1)} = \{v_{(1,1,1)}, v_{(0,2,1)}, v_{(0,1,2)}\},\\
D_{(3,0)} &= \{v_{(4,0)}, v_{(3,1)}\}, \quad D_{(0,3)} = \{v_{(0,4)}, v_{(1,3)}\},\\
D_{(2,1)} &= \{v_{(3,1)}, v_{(2,2)}\}, \quad D_{(1,2)} = \{v_{(2,2)}, v_{(1,3)}\},\\
D_{(1,0,0,0)} &= \{v_{(2,0,0,0)}, v_{(1,1,0,0)}, v_{(1,0,1,0)}, v_{(1,0,0,1)}\},\\
D_{(0,1,0,0)} &= \{v_{(1,1,0,0)}, v_{(0,2,0,0)}, v_{(0,1,1,0)}, v_{(0,1,0,1)}\},\\
D_{(0,0,1,0)} &= \{v_{(1,0,1,0)}, v_{(0,1,1,0)}, v_{(0,0,2,0)}, v_{(0,0,1,1)}\},\\
D_{(0,0,0,1)} &= \{v_{(1,0,0,1)}, v_{(0,1,0,1)}, v_{(0,0,1,1)}, v_{(0,0,0,2)}\}.
\end{aligned}
$$

Also we have $\mathcal{V}_t(\mathcal{H}^{\frac{1}{f}}) = \mathcal{V}(\mathcal{H})$, where

$$
\begin{aligned}
&v_1 = v_{(3,0,0)}, && v_2 = v_{(0,3,0)} = v_{(4,0)},\\
&v_3 = v_{(0,0,3)} = v_{(2,0,0,0)}, && v_4 = v_{(0,2,0,0)} = v_{(0,4)},\\
&v_5 = v_{(0,0,2,0)}, && v_6 = v_{(0,0,0,2)}.
\end{aligned}
$$

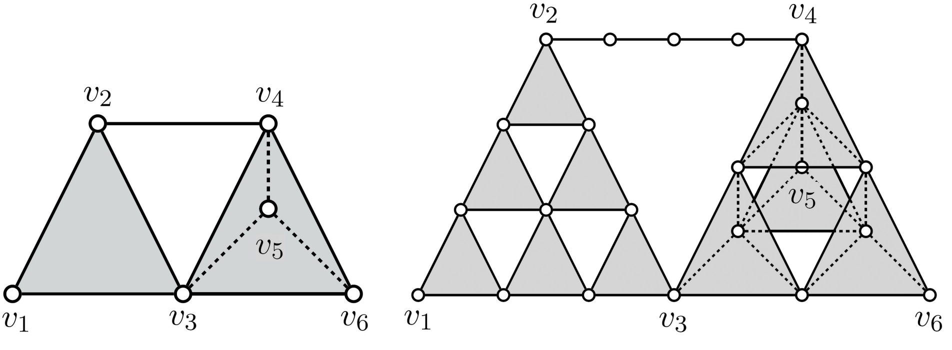

Fig. 2. $\mathcal{H}$ and $\mathcal{H}^{\frac{1}{f}}$

2.2. PROPERTIES

In this section, we calculate some of the basic parameters of a subdivision of a hypergraph in terms of the parameters of the original hypergraph. In the first theorem, we obtain the order and the size of these hypergraphs.

Theorem 2.8. *Let $\mathcal{H}$ be a hypergraph, f be a mapping from $\mathcal{D}(\mathcal{H})$ to $\mathbb{N}$ and $r, k \in \mathbb{N}$. Then*

(i)
$$n(\mathcal{H}^{\frac{1}{f}}) = \sum_{D\in\mathcal{D}} \binom{f(D)+|D|-1}{|D|-1} - \sum_{v\in\mathcal{V}} d(v) + n(\mathcal{H}),$$

(ii)
$$m(\mathcal{H}^{\frac{1}{f}}) = \sum_{D\in\mathcal{D}} \binom{f(D)+|D|-2}{|D|-1},$$

(iii)
$$n(\mathcal{H}^{\frac{1}{r}}) = m(\mathcal{H})\left(\binom{r+k-1}{k-1} - k\right) + n(\mathcal{H}) \quad \text{and} \quad m(\mathcal{H}^{\frac{1}{r}}) = m(\mathcal{H})\binom{r+k-2}{k-1}$$

when $\mathcal{H}$ is k-uniform.

Proof. (i) We know that

$$|\mathcal{V}_t(D^{\frac{1}{f(D)}})| = |D| \quad \text{and} \quad |\mathcal{V}_i(D^{\frac{1}{f(D)}})| = \binom{f(D)+|D|-1}{|D|-1} - |D|$$

for each $D \in \mathcal{D}(\mathcal{H})$. So

$$\begin{aligned} n(\mathcal{H}^{\frac{1}{f}}) &= |\mathcal{V}_i(\mathcal{H}^{\frac{1}{f}})| + |\mathcal{V}_t(\mathcal{H}^{\frac{1}{f}})| \\ &= \sum_{D\in\mathcal{D}} (|\mathcal{V}_i(D^{\frac{1}{f}})|) + n(\mathcal{H}) \\ &= \sum_{D\in\mathcal{D}} \left(\binom{f(D)+|D|-1}{|D|-1} - |D|\right) + n(\mathcal{H}) \\ &= \sum_{D\in\mathcal{D}} \binom{f(D)+|D|-1}{|D|-1} - \sum_{D\in\mathcal{D}} |D| + n(\mathcal{H}) \\ &= \sum_{D\in\mathcal{D}} \binom{f(D)+|D|-1}{|D|-1} - \sum_{v\in\mathcal{V}(\mathcal{H})} d(v) + n(\mathcal{H}). \end{aligned}$$

(ii) This part can be derived from the part (iii) of Theorem 2.2.
(iii) One can deduce it from the previous parts. □

In Theorem 2.9, we calculate the degrees of the vertices of a subdivision of a hypergraph.

Theorem 2.9. *Let $\mathcal{H}$ be a hypergraph, f be a mapping from $\mathcal{D}(\mathcal{H})$ to $\mathbb{N}$. Then*

(i) *the degree of each terminal vertex in $\mathcal{H}^{\frac{1}{f}}$ is equal to its degree in $\mathcal{H}$,*
(ii) *the degree of each internal vertex v_x in $\mathcal{H}^{\frac{1}{f}}$ is equal to the number of non-zero components of x.*

Proof. (i) Suppose that v is a terminal vertex of the edge $D \in \mathcal{D}(\mathcal{H})$. In the subhypergraph $D^{\frac{1}{f}} = (\mathcal{V}(D^{\frac{1}{f(D)}}), \mathcal{D}(D^{\frac{1}{f(D)}}))$ there is exactly one edge that contains v. So $d_{\mathcal{H}^{\frac{1}{f}}}(v) = d_{\mathcal{H}}(v)$.

(ii) Suppose that the internal vertex v_x belongs to the edge $D_v = \{v_x : x \in v + V_{k,1}\}$, where D is underlying edge of v, $|D| = k$, $f(D) = n$ and $v \in V_{k,n-1}$. So there exists $i \in [k]$ such that $x = v + e_i$ and therefore $x_i = v_i + 1$ that shows $x_i \in \mathbb{N}$. Thus, for any positive component x_j of x, we have $x = (x - e_j) + e_j$ and so $x \in (x - e_j) + V_{k,1}$ in which $x - e_j \in V_{k,n-1}$. This implies that

$$d(v_x) = |\{x - e_j + V_{k,1} : x_j \neq 0\}| = |\{j \in [k] : x_j \neq 0\}|. \qquad \square$$

In the next theorem, we find the distance between any two terminal vertices and internal vertices (with the same underlying edge) of a subdivision of a hypergraph. In a hypergraph $\mathcal{H}$, let $P_{\mathcal{H}}(x, y)$ be the set of all paths between the vertices x and y and $f(P) := \sum_{D \in P} f(D)$ for each path P of $\mathcal{H}$ and each mapping f from $\mathcal{D}(\mathcal{H})$ to $\mathbb{N}$. Also for any $D \in \mathcal{V}(\mathcal{H})$, let $D^{\frac{1}{f}}$ be a subhypergraph of $\mathcal{H}^{\frac{1}{n}}$ with vertex set $\mathcal{V}(D^{\frac{1}{f(D)}})$ and edge set $\mathcal{D}(D^{\frac{1}{f(D)}})$.

Theorem 2.10. *Let $\mathcal{H}$ be a hypergraph, f be a mapping from $\mathcal{D}(\mathcal{H})$ to $\mathbb{N}$ and $n \in \mathbb{N}$. Then*

(i) $d_{D^{\frac{1}{f}}}(v_x, v_y) = \frac{1}{2}\sum_{i=1}^{k} |x_i - y_i|$, *where* $x = (x_1, x_2, \ldots, x_k)$, $y = (y_1, y_2, \ldots, y_k)$, $D \in \mathcal{D}(\mathcal{H})$, $|D| = k$ *and* $v_x, v_y \in \mathcal{V}(D^{\frac{1}{f}})$,

(ii) $d_{D^{\frac{1}{f}}}(v_x, v_y) = f(D)$, *where* $v_x, v_y \in \mathcal{V}_t(D^{\frac{1}{f}})$,

(iii) $d_{\mathcal{H}^{\frac{1}{f}}}(v_x, v_y) = \min\{f(P) : P \in P_{\mathcal{H}}(v_x, v_y)\}$, *where* $v_x, v_y \in \mathcal{V}_t(\mathcal{H}^{\frac{1}{f}})$,

(iv) $d_{\mathcal{H}^{\frac{1}{n}}}(v_x, v_y) = n \cdot d_{\mathcal{H}}(v_x, v_y)$ *for any two terminal vertices* v_x *and* v_y.

Proof. (i) We proceed by induction. Since $\sum_{i=1}^{k} x_i = \sum_{i=1}^{k} y_i = f(D)$, $\sum_{i=1}^{k} |x_i - y_i|$ is always even. When $\sum_{i=1}^{k} |x_i - y_i| = 0$, trivially $d_{D^{\frac{1}{f}}}(v_x, v_y) = 0$. Also obviously we have $d_{D^{\frac{1}{f}}}(v_x, v_y) = 1$ when $\sum_{i=1}^{k} |x_i - y_i| = 2$. Now consider $\sum_{i=1}^{k} |x_i - y_i| = 2m \geq 4$. Because of $\sum_{i=1}^{k} x_i = \sum_{i=1}^{k} y_i = f(D)$, there are two indices i and j such that $x_i > y_i$ and $x_j < y_j$. Now considering the vertex v_z such that $z = x - e_i + e_j$, we have $d_{D^{\frac{1}{f}}}(v_x, v_z) = 1$ and $\sum_{i=1}^{k} |z_i - y_i| = 2m - 2$. Therefore, inductively we can conclude that $d_{D^{\frac{1}{f}}}(v_z, v_y) = m - 1$. Hence

$$d_{D^{\frac{1}{f}}}(v_x, v_y) \leq d_{D^{\frac{1}{f}}}(v_x, v_z) + d_{D^{\frac{1}{f}}}(v_y, v_z) = m.$$

To complete the proof, we will show that $d_{D^{\frac{1}{f}}}(v_x, v_y) \geq m$ and so $d_{D^{\frac{1}{f}}}(v_x, v_y) = m$. Suppose that $P = v_{x^0} v_{x^1} v_{x^2} \ldots v_{x^{l-1}} v_{x^l}$ is a $v_x v_y$-path with minimum length l, $x^0 = x$ and $x^l = y$. Therefore, $\sum_{i=1}^{k} |x_i^j - x_i^{j+1}| = 2$ for all $j \in \{0, 1, \ldots, l-1\}$. Now using triangle inequality, we get

$$2l = \sum_{j=0}^{l-1} \sum_{i=1}^{k} |x_i^j - x_i^{j+1}| \geq \sum_{i=1}^{k} |x_i - y_i| = 2m.$$

So $d_{D^{\frac{1}{f}}}(v_x, v_y) = l \geq m$.

(ii) This part can be derived from (i).

(iii) Suppose that $P_0 = v_{x_0} v_{x_1} v_{x_2} \ldots v_{x_{k-1}} v_{x_k}$ is a path between terminal vertices $x = v_{x_0}$ and $y = v_{x_k}$ in $\mathcal{H}^{\frac{1}{f}}$ with length k. This path contains some terminal and some internal vertices. Suppose that $v_{x_0}, v_{x_{i_1}}, v_{x_{i_2}}, \ldots, v_{x_{i_{l-1}}}$ and v_{x_k} are all of the terminal vertices of P_0 such that $i_0 = 0 < i_1 < i_2 < \ldots < i_{l-1} < i_l = k$. Therefore, the induced path of P_0 between $v_{x_{i_j}}$ and $v_{x_{i_{j+1}}}$ contains only two terminal vertices of Γ. So using Part (ii), we have

$$d_\Gamma(v_x, v_y) = k \geq \sum_{j=0}^{l-1} d_\Gamma(v_{x_{i_j}}, v_{x_{i_{j+1}}}) = \sum_{j=1}^{l} f(D_j),$$

where D_j is the edge that contains $v_{x_{i_{j-1}}}$ and $v_{x_{i_j}}$ and $v_x, D_1, v_{x_{i_1}}, D_2, \ldots, v_{x_{i_{l-1}}}, D_l, v_y$ is a path between v_x and v_y in $\mathcal{H}$. Thus $d_\Gamma(v_x, v_y) \geq \min\{f(P) : P \in P_{\mathcal{H}}(v_x, v_y)\}$. Similarly, for each path $P \in P_{\mathcal{H}}(v_x, v_y)$ there is a path $P' \in P_\Gamma(v_x, v_y)$ such that the length of P' is equal to $f(P)$. Therefore, $d_\Gamma(v_x, v_y) \leq f(P)$ for any path $P \in P_{\mathcal{H}}(v_x, v_y)$ and hence $d_\Gamma(v_x, v_y) = \min\{f(P) : P \in P_{\mathcal{H}}(v_x, v_y)\}$.

(iv) This part follows immediately by (iii). □

We need the following technical result in the proof of Theorem 2.12.

Lemma 2.11. *Let D be the only edge of $\mathcal{H} = \mathcal{K}_n^n$, $k \in \mathbb{N}$ and $\Gamma = \mathcal{H}^{\frac{1}{k}}$. Then*

$$\sum_{v_y \in \mathcal{V}_t(D^{\frac{1}{k}})} d_\Gamma(v_x, v_y) = (n-1)k$$

for every vertex $v_x \in \mathcal{V}(\Gamma)$.

Proof. Observe that

$$\begin{aligned}
\sum_{v_y \in \mathcal{V}_t(D^{\frac{1}{k}})} d_\Gamma(v_x, v_y) &= \sum_{v_y \in \mathcal{V}_t(D^{\frac{1}{k}})} \frac{1}{2} \sum_{i=1}^{n} |x_i - y_i| \\
&= \frac{1}{2} \sum_{i=1}^{n} \sum_{v_y \in \mathcal{V}_t(D^{\frac{1}{k}})} |x_i - y_i| \\
&= \frac{1}{2} \sum_{i=1}^{n} (k - x_i + (n-1)x_i) \\
&= \frac{1}{2}\Big(nk + (n-2)\sum_{i=1}^{n} x_i\Big) = (n-1)k.
\end{aligned}$$

□

In the following theorem, we establish tight bounds on the diameter of $\mathcal{H}^{\frac{1}{n}}$ and an exact formula for its girth in terms of the parameters of $\mathcal{H}$.

Theorem 2.12. *Let $\mathcal{H}$ be a hypergraph and $n \in \mathbb{N}$. Then*
(i) $n \cdot diam(\mathcal{H}) \leq diam(\mathcal{H}^{\frac{1}{n}}) \leq n \cdot diam(\mathcal{H}) + 2n(1 - \frac{1}{r(\mathcal{H})})$,
(ii) $g_v(\mathcal{H}^{\frac{1}{n}}) = n \cdot g_v(\mathcal{H})$ *for each terminal vertex* v,
(iii) $g(\mathcal{H}^{\frac{1}{n}}) = 3$ *for any* $n \in \mathbb{N} \setminus \{1\}$ *and any hypergraph* $\mathcal{H}$ *with rank at least three.*

Proof. (i) Suppose that x and y are two vertices of $\mathcal{H}$ such that $d_{\mathcal{H}}(x, y) = diam(\mathcal{H})$. By the last part of Theorem 2.10, we conclude

$$d_{\mathcal{H}^{\frac{1}{n}}}(x, y) = n \cdot d_{\mathcal{H}}(x, y) = n \cdot diam(\mathcal{H}).$$

Therefore, $diam(\mathcal{H}^{\frac{1}{n}}) \geq n \cdot diam(\mathcal{H})$. Now suppose that $d_{\mathcal{H}^{\frac{1}{n}}}(v_x, v_y) = diam(\mathcal{H}^{\frac{1}{n}})$, $v_x \in D_1$ and $v_y \in D_2$, where $D_1, D_2 \in D(\mathcal{H})$. Let P_{ij} be a path of minimum length between v_x and v_y which contains two terminal vertices $x_i \in \mathcal{V}_t(D_1)$ and $y_j \in \mathcal{V}_t(D_2)$. Let S be the sum of the length of the paths P_{ij}, where $1 \leq i \leq |D_1|$ and $1 \leq j \leq |D_2|$. So we get

$$\begin{aligned} S &= |D_2| \sum_{i=1}^{|D_1|} d_{\mathcal{H}^{\frac{1}{n}}}(v_x, x_i) + \sum_{i=1}^{|D_1|} \sum_{j=1}^{|D_2|} d_{\mathcal{H}^{\frac{1}{n}}}(x_i, y_j) + |D_1| \sum_{i=1}^{|D_2|} d_{\mathcal{H}^{\frac{1}{n}}}(y_j, v_y) \\ &\leq |D_2|(|D_1| - 1)n + |D_1||D_2|n \cdot diam(\mathcal{H}) + |D_1|(|D_2| - 1)n. \end{aligned}$$

Hence

$$|D_1||D_2| diam(\mathcal{H}^{\frac{1}{n}}) \leq S \leq |D_2|(|D_1| - 1)n + |D_1||D_2|n \cdot diam(\mathcal{H}) + |D_1|(|D_2| - 1)n.$$

Therefore,

$$\begin{aligned} diam(\mathcal{H}^{\frac{1}{n}}) &\leq \Big(1 - \frac{1}{|D_1|}\Big)n + n \cdot diam(\mathcal{H}) + \Big(1 - \frac{1}{|D_2|}\Big)n \\ &\leq n \cdot diam(\mathcal{H}) + 2n\Big(1 - \frac{1}{r(\mathcal{H})}\Big). \end{aligned}$$

(ii) Assume that $C = v_{x_0} v_{x_1} v_{x_2} \ldots v_{x_{k-1}} v_{x_k}$ is the shortest cycle with length k in $\mathcal{H}^{\frac{1}{n}}$ that contains terminal vertex $v = v_{x_0} = v_{x_k}$. This cycle contains some terminal and some internal vertices. Suppose that $v_{x_0}, v_{x_{i_1}}, v_{x_{i_2}}, \ldots, v_{x_{i_{l-1}}}, v_{x_{i_l}} = v_{x_k}$ are all of the terminal vertices of C such that $i_0 = 0 < i_1 < i_2 < \ldots < i_{l-1} < i_l = k$. Therefore the induced path of C between $v_{x_{i_j}}$ and $v_{x_{i_{j+1}}}$ contains only two terminal vertices of $\mathcal{H}^{\frac{1}{n}}$. So using Theorem 2.10, we have

$$k = \sum_{j=0}^{l-1} d_{\mathcal{H}^{\frac{1}{n}}}(v_{x_{i_j}}, v_{x_{i_{j+1}}}) = \sum_{j=0}^{l-1} n = nl,$$

where $C' = v_{x_{i_0}}, D_1, v_{x_{i_1}}, D_2, \ldots, v_{x_{i_{l-1}}}, D_l, v_{x_{i_l}}$ is a cycle in $\mathcal{H}$ that contains v. Therefore, $g_v(\mathcal{H}^{\frac{1}{n}}) \geq n \cdot g_v(\mathcal{H})$. Now suppose that C is a cycle of $\mathcal{H}$ of length $g_v(\mathcal{H})$ that contains v. Since trivially C can be extended to a cycle of length $n \cdot g_v(\mathcal{H})$ in $\mathcal{H}^{\frac{1}{n}}$ that contains v, therefore, $g_v(\mathcal{H}^{\frac{1}{n}}) \leq n \cdot g_v(\mathcal{H})$ and so $g_v(\mathcal{H}^{\frac{1}{n}}) = n \cdot g_v(\mathcal{H})$.

(iii) Suppose that D is an edge of size $k \geq 3$ in $\mathcal{H}$. Consider three vertices $u_1 = v_{(n-1,1,0,0,\ldots,0)}$, $u_2 = v_{(n-1,0,1,0,\ldots,0)}$ and $u_3 = v_{(n-2,1,1,0,\ldots,0)}$ and three edges $D_1 = D_{(n-1,0,0,\ldots,0)}$, $D_2 = D_{(n-2,0,1,0,\ldots,0)}$ and $D_3 = D_{(n-2,1,0,\ldots,0)}$ with underlying edge D. Clearly $C = u_1D_1u_2D_2u_3D_3u_1$ is a 3-cycle in $\mathcal{H}^{\frac{1}{n}}$ and so $g(\mathcal{H}^{\frac{1}{n}}) = 3$. □

Note that both bounds in part (i) of Theorem 2.12 are achievable. Suppose that $r, k, m \in \mathbb{N}$, $r \geq 2$ and $\mathcal{P}_n^r$ is the path with $n = k(r-1)+1$ vertices $x_1, x_2, \ldots, x_n$ and k edges $D_1, D_2, \ldots, D_k$ such that $D_1 = \{x_1, x_2, \ldots, x_r\}$ and $D_{i+1} = \{x_{ir-(i-1)}, x_{ir-i+2}, \ldots, x_{(i+1)r-i}\}$ whenever $1 \leq i \leq k-1$. Then P_n^r has diameter k and $(P_n^r)^{\frac{1}{m}}$ has diameter mk. To achieve the upper bound, consider complete r-uniform hypergraph $\mathcal{K}_n^r$ whenever $n \geq 2r$. We know that $diam(\mathcal{K}_n^r) = 1$ whereas

$$diam((\mathcal{K}_n^r)^{\frac{1}{kr}}) = kr + 2k(r-1) = kr + 2kr\Big(1 - \frac{1}{r(\mathcal{K}_n^r)}\Big),$$

that is the distance of two central internal vertices $x_{(k,k,\ldots,k)} \in D_1^{\frac{1}{kr}}$ and $y_{(k,k,\ldots,k)} \in D_2^{\frac{1}{kr}}$ with nonadjacent underlying edges D_1 and D_2.

3. COLORING OF SUBDIVISIONS OF A HYPERGRAPH

Our goal in this section is to discuss and study weak and strong colorings of the subdivisions of a hypergraph. At first, we consider the weak coloring of a hypergraph. It was mentioned in [11], when G is a connected graph and n is a positive integer greater than 1, then at most three colors are enough to achieve a proper coloring of $G^{\frac{1}{n}}$. Precisely,

$$\chi(G^{\frac{1}{n}}) = \begin{cases} 3, & n \equiv 1 (mod\, 2) \text{ and } \chi(G) \geq 3, \\ 2, & \text{otherwise.} \end{cases}$$

One can easily prove that any fully subdivided graph is 3-colorable. In fact, for any graph G and any mapping $f : E(G) \to \mathbb{N} \setminus \{1\}$, the set of terminal vertices of $G^{\frac{1}{f}}$ is an independent set and so we can properly color them with one color and the set of internal vertices of $G^{\frac{1}{f}}$ induces a disjoint union of paths and so we can properly color them with two colors. Therefore, $\chi(G^{\frac{1}{f}}) \leq 3$.

In Theorem 3.2, we show that the chromatic number of the subdivision of a hypergraph is equal to the chromatic number of one of its subhypergraphs. Before that, we prove the following useful lemma.

Lemma 3.1. *Let $n, k, s \in \mathbb{N}$, $n \geq 3$, $k \geq 2$ and $c_t : \mathcal{V}_t(G) \to \{0, 1, \ldots, s-1\}$ be a partial coloring of the terminal vertices of $G = (\mathcal{K}_n^n)^{\frac{1}{k}}$. Then c_t can be extended to a proper weak coloring of G such that the color of any internal vertex is 0 or 1.*

Proof. We know that $\mathcal{K}_n^n$ has one edge D of size n. We define $c(v_x) = c_t(v_x)$ for each terminal vertex v_x of $D^{\frac{1}{k}}$. In addition, we assign the color $(c(v_0) + d_G(v_u, v_0))(mod\ 2)$

to the internal vertex v_u of $D^{\frac{1}{k}}$, where $v_0 = v_{(k,0,\ldots,0)}$. Easily we can show that this coloring is proper. Firstly, consider the edge

$$D_{(k-1,0,\ldots,0)} = \{v_x : \ x \in (k-1,0,\ldots,0) + V_{n,1}\}$$

that contains v_0. In this edge, $c(v_u) \neq c(v_0)$ for any internal vertex $v_u \in D_{(k-1,0,\ldots,0)} \setminus \{v_0\}$, because $d_G(v_u, v_0) = 1$. Assume that $u + V_{n,1}$ is an edge of G, where $u = (u_1, u_2, \ldots, u_n) \in V_{n,k-1}$ and $u \neq (k-1, 0, \ldots, 0)$. Note that, because $k \geq 2$, any edge of G has at most one terminal vertex. Furthermore, any edge has at least two internal vertices, because $n \geq 3$. Since v_{u+e_1} is the internal vertex of D_u and

$$d_G(v_{u+e_1}, v_0) = k - u_1 - 1 \neq k - u_1 = d_G(v_{u+e_j}, v_0)$$

for any $j \neq 1$, we deduce that at least two colors appear on this edge of $D^{\frac{1}{k}}$. □

Theorem 3.2. *Let $\mathcal{H}$ be a non-empty hypergraph, f be a mapping from $\mathcal{D}(\mathcal{H})$ to $\mathbb{N}$ and*

$$\mathcal{D}_1 = \{D \in \mathcal{D} : f(D) = 1 \text{ or } |D| = 2\}.$$

Then

$$\chi(\mathcal{H}^{\frac{1}{f}}) = \max\{2, \chi(\mathcal{H}_1^{\frac{1}{f}})\},$$

where $\mathcal{H}_1$ is a subhypergraph of $\mathcal{H}$ with $\mathcal{V}(\mathcal{H}_1) = \mathcal{V}(\mathcal{H})$ and $\mathcal{D}(\mathcal{H}_1) = \mathcal{D}_1$.

Proof. Since $\mathcal{H}_1^{\frac{1}{f}}$ is a subhypergraph of $\mathcal{H}^{\frac{1}{f}}$, so $\chi(\mathcal{H}_1^{\frac{1}{f}}) \leq \chi(\mathcal{H}^{\frac{1}{f}})$. Now suppose that $\chi(\mathcal{H}_1^{\frac{1}{f}}) = m$ and $c : \mathcal{V}(\mathcal{H}_1^{\frac{1}{f}}) \to \{0, 1, \ldots, m-1\}$ is a proper coloring of $\mathcal{H}_1^{\frac{1}{f}}$. We claim that this coloring is extendable to a proper coloring of $\mathcal{H}^{\frac{1}{f}}$ by using two colors 0 and 1 for the uncolored vertices of $\mathcal{H}^{\frac{1}{f}}$, which completes the proof.

Suppose that D is an edge of $\mathcal{H}$ with $|D| \geq 3$ and $f(D) = k \geq 2$. Consider $D^{\frac{1}{k}}$. By Lemma 3.1, we can extend the partial coloring c of the terminal vertices of $D^{\frac{1}{k}}$ to a proper coloring c_D of $D^{\frac{1}{k}}$. Therefore, applying Lemma 3.1 for any edge of $\mathcal{D}(\mathcal{H}) \setminus \mathcal{D}_1$, leads us to a proper coloring of $\mathcal{H}^{\frac{1}{f}}$ with $\max\{2, \chi(\mathcal{H}_1^{\frac{1}{f}})\}$ colors. □

In the following corollary, we show that three colors are enough to achieve a proper coloring of any fully subdivided hypergraph.

Corollary 3.3. *Let $\mathcal{H}$ be a non-empty hypergraph and f be a mapping from $\mathcal{D}(\mathcal{H})$ to $\mathbb{N} \setminus \{1\}$. Then $\chi(\mathcal{H}^{\frac{1}{f}}) \leq 3$.*

Proof. Here we apply Theorem 3.2. Because $f(D) \geq 2$ for any edge $D \in \mathcal{D}(\mathcal{H})$, then $\mathcal{D}_1 = \{D \in \mathcal{D}\colon |D| = 2\}$. Therefore, $\mathcal{H}_1$ is a graph and so $\chi((\mathcal{H}_1)^{\frac{1}{f}}) \leq 3$. Now by applying Theorem 3.2, we conclude

$$\chi(\mathcal{H}^{\frac{1}{f}}) = \max\{2, \chi(\mathcal{H}_1^{\frac{1}{f}})\} \leq 3. \qquad \square$$

Definition 3.4. To *contract* an edge D of a hypergraph $\mathcal{H}$ is to delete the edge and then identify its vertices. The resulting hypergraph is denoted by $\mathcal{H}/D$. Let $\mathcal{D}'$ be a subset of $\mathcal{D}(\mathcal{H})$. If we contract all edges of $\mathcal{D}'$, then the resulting hypergraph is denoted by $\mathcal{H}/\mathcal{D}'$.

Theorem 3.5. *Let $\mathcal{H}$ be a non-empty hypergraph, f be a mapping from $\mathcal{D}(\mathcal{H})$ to $\mathbb{N}\setminus\{1\}$ and $\mathcal{H}_0$ be the subhypergraph of $\mathcal{H}$ induced by*

$$\mathcal{D}_0 = \{D \in \mathcal{D}(\mathcal{H}) : f(D) \equiv 0(mod\, 2), |D| = 2\}.$$

Then $\chi(\mathcal{H}^{\frac{1}{f}}) = 2$ if and only if:

(i) *each 2-edge of $\mathcal{D}(\mathcal{H})\setminus\mathcal{D}_0$ has at most one vertex in common with any connected component of $\mathcal{H}_0$ and*
(ii) *$\chi(\mathcal{H}') \leq 2$, where $\mathcal{H}'$ is constructed from $\mathcal{H}$ by removing all edges of size at least 3 and contracting all edges of $\mathcal{D}_0$.*

Proof. At first, suppose that $\chi(\mathcal{H}^{\frac{1}{f}}) = 2$ and $c : \mathcal{V}(\mathcal{H}^{\frac{1}{f}}) \to \{0,1\}$ is a proper weak 2-coloring of $\mathcal{H}^{\frac{1}{f}}$. If some 2-edge such as $D \in \mathcal{D}(\mathcal{H})\setminus\mathcal{D}_0$ has two vertices in common with the component C of $\mathcal{H}_0$, then there is a 2-uniform odd cycle in $\mathcal{H}^{\frac{1}{f}}$ which shows that $\chi(\mathcal{H}^{\frac{1}{f}}) \geq 3$, a contradiction. Therefore, condition (i) holds. Now we obtain a proper weak 2-coloring for $\mathcal{H}'$ by use of c. Because for any $D \in \mathcal{D}_0$, $D^{\frac{1}{f(D)}}$ is a path with even length so the colors of two terminal vertices of $D^{\frac{1}{f(D)}}$ are the same and for any 2-edge $D \in \mathcal{D}(\mathcal{H})\setminus\mathcal{D}_0$, the colors of two terminal vertices of $D^{\frac{1}{f(D)}}$ are different. So in contraction of each edge of $\mathcal{D}_0$, we identify some of vertices with the same color. Additionally, to construct $\mathcal{H}'$, we remove all edges of size at least three from $\mathcal{H}$ and so in colorig of $\mathcal{H}'$, we have no condition on the terminal vertices of these edges. Therefore, if we assign the color $c(v_x)$ to the vertex v_x of $\mathcal{H}'$ (that is, a terminal vertex of $\mathcal{H}^{\frac{1}{f}}$), the resulting coloring is proper.

Conversely, assume that $\chi(\mathcal{H}') \leq 2$ and $c' : \mathcal{V}(\mathcal{H}') \to \{0,1\}$ is a proper weak 2-coloring of $\mathcal{H}'$. To obtain a proper weak 2-coloring c of $\mathcal{H}^{\frac{1}{f}}$, at first we define a proper weak 2-coloring c'' of $\mathcal{H}_1^{\frac{1}{f}}$ as follows, where $\mathcal{H}_1$ was defined in Theorem 3.2: If in contraction of the edge D, two vertices of D are identified to a vertex v_D in $\mathcal{H}'$, then we define $c''(v) = c'(v_D)$ for any vertex v of D. Also for any other terminal vertex v of $\mathcal{H}_1^{\frac{1}{f}}$ we define $c''(v) = c'(v)$. Therefore, in c'', if $2 \mid f(D)$, two terminal vertices of $D^{\frac{1}{f(D)}}$ have the same color and we can color properly the internal vertices of $D^{\frac{1}{f(D)}}$ by using alternately the colors for internal vertices and if $2 \nmid f(D)$, two terminal vertices of $D^{\frac{1}{f(D)}}$ have different colors and again we can color properly the internal vertices of $D^{\frac{1}{f(D)}}$ by using alternately the colors for internal vertices. Therefore, $\chi(\mathcal{H}_1^{\frac{1}{f}}) \leq 2$ and by Theorem 3.2, we have $\chi(\mathcal{H}^{\frac{1}{f}}) = 2$. $\square$

Corollary 3.6. *Let $\mathcal{H}$ be a non-empty hypergraph and f be a mapping from $\mathcal{D}(\mathcal{H})$ to $\mathbb{N} \setminus \{1\}$ such that $2 \mid f(D)$ for any 2-edge D. Then $\chi(\mathcal{H}^{\frac{1}{f}}) = 2$.*

Corollary 3.7. *Let $\mathcal{H}$ be a non-empty hypergraph and $n \in \mathbb{N} \setminus \{1\}$. Then*

$$\chi(\mathcal{H}^{\frac{1}{n}}) = \begin{cases} 3, & 2 \nmid n \text{ and } \chi(\mathcal{H}') \geq 3, \\ 2, & \text{otherwise,} \end{cases}$$

where $\mathcal{H}'$ is constructed from $\mathcal{H}$ by removing all edges of size at least 3.

Observe that if $r(\mathcal{H}) = 2$ and $\mathcal{H}$ is a graph, then $\mathcal{H}_1 = \mathcal{H}$ and any proper weak coloring is a proper (graph) coloring. In this case, Corollary 3.7 generalizes the previous similar result that was published in [11].

Now we study strong chromatic number of fully subdivided hypergraphs. Obviously, for any hypergraph $\mathcal{H}$, $\gamma(\mathcal{H}) \geq r(\mathcal{H})$. We show that the strong chromatic number of $\mathcal{H}^{\frac{1}{f}}$ is equal to $r(\mathcal{H})$ or $r(\mathcal{H}) + 1$ for any full subdivision of $\mathcal{H}$.

Theorem 3.8. *Let $\mathcal{H}$ be a hypergraph with rank r and f be a mapping from $\mathcal{D}(\mathcal{H})$ to $\mathbb{N} \setminus \{1\}$. Then $\gamma(\mathcal{H}^{\frac{1}{f}}) \leq r + 1$. Specially $\gamma(\mathcal{H}^{\frac{1}{f}}) = r$ when $r \mid f(D)$ for each $D \in \mathcal{D}(\mathcal{H})$.*

Proof. Suppose that D is an edge of $\mathcal{H}$ of size s and $f(D) = k$ and consider the k-subdivision of D. We find a proper $(r + 1)$-coloring c_D with color set $\{0, 1, 2, \ldots, r\}$ for $D^{\frac{1}{k}}$ such that all its terminal vertices have the same color. We assign the color $c_D(v_x) = \sum_{i=1}^{s} ix_i \pmod r$ to the internal vertex $v_x = v_{(x_1,x_2,\ldots,x_s)}$ and the color r to each terminal vertex. Now we show that this coloring is proper. Suppose that $v_x = v_{(x_1,x_2,\ldots,x_s)}$ and $v_y = v_{(y_1,y_2,\ldots,y_s)}$ are adjacent vertices of $D^{\frac{1}{k}}$. Therefore, $x = u_0 + e_i$ and $y = u_0 + e_j$, where $i \neq j$ and $1 \leq i, j \leq s$. Then

$$c_D(x) - c_D(y) = i - j \pmod r \neq 0.$$

Therefore, c_D is a proper coloring of $D^{\frac{1}{k}}$. Because in this coloring of edges, all terminal vertices have the same color r, we can define a proper coloring c of $\mathcal{H}^{\frac{1}{f}}$ as follows. We use the color r for all terminal vertices and the color $c_D(v_x)$ for each internal vertex v_x of the edge $D^{\frac{1}{f(D)}}$.

Now suppose that $r \mid f(D)$ for every $D \in \mathcal{D}(\mathcal{H})$. In this case we assign the color $c_D(x) = \sum_{i=1}^{s} iu_i \pmod r$ to each vertex $v_u = v_{(u_1,u_2,\ldots,u_r)}$ of $D^{\frac{1}{f(D)}}$, where $|D| = s$. Since $r \mid f(D)$, the color of any terminal vertex is 0 and similar to the previous case, we can show that this coloring is proper. Finally, we can define a proper coloring of $\mathcal{H}$ by merging these colorings of edges. □

Corollary 3.9. *Let $\mathcal{H}$ be a hypergraph with rank r and $n \in \mathbb{N} \setminus \{1\}$. Then $\gamma(\mathcal{H}^{\frac{1}{n}}) \leq r+1$. Specially $\gamma(\mathcal{H}^{\frac{1}{n}}) = r$ when $r \mid n$.*

In Theorem 3.8 and Corollary 3.9, we proved that $r(\mathcal{H})+1$ colors are enough for the strong coloring of any full subdivision of a hypergraph and characterized some of the subdivisions of a hypergraph $\mathcal{H}$ such as $\mathcal{H}^{\frac{1}{f}}$ with strong chromatic number equal to $r(\mathcal{H})$. Therefore, the following problem arises naturally.

Problem 3.10. *Let $\mathcal{H}$ be a hypergraph. Characterize all subdivisions f of $\mathcal{H}$ with $\gamma(\mathcal{H}^{\frac{1}{f}}) = r(\mathcal{H}^{\frac{1}{f}})$.*

Also in the following two special cases, we have easier problems:

Problem 3.11. *Let $\mathcal{H}$ be a hypergraph. Characterize all fully subdivided hypergraph $\mathcal{H}^{\frac{1}{f}}$ with $\gamma(\mathcal{H}^{\frac{1}{f}}) = r(\mathcal{H}^{\frac{1}{f}})$.*

Problem 3.12. *Let $\mathcal{H}$ be a hypergraph. Characterize all $n \in \mathbb{N}$ for which we have $\gamma(\mathcal{H}^{\frac{1}{n}}) = r(\mathcal{H}^{\frac{1}{n}})$.*

In the next theorem, we obtain sufficient and necessary condition for a fully subdivided hypergraph of rank three to be strong 3-colorable. In order to prove the next theorem, we need the following lemma.

Lemma 3.13. *Let $k \in \mathbb{N} \setminus \{1\}$ and $c : \mathcal{V}_t((\mathcal{K}_3^3)^{\frac{1}{k}}) \to \{1, 2, 3\}$ be a partial coloring of $(\mathcal{K}_3^3)^{\frac{1}{k}}$.*

(i) *If $3 \mid k$ then c can be extended to a proper strong 3-coloring of $(\mathcal{K}_3^3)^{\frac{1}{k}}$ if and only if the colors of all terminal vertices are the same.*
(ii) *If $3 \nmid k$ then c can be extended to a proper strong 3-coloring of $(\mathcal{K}_3^3)^{\frac{1}{k}}$ if and only if the colors of any two terminal vertices are different.*

Proof. (i) Assume that $3 \mid k$. At first, suppose that $c(v_x) = c_0$ for any terminal vertex v_x of $(\mathcal{K}_3^3)^{\frac{1}{k}}$. To extend this partial coloring to a proper strong 3-coloring c', we color the vertex $v_{(x_1,x_2,x_3)}$ with color $c_0 + x_1 + 2x_2 \pmod 3$. So on each edge $D_{(y_1,y_2,y_3)}$ we have $c'(v_{(y_1+1,y_2,y_3)}) = c_0 + y_1 + 1 + 2y_2 \pmod 3$, $c'(v_{(y_1,y_2+1,y_3)}) = c_0 + y_1 + 2y_2 + 2 \pmod 3$ and $c'(v_{(y_1,y_2,y_3+1)}) = c_0 + y_1 + 2y_2 \pmod 3$ which show that c' is a proper strong 3-coloring. In addition, $c'(v_{(k,0,0)}) = c_0 + k \pmod 3 = c_0$, $c'(v_{(0,k,0)}) = c_0 + 2k \pmod 3 = c_0$ and $c'(v_{(0,0,k)}) = c_0$ which show that c' is an extension of c.
Conversely, suppose that $c(v_{(k,0,0)}) = c_1$, $c(v_{(0,k,0)}) = c_2$, $c(v_{(0,0,k)}) = c_3$ and a proper strong 3-coloring c' is an extension of c. Consider the shortest path between terminal vertices $v_{(k,0,0)}$ and $v_{(0,k,0)}$ whose vertices are

$$v_{(k,0,0)}, v_{(k-1,1,0)}, v_{(k-2,2,0)}, \ldots, v_{(1,k-1,0)}, v_{(0,k,0)}.$$

We show that the colors of any two vertices $v_{(k-i,i,0)}$ and $v_{(k-i-2,i+2,0)}$ of this path are different, where $0 \leq i \leq k-2$ and therefore,

$$c_1 = c(v_{(k,0,0)}) = c(v_{(k-3,3,0)}) = \ldots = c(v_{(0,k,0)}) = c_2.$$

Suppose that $c(v_{(k-i,i,0)}) = c(v_{(k-i-2,i+2,0)}) = a$ for some $i \in \{0, 1, \ldots, k-2\}$. Then $c(v_{(k-i-1,i+1,0)}) = b \neq a$ and so $a \neq c(v_{(k-i-1,i,1)}) \neq b$ and $a \neq c(v_{(k-i-2,i+1,1)}) \neq b$. This concludes $c(v_{(k-i-1,i,1)}) = c(v_{(k-i-2,i+1,1)})$. But $v_{(k-i-1,i,1)}$ and $v_{(k-i-2,i+1,1)}$ are adjacent and so c' is not proper, a contradiction (Figure 3). Similarly, $c_1 = c(v_{(k,0,0)}) = c(v_{(0,0,k)}) = c_3$. Therefore, $c_1 = c_2 = c_3$.

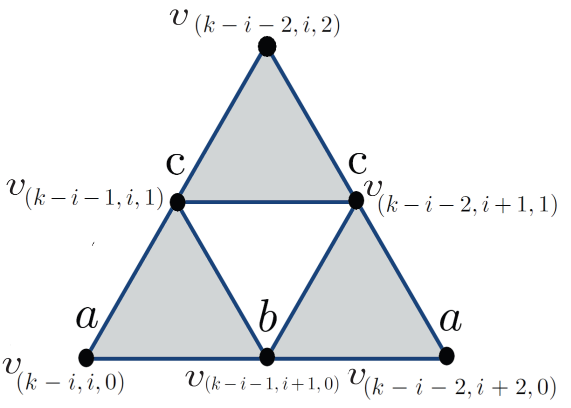

Fig. 3. $\mathcal{H}$ and $\mathcal{H}^{\frac{1}{f}}$

(ii) Assume that $k = 3q + r$, where $r \in \{1, 2\}$. At first, without loss of generality, suppose that $c(v_{(k,0,0)}) = 1$, $c(v_{(0,k,0)}) = 2$ and $c(v_{(0,0,k)}) = 0$. To extend this partial coloring to a proper strong 3-coloring c', we assign the color $rx_1 + 2rx_2 \pmod 3$ to the vertex $v_{(x_1,x_2,x_3)}$. So on each edge $D_{(y_1,y_2,y_3)}$ we have

$$c'(v_{(y_1+1,y_2,y_3)}) = ry_1 + r + 2ry_2 \pmod 3,$$
$$c'(v_{(y_1,y_2+1,y_3)}) = ry_1 + 2ry_2 + 2r \pmod 3$$

and

$$c'(v_{(y_1,y_2,y_3+1)}) = ry_1 + 2ry_2 \pmod 3$$

which show that c' is a proper strong 3-coloring. Additionally,

$$c'(v_{(k,0,0)}) = kr \pmod 3 = r^2 \pmod 3 = 1,$$
$$c'(v_{(0,k,0)}) = 2rk \pmod 3 = 2r^2 = 2 \pmod 3$$

and

$$c'(v_{(0,0,k)}) = 0$$

which show that c' is an extension of c.

Conversely, suppose that $c(v_{(k,0,0)}) = c_1$, $c(v_{(0,k,0)}) = c_2$, $c(v_{(0,0,k)}) = c_3$ and proper strong 3-coloring c' is an extension of c. Consider the shortest path between terminal vertices $v_{(k,0,0)}$ and $v_{(0,k,0)}$ whose vertices are

$$v_{(k,0,0)}, v_{(k-1,1,0)}, v_{(k-2,2,0)}, \ldots, v_{(1,k-1,0)}, v_{(0,k,0)}.$$

As we see, the colors of any two vertices $v_{(k-i,i,0)}$ and $v_{(k-i-2,i+2,0)}$ of this path are different, where $0 \le i \le k-2$. Therefore

$$c_1 = c(v_{(k,0,0)}) = c(v_{(k-3,3,0)}) = \ldots = c(v_{(r,k-r,0)}) \neq c(v_{(0,k,0)}) = c_2.$$

Similarly $c_1 \neq c_3$ and $c_2 \neq c_3$. $\square$

Theorem 3.14. *Let $\mathcal{H}$ be a hypergraph of rank 3, f be a mapping from $\mathcal{D}(\mathcal{H})$ to $\mathbb{N}\setminus\{1\}$ and $\mathcal{H}_0$ be the subhypergraph of $\mathcal{H}$ induced by*

$$\mathcal{D}_0 = \{D \in \mathcal{D}(\mathcal{H}) : f(D) \equiv 0 (mod\, 3), |D| = 3\}.$$

Then $\gamma(\mathcal{H}^{\frac{1}{f}}) = 3$ if and only if the following assertions hold:

(i) *each 3-edge of $\mathcal{D}(\mathcal{H}) \setminus \mathcal{D}_0$ has at most one vertex in common with any connected component of $\mathcal{H}_0$,*
(ii) *$\gamma(\mathcal{H}') \leq 3$, where $\mathcal{H}'$ is constructed from $\mathcal{H}$ by removing all edges of size 2 and contracting all edges of $\mathcal{D}_0$.*

Proof. Firstly, suppose that $\gamma(\mathcal{H}^{\frac{1}{f}}) = 3$ and $c : \mathcal{V}(\mathcal{H}^{\frac{1}{f}}) \to \{0, 1, 2\}$ is a proper strong coloring of $\mathcal{H}^{\frac{1}{f}}$. Let C be a connected component of $\mathcal{H}_0$ and D be a 3-edge of $\mathcal{D}(\mathcal{H}) \setminus \mathcal{D}_0$. By Lemma 3.13, we know that the colors of all terminal vertices of $D^{\frac{1}{f(D)}}$ are different. In addition, Lemma 3.13 shows that the colors of all terminal vertices of C are the same. So D has at most one vertex in common with C. Now we obtain a proper strong coloring for $\mathcal{H}'$ by use of c. For any $D \in \mathcal{D}_0$, the colors of all terminal vertices of $D^{\frac{1}{f(D)}}$ are the same. So in contraction of each edge of $\mathcal{D}_0$, we identify some of the vertices with the same color. In addition, to construct $\mathcal{H}'$, we remove all edges of size 2 from $\mathcal{H}$ and so in colorig of $(\mathcal{H}')^{\frac{1}{f}}$, we have no condition on the vertices of these edges. Therefore, if we assign the color $c(v_x)$ to the vertex v_x of $\mathcal{H}'$ (that is, a terminal vertex of $\mathcal{H}^{\frac{1}{f}}$), the resulting coloring is proper.

Conversely, assume that $\gamma(\mathcal{H}') \leq 3$ and $c' : \mathcal{V}(\mathcal{H}') \to \{0, 1, 2\}$ is a proper strong coloring of $\mathcal{H}'$. To obtain a proper strong 3-coloring c of $\mathcal{H}^{\frac{1}{f}}$, at first we define a partial coloring c'' of the terminal vertices of $\mathcal{H}^{\frac{1}{f}}$ as follows: If in contraction of the edge D, all vertices of D are identified to a vertex v_D in $\mathcal{H}'$, then we define $c''(v) = c'(v_D)$ for any vertex v of D. Also for any other terminal vertex v of $\mathcal{H}^{\frac{1}{f}}$ we define $c''(v) = c'(v)$. Therefore, in partial coloring c'', if $3 \mid f(D)$, all terminal vertices of $D^{\frac{1}{f(D)}}$ have the same color and if $3 \nmid f(D)$, any two terminal vertices of $D^{\frac{1}{f(D)}}$ have different colors. Now for any 3-edge D, we can extend c'' to a proper strong 3-coloring by Lemma 3.13.

Finally, we consider the internal vertices of the edges of size two. Suppose that $D = \{v, u\}$ is an edge of $\mathcal{H}$ and $f(D) = k \geq 2$. So $D^{\frac{1}{f(D)}}$ is a k-path between the terminal vertices v and u. If you consider two cases $c''(v) = c''(u)$ and $c''(v) \neq c''(u)$, it is not difficult to extend c'' to a proper strong 3-coloring of $D^{\frac{1}{k}}$. Therefore, we can define a proper strong 3-coloring of $\mathcal{H}^{\frac{1}{f}}$ by merging these colorings of subdivided edges. Therefore, $\gamma(\mathcal{H}^{\frac{1}{f}}) = 3$. □

Corollary 3.15. *Let $\mathcal{H}$ be a hypergraph with rank three and $n \in \mathbb{N}\setminus\{1\}$. Then*

$$\gamma(\mathcal{H}^{\frac{1}{n}}) = \begin{cases} 4, & 3 \nmid n \text{ and } \gamma(\mathcal{H}) \geq 4, \\ 3, & \text{otherwise.} \end{cases}$$

Proof. We consider the following three cases.
Case (i). $3 \mid n$. Applying Corollary 3.9, we deduce that $\gamma(\mathcal{H}^{\frac{1}{n}}) = 3$.

Case (ii). $3 \nmid n$ and $\gamma(\mathcal{H}) = 3$. In this case $\mathcal{D}_0 = \varnothing$ and so $\mathcal{H}'$ is a subhypergraph of $\mathcal{H}$ with rank 3. Therefore, $3 \leq \gamma(\mathcal{H}') \leq \gamma(\mathcal{H}) = 3$ and so $\gamma(\mathcal{H}') = 3$. Now by Theorem 3.14, we conclude that $\gamma(\mathcal{H}^{\frac{1}{n}}) = 3$.

Case (iii). $3 \nmid n$ and $\gamma(\mathcal{H}) \geq 4$. In this case $\mathcal{D}_0 = \varnothing$ and so $\mathcal{H}'$ is a subhypergraph of $\mathcal{H}$ with rank 3. If $\gamma(\mathcal{H}') = 3$, similar to the proof of Theorem 3.14, we can properly color the internal vertices of each 2-edge and extend any proper 3-coloring of $\mathcal{H}'$ to a proper 3-coloring of $\mathcal{H}$, a contradiction. Hence $\gamma(\mathcal{H}') \geq 4$. Again by Theorem 3.14, we conclude that $\gamma(\mathcal{H}^{\frac{1}{n}}) \geq 4$. In addition, by Corollary 3.9, we have $\gamma(\mathcal{H}^{\frac{1}{n}}) \leq 4$. Therefore, $\gamma(\mathcal{H}^{\frac{1}{n}}) = 4$. $\square$

Acknowledgements

The author is grateful to the referee for suggestions which improved the paper. Also the author would like to thank the School of Mathematics, Institute for Research in Fundamental Sciences (IPM) for support. This research was in part supported by a grant from IPM (No.93050013).

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Received: September 19, 2019.
Revised: January 14, 2020.
Accepted: January 16, 2020.