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L^{∞}-ERROR ESTIMATES OF FINITE ELEMENT METHODS WITH EULER TIME DISCRETIZATION SCHEME FOR AN EVOLUTIONARY HJB EQUATIONS WITH NONLINEAR SOURCE TERMS

***Salah Boulaaras*[1,2], *Med Amine Bencheikh Le Hocine*[3,4], *Mohamed Haiour*[3]**

[1] *Department of Mathematics, College of Science and Arts, Ar-Ras, Qassim University Kingdom of Saudi Arabia*
[2] *Laboratory of Fundamental and Applied Mathematics of Oran (LMFAO), University of Oran 1 Ahmed Benbella, Algeria, phone +966559618327*
[3] *Department of Mathematics, Faculty of Science, University of Annaba, Box. 12 Annaba 23000, Algeria*
[4] *Tamanghesset University Center, Box. 10034, Sersouf, Tamanghesset 11000, Algeria*
S.Boularas@qu.edu.sa or saleh_boulaares@yahoo.fr, kawlamine@gmal.com, haiourm@yahoo.fr

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Abstract. The main purpose of this paper is to analyze the convergence of the proposed algorithm of the finite element methods coupled with a Euler discretization scheme. Also, an optimal error estimate with an asymptotic behavior in uniform norm are given for an evolutionary nonlinear Hamilton Jacobi Bellman (HJB) equation with respect to the Dirichlet boundary conditions.

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1. Introduction

In this paper, we extend our work [1] and continue to analyze the convergence of the proposed algorithm of the finite element methods coupled with a Euler discretization scheme. In addition, an optimal error estimate with an asymptotic behavior in uniform norm are given, for the following evolutionary nonlinear HJB equation:

$$\begin{cases} u_t + \max\limits_{1\leq\ i\leq\ M}\left(A^i u - f^i(u)\right) = 0 \ \text{ in } \Sigma, \\ u^i = 0 \text{ on } \Gamma, \\ u^i(x,0) = u_0 = \varphi(x) \text{ in } \Omega, \end{cases} \tag{1}$$

where Ω is a bounded open domain of R^N, $N \geq 1$, with boundary Γ sufficiently smooth and Σ set in $\mathrm{R}^n \times \mathrm{R}$, $\Sigma = \Omega \times [0,T]$ with $T < +\infty$, the $f^i(.), (i = 1,...,M)$ are given smooth positive functions, and the $A^i, (i = 1,...,M)$ are second-order, uniformly elliptic operators defined over $(H^1(\Omega))^M$

$$A^i u = -\sum_{j,k=1}^{N} \frac{\partial}{\partial x_j} a^i_{jk}(x) \frac{\partial u^i}{\partial x_k} + \sum_{k=1}^{N} b^i_k(x) \frac{\partial u^i}{\partial x_k} + a^i_0(x) u^i, \tag{2}$$

and whose coefficients
$a^i_{j,k}(x),\ b^i_k(x),\ a^i_0(x) \in \left(L^\infty(\Omega) \cap C^2(\overline{\Omega})\right)^M, x \in \overline{\Omega}, 1 \leq k, j \leq N$
are sufficiently smooth coefficients and satisfy the following conditions

$$a^i_{jk}(x) = a^i_{kj}(x); \quad a^i_0(x) \geq \beta > 0,\ \beta \text{ is a constant}, \tag{3}$$

with

$$\sum_{j,k=1}^{N} a^i_{jk}(x) \xi_j \xi_k \geq \gamma |\xi|^2;\ \xi \in \mathrm{R}^n,\ \gamma > 0,\ x \in \overline{\Omega}, \tag{4}$$

and the bilinear forms associated with A^i, for $u, v \in H^1_0(\Omega)$

$$a^i(u,v) = \int_\Omega \left(\sum_{j,k=1}^{N} a^i_{jk}(x) \frac{\partial u^i}{\partial x_j} \frac{\partial v^i}{\partial x_k} + \sum_{j=1}^{N} b^i_k(x) \frac{\partial u^i}{\partial x_j} v^i + a^i_0(x) u^i v^i \right) dx. \tag{5}$$

f^i is a regular function satisfying

$$f^i \in \left(L^2(0,T,L^\infty\Omega) \cap C^2(0,T,H^{-1}(\Omega))\right)^M,\ f^i \geq 0. \tag{6}$$

We shall also need the following norm

$$\forall\, W = (w_1, w_2, ..., w^M) \in \prod_{i=1}^{M} L^\infty(\Omega),\ \|W\|_\infty = \max_{1 \leq i \leq M} \|w^i\|_\infty.$$

Let $(.,.)$ be the scalar product in $L^2(\Omega)$.

In (cf. [1]), we applied a new time-space discretization using the semi-implicit time scheme combined with a finite element approximation, we found (1) can be transformed into the following full-discrete HJB equation

$$\begin{cases} \max\limits_{1\le i\le M}\left(B^i u_h^{i,k} - f^i\left(u_h^{i,k}\right)\right) = 0, \\ u = 0 \text{ on } \partial\Omega, \\ u(x,0) = u_0 \text{ in } \Omega, \end{cases} \tag{7}$$

where $f^i(.) = ?$, $B^i = A^i + \mu I$ such that A^i defined on (2), $\mu = \frac{1}{\Delta t} = \frac{n}{T}$, respectively.

In [1, 2] we proved the theorem of the geometrical convergence and the existence and uniqueness of the solution of both the continuous and the discrete HJB equation of the stationary case using Bensoussan's algorithm. Also, in (cf. [1, 2]) the system of parabolic quasi variational inequalities (PQVIs) can be transformed into a system of the following full-discrete system of strongly coercive elliptic quasi variational inequalities (QVIs): find $\left(u_h^{1,k}, u_h^{2,k} \ldots u_h^{M,k}\right) \in V_h^{i,k}$ solution of

$$\begin{cases} b^i\left(u_h^{i,k}, v_h^i - u_h^{i,k}\right) \ge \left(f^i\left(u_h^{i,k}\right) + \mu u_h^{i,k-1}, v_h^i - u_h^{i,k}\right),\ v_h^i \in V_h^{i,k}, \\ u_h^{i,k} \le r_h M u_h^{i,k-1},\ i = 1,2,\ldots,M, \end{cases} \tag{8}$$

with

$$\begin{cases} b^i\left(u_h^{i,k}, v_h^i - u_h^{i,k}\right) = \mu\left(u_h^{i,k}, v_h^i - u_h^{i,k}\right) + a^i\left(u_h^{i,k}, v_h^i - u_h^{i,k}\right),\ v_h^i,\ u_h^{i,k} \in V_h^i, \\ \mu = \frac{1}{\Delta t} = \frac{n}{T},\ k = 1,\ldots,n,\ i = 1,\ldots,M, \end{cases} \tag{9}$$

where the discrete spaces V_h^i of finite element given by

$$V_h^i = \left\{ \begin{array}{l} u_h^{i,k} \in \left(L^2\left(0,T,H_0^1(\Omega)\right) \cap C\left(0,T,H_0^1(\overline{\Omega})\right)\right)^M, \text{ such that} \\ u_h^{i,k}|_{K_i} \in P_1,\ K_i \in \tau_h^i, \text{ and } u^i(.,0) = u_0^i \text{ in } \Omega,\ u^i = 0 \text{ on } \partial\Omega, \end{array} \right\} \tag{10}$$

where r_h is the usual interpolation operator defined by

$$r_h v = \sum_{i=1}^{m(h)} v(K_i)\varphi_i(x) \tag{11}$$

and τ_h denote the set of all those elements, $h > 0$ is the mesh size and it is regular and quasi-uniform. Moreover, the usual basis of affine functions φ_l, $l = \{1,\ldots,m(h)\}$ defined by $\varphi_l(K_s) = \delta_{ls}$, where K_s is a sum of triangulation mesh and B^i be the M-matrices [3] with generic entries

$$\left(B^i\right)_{ls} = b^i\left(\varphi_l,\ \varphi_s\right) \quad 1 \le i \le M,\ 1 \le l,\ s \le m(h) \tag{12}$$

and M is an operator defined by

$$Mu^i = L + \psi\left(u^i\right), \tag{13}$$

with $L > 0$ and $\psi\left(u^i\right)$ is a continuous operator from $L^\infty(\Omega)$ into itself satisfying the following assumptions

1. $\psi\left(u^i\right) \le \psi\left(\tilde{u}^i\right)$ whenever $u^i \le \tilde{u}^i$, almost everywhere in Ω, $i = 1,\dots,M.$
2. $\psi\left(u^i + \gamma\right) \le u^i + \gamma^i, \gamma \ge 0.$

The class of the system of QVIs with coercive bilinear form includes at least two well-known important problems: the system of variational inequality of feedback obstacle (VIs) (when $\psi\left(u^i\right) = u^{i+1}$ and $u^1 = u^{M+1}$), and the system of quasi--variational inequality related to management of energy production (when $\psi\left(u^i\right)$ is identically equal to $\inf\limits_{i \neq \mu} u^\mu$, $\mu > 0$, (cf. [4]).

The evolutionary HJB equations (1) have many applications in science, engineering and economics; see for example [4] and references therein. They can arise in solving optimal control problems by dynamic programming techniques. Many nonlinear option pricing problems can also be formulated as optimal control problems, leading to HJB equations.

In the last few decades, many numerical schemes have been proposed for solving the stationary HJB equations; see for example [3, 5, 6] and references therein. Lions and Mercier [7] presented two iterative algorithms for solving HJB equations. At each iteration, a linear complementary subproblem or a linear equation system subproblem is solved. Boulbrachene and Haiour [8], by means of a subsolution method, conducted a finite element approximation study for the first time, for the stationary of the problem (1) and by using Bensoussan-Lions algorithm [4], a quasi-optimal error estimate in the L^∞ - norm has been derived according the following result

$$\max_{1 \le i \le M} \left\| u_h^i - u^i \right\|_\infty \le Ch^2 \left|\log h\right|^2.$$

In [2], exploiting the above arguments, where we analyzed the theta time scheme combined with a finite element spatial approximation for an evolutionary HJB equation with linear source terms and we derived the following error estimate

$$\left\| U_h^{;n} - U^\infty \right\|_\infty \le C^* \left[h^2 \left|\log h\right|^2 + \left(\frac{1}{1 + \beta\theta(\Delta t)} \right)^n \right],$$

with C^* a constant independent of both h (step of the space discretization) and k (step of the time discretization), where $U_h^{\cdot} = \left(u_h^{1,1},\ldots,u_h^{M,n}\right)$, the discrete solution calculated at the moment-end $T = n\Delta t$, $\theta \in [0,1]$ and U^∞, the asymptotic continuous solution with respect to the right hand side condition. In addition, we extended the above result [1] to nonlinear case but with the new generalized space-time discretization stands using the theta scheme and we obtained the following result:

$$\left\|U_h^{;p} - U^\infty\right\|_\infty \le C\left[h^2\left|\log h\right|^3 + \left(\frac{1+(\Delta t)c}{1+(\Delta t)\beta}\right)^n\right],$$

where c is the rate of contraction of the nonlinear source term satisfying

$$c < \beta, \tag{14}$$

with

$$a_0^i(x) \ge \beta > 0,\ \beta \text{ is a constant.} \tag{15}$$

In this paper, an L^∞-error estimate is established combining the geometric convergence of discrete iterative schemes using the known L^∞-error estimates for stationary and evolutionary free boundary problems (cf., e.g., [2, 8]) which play a major role in the finite element error analysis section. Finally the asymptotic behavior in uniform norm is deduced which investigated the evolutionary free boundary problem similar to that in [1].

The structure of this paper is as follows. In Section 2 and 3, we consider the discrete system of quasi-variational inequalities, discretize the iterative scheme by the standard finite element method combined with a theta scheme and an algorithm iterative discrete scheme is introduced. Then its geometric convergence is proved with respect to L^∞-stability of the solution and the right-hand side and its characterization as the least upper bound of the subsolutions set (see also [1, 8]). It is worth mentioning that this approach is entirely different from the one developed for the evolutionary problem. Also, it is used for the first time for a system of stationary QVIs. In Section 4, a fundamental lemma and given optimal error estimates with an asymptotic behavior in uniform norm are proved for the presented problem. Finally, we make some comments on the approach and the results presented in this paper.

2. The discrete coercive system of QVIs

Definition 1: $\zeta_h^k = \left(\zeta_h^{1,k},\ldots,\zeta_h^{M,k}\right)$ is said to be a subsolution for the system of QVIs (8) if

$$\begin{cases} b^i\left(\zeta_h^{i,k}, \varphi_s\right) \leq \left(f^i\left(\zeta_h^{i,k}\right) + \mu\zeta_h^{i,k-1}, \varphi_s\right), \ \forall \varphi_s, \quad s = 1,\ldots,m(h), \\ \zeta_h^{i,k} \leq r_h M \zeta_h^{i,k}. \end{cases} \tag{16}$$

Notation 1: Let X_h be the set of discrete subsolutions. Then, we have the following theorem.

Theorem 1: [1] Under the discrete maximum principle, the solution of the system of QVI (8) is the maximum element of X_h.

2.1. Existence and uniqueness

In [2], we have proved the existence and uniqueness of the discrete QVIs (8) using the algorithm based on semi-implicit time scheme combined with a finite element method, which has already been used in our previous research regarding the evolutionary free boundary problems (see [1, 2]).

For that, let us first introduce the initial vector $\overline{U}_h^0 = \left(\overline{u}_h^{1,0},\ldots,\overline{u}_h^{M,0}\right)$, where $\overline{u}_h^{i,0}$ for $i = 1,\ldots,M$ is solution of

$$b^i\left(\overline{u}_h^{i,0}, v_h\right) = \left(f^i\left(\overline{u}_h^{i,0}\right), v_h\right), \ \forall v_h \in V_h^i, \tag{17}$$

where

$$b^i\left(\overline{u}_h^{i,0}, v_h\right) = a^i\left(\overline{u}_h^{i,0}, v_h\right) + \mu\left(\overline{u}_h^{i,0}, v_h\right).$$

Let $\mathbf{H}^+ = \prod_{i=1}^{M} L_+^\infty(\Omega)$, where $L_+^\infty(\Omega)$ denotes the positive cone of $L^\infty(\Omega)$.

Now, we consider the mapping

$$\begin{aligned} T_h :\ & \mathbf{H}^+ \to (V_h)^M \\ & W \to TW = \xi_h^{i,k} = \left(\xi_h^{1,k},\ldots,\xi_h^{M,k}\right) \\ & \qquad = \partial_h\left(f^i\left(w^{i,k-1}\right) + \mu w^{i,k-1}, r_h M w^{i+1,k-1}\right), \end{aligned} \tag{18}$$

where $\xi_h^{i,k}$ is solution of the following problem

$$\begin{cases} b^i\left(\xi_h^{i,k}, v_h - \xi_h^{i,k}\right) \geq \left(F^{i,k}, v_h - \xi_h^{i,k}\right), \\ \xi_h^{i,k} \leq r_h M w^{i+1,k-1}, \ i = 1,\ldots,M, \ k = 1,\ldots,n, \end{cases} \tag{19}$$

where

$$F^{i,k} = f^i\left(w^{i,k-1}\right) + \mu w^{i,k-1}.$$

2.2. A discrete iterative scheme

Starting from $\hat{U}_h^0 = \overline{U}_h^0$ solution of (17), we define

$$\hat{U}_h^1 = T\hat{U}_h^0 \tag{20}$$

and

$$\hat{U}_h^k = T\hat{U}_h^{k-1},\ k = 2,3,..., \tag{21}$$

where $\hat{U}_h^k = \left(\hat{u}_h^{1,k},...,\hat{u}_h^{M,k}\right)$ is the subsolution of the problem (8).

Theorem 2: [1] The sequences $\left(\hat{U}_h^k\right)$ well defined in K and converge to the unique solution of system of inequalities (8).
where

$$K = \left\{W \in \mathbf{H}^+ \text{ such that } 0 \le W \le \hat{U}^0\right\}. \tag{22}$$

2.3. Regularity of sequences of HJB (21)

Theorem 3: (Lewy Stampacchia inequality) [4] Let A be an elliptic operator defined in (2) and u the solution of an elliptic variational inequalities (VIs) with a simple obstacle $\psi \in H^1(\Omega)$, $\psi > 0$ in $\partial\Omega$ and the right hand side $f \in L^\infty(\Omega)$ such that $Au \ge g$ in the sense of H^{-1}, where $g \in L^2(\Omega)$. Then

$$f \ge A\psi \ge f \wedge g, \tag{23}$$

$$Au \in L^\infty(\Omega) \tag{24}$$

and

$$u \in W^{2,p}(\Omega). \tag{25}$$

Lemma 1: For $i = 1,...,M$, we have

$$\max_{1\le i\le M}\left(\left\|\hat{u}_h^{i,k}\right\|_{W^{2,p}(\Omega)}\right) \le C,\ \ 2 \le p < \infty, \tag{26}$$

where $\hat{u}_h^{i,k}$ is a subsolution of the problem (8).

Proof: It is clear that

$$u_h^{i,1} = \partial_h\left(f^i\left(w^{i,0}\right) + \mu w^{i,0}, r_h M w^{i+1,0}\right)$$

is the solution of (8) with the obstacle $r_h M w^{i+1,0} = r_h\psi$ and the right hand side $f^i\left(w^{i,0}\right) + \mu w^{i,0}$ and $u_h^0 \in W^{2,p}(\Omega)$.

Since

$$\|\psi\|_{W^{2,p}(\Omega)} \le C_1,$$

then, we have

$$B^i\psi \ge -C_1 + \lambda\psi,$$

where

$$B^i = A^i + \lambda I.$$

Using Lewy Stampacchia inequality, we get

$$\left\|u_h^{i,1}\right\|_{W^{2,p}(\Omega)} \le C_2,$$

where $C_2 = \max(-C_{1,}\lambda C_3)$, with $\|\psi\|_{W^{2,p}(\Omega)} \le C_3$.
Now, we assume that

$$\left\|u_h^{i,k-1}\right\|_{W^{2,p}(\Omega)} \le C_4,$$

then $\psi = r_h M w^{i+1,k}$ verify

$$\|\psi\|_{W^{2,p}(\Omega)} \le C_4$$

and we have $B^i\psi \ge -C_4$ in $H^{-1}(\Omega)$, then

$$\left\|u_h^{i,k}\right\|_{W^{2,p}(\Omega)} \le C_3.$$

3. Geometrical convergence of the discrete algorithm

Lemma 2: [1] For $0 \le \lambda \le \inf\left(\frac{L}{\left\|\hat{u}^0\right\|_\infty}, 1\right)$, where L is a positive constant defined in (13), then we have

$$T_h(0) \ge \lambda \left\|\hat{U}_h^0\right\|_\infty. \tag{27}$$

Proposition 1: [1] Let $\omega \in [0,1]$ such that

$$W - V \le \omega W, \quad \forall V, W \in K, \tag{28}$$

then, we have

$$T_h V - T_h W \le \omega(1-\lambda) T_h V. \tag{29}$$

Proposition 2: Under the assumptions and previous notations, we have

$$\left\| \hat{U}_h^k - U_h^{\infty} \right\|_{\infty} \le (1-\lambda)^k \left\| \hat{U}_h^{;0} \right\|, \tag{30}$$

where $\hat{U}_h^k = \left(\hat{u}_h^{1,k},...,\hat{u}_h^{M,k}\right)$ is the subsolution of (8), and U_h^{∞} is an asymptotic semi-discrete solution of (1) using the standard finite element method.

Proof: Using Theorem 2, we have

$$0 \le U_h^{\infty} \le \hat{U}_h^0,$$

then

$$0 \le \hat{U}_h^0 - U_h^{\infty} \le \hat{U}_h^0.$$

Using Proposition 1 with $\omega = 1$, we get

$$0 \le T_h \hat{U}_h^0 - T_h U_h^{\infty} \le (1-\lambda) T \hat{U}_h^0,$$

or

$$0 \le \hat{U}_h^1 - U_h^{\infty} \le (1-\lambda) \hat{U}_h^1.$$

Using Proposition 1 again with $\omega = 1-\lambda$, we get

$$0 \le T_h \hat{U}_h^1 - T_h U_h^{\infty} \le (1-\lambda)(1-\lambda) T \hat{U}_h^1,$$

i.e.

$$0 \le \hat{U}_h^2 - U_h^{\infty} \le (1-\lambda)^2 \hat{U}_h^2$$

and by induction

$$0 \le \hat{U}_h^k - U_h^{\infty} \le (1-\lambda)^k \hat{U}_h^k.$$

Under the third step of the proof of Theorem 3 in [1], we deduce that

$$0 \le \hat{U}_h^k - U_h^{\infty} \le (1-\lambda)^k \hat{U}_h^0.$$

4. Optimal error estimates and asymptotic behavior

Before discussing the results, it is interesting to introduce the result of the following problems

$$\begin{cases} \widetilde{U}_h^1 = T_h \hat{U}_h^0 = \hat{U}_h^1, \\ \widetilde{U}_h^k = T_h \hat{U}_h^{k-1}, \ k = 2,3,..., \end{cases} \tag{31}$$

where $\hat{U}_h^0 = \overline{U}_h^0$ solution of (17) and $\hat{U}_h^{k-1}$ is the subsolution of (8).

Theorem 4: For all $k = 1,...,n$ and C is a constant independent with n, we have the following estimate

$$\left\| \hat{U}^k - \widetilde{U}_h^k \right\|_\infty \le Ch^2 \left| \log h \right|^2, \tag{32}$$

where $\hat{U}^k = \left(\hat{u}^{1,k},...,\hat{u}^{M,k} \right)$ is the subsolution of a semi-discrete problem in time using the semi-implicit scheme.

Proof: The proof is similar to that in [3].

The following lemma will play a crucial role in obtaining the approximation error:

Lemma 3: For all $k = 1,...,n$ and C independent by k, we have the following estimate

$$\left\| \hat{U}^k - \hat{U}_h^k \right\|_\infty \le \sum_{p=0}^{k} \left\| \hat{U}^p - \widetilde{U}_h^p \right\|_\infty. \tag{33}$$

Proof: By induction, we have

$$\hat{U}^1 = T\hat{U}^0, \ \hat{U}_h^1 = T_h \hat{U}_h^0, \ \widetilde{U}_h^1 = T_h \hat{U}^0,$$

then

$$\left\| \hat{U}^1 - \hat{U}_h^1 \right\|_\infty \le \left\| \hat{U}^1 - \widetilde{U}_h^1 \right\|_\infty + \left\| \widetilde{U}^1 - \hat{U}_h^1 \right\|_\infty \le \left\| \hat{U}^1 - \widetilde{U}_h^1 \right\|_\infty + \left\| T_h \hat{U}^0 - T_h \hat{U}_h^0 \right\|_\infty.$$

Since T_h is Lipschitz, thus

$$\left\| \hat{U}^1 - \hat{U}_h^1 \right\|_\infty \le \left\| \hat{U}^1 - \widetilde{U}_h^1 \right\|_\infty + \left\| \hat{U}^0 - \hat{U}_h^0 \right\|_\infty \le \sum_{p=0}^{1} \left\| \hat{U}^k - \widetilde{U}_h^k \right\|_\infty.$$

Assume that

$$\left\| \hat{U}^{k-1} - \hat{U}_h^{k-1} \right\|_\infty \le \sum_{p=0}^{k-1} \left\| \hat{U}^{p-1} - \widetilde{U}_h^{p-1} \right\|_\infty,$$

then, we have

$$\begin{aligned} \left\| \hat{U}^k - \hat{U}_h^k \right\|_\infty &\le \left\| \hat{U}^k - \widetilde{U}_h^k \right\|_\infty + \left\| \widetilde{U}^k - \hat{U}_h^k \right\|_\infty \le \left\| \hat{U}^k - \widetilde{U}_h^k \right\|_\infty + \left\| T_h \hat{U}^{k-1} - T_h \hat{U}_h^{k-1} \right\|_\infty \\ &\le \left\| \hat{U}^k - \widetilde{U}_h^k \right\|_\infty + \left\| \hat{U}^{k-1} - \hat{U}_h^{k-1} \right\|_\infty. \end{aligned}$$

Using the induction assumption, we get

$$\left\|\hat{U}^k-\hat{U}_h^k\right\|_\infty \le \left\|\hat{U}^k-\tilde{U}_h^k\right\|_\infty + \sum_{p=0}^{k-1}\left\|\hat{U}^{k-1}-\tilde{U}_h^{k-1}\right\|_\infty \le \sum_{p=0}^{k}\left\|\hat{U}^k-\tilde{U}_h^k\right\|_\infty.$$

5. L^∞-optimal error estimate

Theorem 5: For all $k=1,...,n$ and C independent by n, we have the following estimate

$$\left\|U^k-U_h^k\right\|_\infty \le Ch^2\left|\log h\right|^3. \tag{34}$$

Proof: We have

$$\begin{aligned}\left\|U^k-\hat{U}_h^k\right\|_\infty &\le \left\|U^k-\hat{U}^k\right\|_\infty+\left\|\hat{U}^k-\hat{U}_h^k\right\|_\infty \le \left\|U^k-\hat{U}^k\right\|_\infty+\left\|\hat{U}^k-\hat{U}_h^k\right\|_\infty+\left\|\hat{U}_h^k-\hat{U}_h^k\right\|_\infty \\ &\le \left\|U^k-\hat{U}^k\right\|_\infty+\left\|\hat{U}_h^k-\hat{U}_h^k\right\|_\infty+\sum_{p=0}^{k}\left\|\hat{U}^p-\hat{U}_h^p\right\|_\infty \\ &\le \left\|U^k-\hat{U}^k\right\|_\infty+\left\|\hat{U}_h^k-\hat{U}_h^k\right\|_\infty+\left\|\hat{U}^0-\hat{U}_h^0\right\|_\infty+\sum_{p=1}^{k}\left\|\hat{U}^p-\hat{U}_h^p\right\|_\infty.\end{aligned}$$

From the initial data in (1), we have $\hat{U}^0=U(x,0)=\varphi(x)$ and $\hat{U}_h^0=\varphi^h$, then, it can be used the following standard error estimate [3, 6] which investigated the stationary case

$$\left\|\hat{U}^0-\hat{U}_h^0\right\|_\infty \le Ch^2\left|\log h\right|^2. \tag{35}$$

Using the estimates (30), (32) and (35), thus

$$\left\|U^k-\hat{U}_h^k\right\|_\infty \le Ch^2\left|\log h\right|^2+kCh^2\left|\log h\right|^2+(1-\lambda)^k\left\|\hat{U}^0\right\|+(1-\lambda)^k\left\|\hat{U}_h^0\right\|.$$

Setting

$$(1-\lambda)^k=h^2,$$

then

$$k=\frac{2\left|\log h\right|}{\left|\log(1-\lambda)\right|}.$$

Therefore, it can be deduced

$$\left\|U^k-\hat{U}_h^k\right\|_\infty \le Ch^2\left|\log h\right|^3.$$

Proposition 3: [1] Under the assumption (14), we have for all $k=1,...,n$ the following estimates

$$\left\| U_h^k - U_h^\infty \right\|_\infty \le \left(\frac{1+(\Delta t)c}{1+(\Delta t)\beta} \right)^k \left\| U_h^\infty - U_{h_0} \right\|_\infty \tag{36}$$

Now we evaluate the variation in $\left(L^\infty(\Omega)\right)^M$ - norm between $\widetilde{U}_h(T,x)$, the discrete solution calculated at the moment $T=n\Delta t$ and U^∞, the asymptotic continuous solution of (1).

Theorem 6: Under the results of Proposition 3 and Theorem 5, we have for

$$\left\| U_h^n - U^\infty \right\|_\infty \le C\left[h^2 \left|\log h\right|^3 + \left(\frac{1+(\Delta t)c}{1+(\Delta t)\beta} \right)^n \right] \tag{37}$$

Proof: Using Theorem 5 and Proposition 3, it can be easily obtained

$$\left\| U_h^n - U^\infty \right\|_\infty \le \left\| U_h^n - U_h^\infty \right\|_\infty + \left\| U_h^\infty - U^\infty \right\|_\infty \le C\left[h^2 \left|\log h\right|^3 + \left(\frac{1+(\Delta t)c}{1+(\Delta t)\beta} \right)^n \right]$$

which completes the proof.

6. Conclusions

In this paper, the regularity and convergence of the presented algorithm sequences of the finite element methods coupled with the Euler time discretization scheme are analyzed. Also, an optimal error estimate with asymptotic behavior in a uniform norm are given for an evolutionary HJB equation with respect to the same proposed boundary conditions in [2]. A next paper will propose a decomposition methods for solving these problems. The convergence of the new scheme will be established and the numerical example will be shown to prove that the new presented scheme is efficient.

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