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Multistage bundle projectionon secondary unprojecting trace and node subspaces

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To memory of Prof. Ph.D. Eng. Stanisław Polański (15.08.1919 – 10.12.1997)
the Professor of Silesian University of Technology,
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Abstract. Assumptions and chosen properties of the presented kind of the multistage bundle projection which was named *multistage bundle projection with secondary non-projected tracely node subspaces* (*MBP II*) are another important contribution to the theory of one-project mappings of the projective space P_n onto a plane. Presented projection is realized by stages. In the particular stages of this projection we adopt subspaces belonging to a pencil trace system as projection planes. Moreover, it is important, that in the presented analysis the secondary projects of node subspaces are the un-projected trace subspaces. Presented mapping significantly extends constructive possibilities in the field of images of n-dimensional subspaces independently on their types.
Keywords: mathematics, descriptive geometry, n-dimensional geometry, projective space, multistage bundle projection

1. Introduction

1.1. Double-stage bundle projection

Fundamental geometrical properties of four and high dimensional spaces [1-3, 6, 7, 21] are the base to formulate and define a family of multistage bundle projections [9-16, 18-20]. The main idea of a projection from four and high dimension space to a 2D plain is to make it by some stages [13-16].

The essence of the method of a single-projecting mapping in four-dimensional projection space, called double-stage bundle projection, is based on the following steps:

1. A decomposition of the bundle projection f_0 from the straight line s into the plane π onto two simple projections f_{I} and f_1 with centers at the different points S_{I}, $S_1 \in s$ and with the projecting hyperplane $\Pi = S_1 \circ \pi$ and the projecting plane π.

$$S_{\mathrm{I}}, S_1 \in s, \tag{1}$$

$$\Pi = S_1 \circ \pi, \tag{2}$$

$$f_0 = f_{\mathrm{I}} \circ f_1, \tag{3}$$

$$f_{\mathrm{I}} = \{S_{\mathrm{I}}, \Pi\}, \tag{4}$$

$$f_1 = \{S_1, \pi\}; \tag{5}$$

2. Adopting the double-spaces and hyper trace system U_{I} of the projection f_{I} in the field $S_{\mathrm{I}} \circ \pi = P_4$. The system U_{I} is determined by the system subspaces Φ_{I} and Φ_{II} with their node subspace $\varphi = \Phi_{\mathrm{I}} \cap \Phi_{\mathrm{II}}$. The subspaces Φ_{I} and Φ_{II} are the hyperspaces with a projecting row equaling 0.

$$U_{\mathrm{I}} = \{\Phi_{\mathrm{I}}, \Phi_{\mathrm{II}}\}, \tag{6}$$

$$\varphi = \Phi_{\mathrm{I}} \cap \Phi_{\mathrm{II}}, \tag{7}$$

$$\varphi \cap s = \emptyset. \tag{8}$$

3. Adopting a double-spaces and the trace system U_1 of the projection $f_{\mathrm{I}1}$ in the field Π with the node line h and with the system planes χ_1 and χ_2 such that $\chi_1 \cap \chi_2 = h$.

$$U_1 = \{\chi_1, \chi_2\}, \tag{9}$$

$$\chi_1 \cap \chi_2 = h. \tag{10}$$

4. A determination of the projection $\varphi^{\mathrm{I}} = f_{\mathrm{I}}(\varphi)$ in the system U_1 as a plane not containing the line h and determined by the given spaces $\varphi_1{}^{\mathrm{I}}$ and $\varphi_2{}^{\mathrm{I}}$ in the planes χ_1, χ_2.

$$\varphi^{\mathrm{I}} = f_{\mathrm{I}}(\varphi), \tag{11}$$

$$\varphi^{\mathrm{I}} = \varphi_1{}^{\mathrm{I}} \circ \varphi_2{}^{\mathrm{I}}, \tag{12}$$

$$\varphi_1{}^{\mathrm{I}} \subset \chi_1, \tag{13}$$

$$\varphi_2{}^{\mathrm{I}} \subset \chi_2, \tag{14}$$

$$h \not\subset \varphi^{\mathrm{I}}. \tag{15}$$

1.2. Multistage projection bundle

1.2.1. Additional assumptions

Assumptions formulated in this way are so general that they have a great margin for additional assumptions specifying the projecting apparatus [19, 20]. For example, we can take additionally the following conditions:

$$h \subset \pi, \tag{16}$$

$$\chi_1 = \pi \text{ or } \chi_2 = \pi, \tag{17}$$

$$\Phi_1 = \Pi \text{ or } \Phi_2 = \Pi. \tag{18}$$

The above conditions do not change descriptive structures. Moreover, changing in this definition the assumptions on adopting the projections φ^{I} in the system U_1 into a more general one we can simplify the method and create a very convenient base for generalization of this method in P_n, where $n \geq 4$.

In order to do that, it suffices to assume that the project φ^{I} is a subspace tracely determined in U_1. In this way we obtain two versions of a double-stage bundle projection.

The one of them is the method presented in [13] where the plane φ^{I} in the system U_1 is a tracely determined subspace but φ^{I} is not a trace subspace. The plane φ^{I} is different from the system planes χ_1 and χ_2.

In the second version of double-stage bundle projection we can assume that φ^{I} is the system plane χ_1 or χ_2. It is possible to do that because trace subspaces are also tracely determined subspaces.

1.2.2. Kinds of multistage bundle projection

A common characteristic feature of both marked versions of double-stage bundle projection is that φ^{I} is an **unprojection subspace** in the field of the secondary projection f_1. It follows from the conditions mentioned in point 2.

The node subspace φ which is an unprojection one will be called a node secondary unprojected subspace. Generalization of double-stage bundle projection leads us to **multistage bundle projection with node secondary unprojection subspaces**. This generalization is a one-project mapping of the projective space P_4 onto a plane. A multistage bundle projection includes two marked versions of double-stage bundle projection. The multistage one does not include the version of a double-stage bundle projection which is a single-projection mapping of P_4 to a plane, discussed in [19]. The node subspace of the trace system U_{I} in [19] is a secondary projected plane and moreover in the system U_1 is a tracely undetermined subspace.

In this article we present properties of a multistage bundle projection. In the presented version, the secondary projects of node subspaces are unprojected trace subspaces. Our multistage projection is one-project mapping of the projective space P_n onto a plane and it will be called multistage bundle projection with the secondary unprojected trace node subspaces. This is another contribution in the theory of one-project mappings.

2. Multistage bundle projection with secondary non-projected tracely node subspaces

2.1. Multistage projective bundle

In the introduced transformation of the space P_n, the projective plane Π is a plane and the project center is the subspace Σ of the dimension $s = n–3$, disconnected with the projective plane. A field of the bundle projection R with the apparatus $\{\Sigma, \Pi\}$ is the space $\Delta = P_n$. We get a decomposition of the projection R onto simple stage projections R_i taking in the projection center Σ, the sequence $(\Sigma_i)_1{}^{n-2}$ of the center projection Σ_i, which is an ordered set of $n–2$ linearly independent points. In these projections, the subspaces Π_i of the dimension $p_i = n–i$ are the projective planes. Each of these planes is a junction of the projective planes $\Pi = \Pi_{n-2}$ and all projection centers following in the sequence $(\Sigma_i)_1{}^{n-2}$ after the projection center Σ_{I}.

Defined in this way the sequence $(R_i)_1{}^{n-2}$ of stage simple projections with the fields Δ_i, which are subspaces of the dimensions $(n–i+1)$, can be enlarged by two additional bundle projections R_{o} and R_{n-1}. For R_{o}, we take as the center Σ_{o} the empty set and as the projective plane Π_{o} we take the space P_n which is also a field of this projection. The projection R_{o} defined in this way is an identity of P_n. We can treat each subspace A as its projection A^{o}. For R_{n-1} we take as a projective plane the straight line $\Pi_{n-1} \subset \Pi_{n-2} = \Pi$ and as the center the point $\Sigma_{n-1} \subset \Pi_{n-2} \backslash \Pi_{n-1}$. The plane Π_{n-2} is a field of the above projection. In the sequence of the projections $(R_i)_{\mathrm{o}}{}^{n-1}$, each projective plane $\Pi_i \neq \Pi_{\mathrm{o}}$ is contained in the projective plane Π_{i-1}, as it is contained in all the projective planes prior to it in the sequence $(\Pi_i)_{\mathrm{o}}{}^{n-1}$ and it contains all the following projective planes. Moreover, each projective plane $\Pi_i \neq \Pi_{n-1}$ is the field $\mathit{\Delta}_{i+1}$ of the projection R_{i+1}. By these assumptions, each superposition $R_i \circ R_{i-1} \circ ... \circ R_o$ is **the phase bundle projection $\boldsymbol{R_{o/i}}$** from the subspaces $\Sigma_i \oplus \Sigma_{i-1} \oplus_{...} \oplus \Sigma_o$ onto the subspace Π_i. The transformation R of the space P_n onto the plane $\Pi_{n-2} = \Pi$ is the phase bundle projection $R_{o/n-2.}$ The transformation R is realized by the stage projection R_i ($i = 0, \ldots, n-2$) and it is called **multistage bundle projection**.

2.2. Projection's invertibility – Double-subspaces trace systems

We can get the invertibility of R by taking, in the field of each stage projection R_i ($i = 1, \ldots, n-1$), the double-subspaces trace system $U_i = \{\{\Phi_{Ii}, \Phi_{IIi}\}, \{\Psi_i\}\}$, where Φ_{Ii} and Φ_{IIi} are $(n-i)$-dimensional system subspaces and $\Psi_i = \Phi_{Ii}\ \ \Phi_{IIi}$ is $(n-i-1)$-dimensional node subspace.

In the sequence $(U_i)_1{}^{n-1}$ of trace system, the first and the last system are the projection U_1 and U_{n-1}, respectively. The pairs of hyperplanes and pairs of straight lines are the system subspaces of U_1 and $U_{n-1.}$ System and node subspaces of a trace system are non-projective subspaces in the field of the projection R_1. These subspaces are univalently transformed into the projective plane Π_i; dimensions of projective planes and system subspaces are equal to each other.

Each $(n-i-1)$-dimensional node subspaces Ψ_i is contained in the field Δ_{i+1} of the secondary projection R_{i+1} and it can be a projective or non-projective subspace.

We assume here that each node subspace Ψ_i is non-projective subspace in the secondary projection. In this way we get *multistage bundle projection with secondary non-projective node subspaces (shortly named MBP I)*. Each node subspace Ψ_i, as a tracely determined one, can be a trace subspace; it means that Ψ_i is identical with the system subspace Φ_{Ii} or Φ_{IIi} of the same dimension.

In the solution presented here we assume additionally that $\Psi_i = \Phi_{Ii+1} = \Pi_{i+1}$. We obtain, in this way, a target method of single-projection mapping of the space P_n onto a plane. This method is called *multistage bundle projection with secondary non-projected tracely node subspaces (shortly named MBP II)* and it is denoted by $R_0{}^{n-2}$.

2.3. Apparatus of the projection

In compliance with our assumptions, the node subspace Ψ_{n-2} is a straight line identical with the system line Φ_{In-1}. However, on this $(n-1)$-stage, the apparatus of the mapping is useless. For this reason, in a descriptive notation we leave only the node space Ψ_{n-2} as an identical straight line with the fixed system line Φ_{In-1}. Thus, the apparatus of the mapping $R_0{}^{n-2}$ is a pair of the sequences $\{(R_i)_o{}^{n-2}, (U_i)_1{}^{n-1}\}$. The assumption that each system subspace Φ_{Ii} is identical with the projection plane

Π_i is not necessary. This assumption serves only to simplify the mapping apparatus, because we obtain the same results as when $\Phi_{\mathrm{I}i} \neq \Pi_i$. The assumption $\Phi_{\mathrm{I}i} = \Pi_i$ implies that each system subspace $\Phi_{\mathrm{I}i}$, as well as contained in it the node subspace Ψ_i, is identical with the stage projections $\Phi_{\mathrm{I}i}{}^i$ and $\Psi_i{}^i$.

We adopt the projective plane $\Pi = \Pi_{n-2}$ (a plane) and contained in it the node line Ψ_{n-2}. We create the mapping system $R_0{}^{n-2}$ taking in turn:

1) the projection center $\Sigma_{n-2}\ \ \Pi_{n-2}$ which is a point;
2) the projective plane $\Pi_i = \Delta_{i+1}$ ($i = n-3, \ldots, 1$) of the dimension $n-i$;
3) the projective plane $\Pi_0 = P_n$ and the projection center $\Sigma_0 \neq \emptyset$.

The center Σ_{n-2} together with the projective plane Π_{n-2} forms the apparatus of the multistage bundle projection R_{n-2}. The subspace $\Delta_{n-2} = \Pi_{n-2}\ \ \Sigma_{n-2}$ is the field of the projection mapping R_{n-2}.

The system planes $\Phi_{\mathrm{I}n-2} = \Pi_{n-2}$ and $\Phi_{\mathrm{II}n-2}\ \ \Delta_{n-2}$ intersect in the node line Ψ_{n-2} and they form the trace system U_{n-2}.

The apparatus of the multistage projection mapping R_i is formed by the projective plane $\Pi_i = \Delta_{i+1}$ ($i = n-3, \ldots, 1$) and the center of projection $\Sigma_i\ \ \Pi_i$, which is a point. The subspace Δ_i of the dimension $n-i+1$ is a field of the projection mapping R_i. The system subspaces $\Phi_{\mathrm{I}i} = \Pi_i$ and $\Phi_{\mathrm{II}i}\ \ \Delta_i$ of the dimension $(n-i)$ intersect in the node subspace $\Psi_i\ \Pi_i$ of the dimension $(n-i-1)$. The subspaces $\Phi_{\mathrm{I}i}$ and $\Phi_{\mathrm{II}i}$ form the trace system U_i.

The projective plane $\Pi_0 = P_n$ and the center of projection $\Sigma_0 \neq \emptyset$ form an apparatus of the projection mapping R_0.

3. Types of trace-determinable subspaces in MBP II

3.1. Division with respect to location in trace system

In each double-subspace trace system projection U_i, the set of tracely determined subspaces contain subspaces of dimensions 0, 1, …, $(n-i+1)$. In this group we determine two subsets. The first set contains *trace subspaces* contained in the system subspace $\Phi_{\mathrm{I}i}$ or $\Phi_{\mathrm{II}i}$.

The second one contains *tracely-symmetric subspaces* whose traces are subspaces with equal dimensions. The common part of these sets is the set of all subspaces contained in node system spaces U_i. Each trace subspace and tracely-symmetric subspace of the system U_i, with the dimension different from $n-i+1$, is a stage non-projected subspace. The traces of the 1st, 2nd, 3rd,…, ith projection (in the system U_i) of a tracely-symmetric subspace will be called **primary traces of this subspace**. Traces of subspaces, which are projections of these traces, will be called **secondary traces**. Secondary trace subspaces as well as nodes, which are their common part, are contained in system subspaces. System subspaces are stage non-projected subspaces.

It follows from here that secondary traces of subspaces and its nodes are univalently mapped onto their projections in the projective plane Π.

The sole subspaces separable from the projection center Σ inasmuch all stage projections of these subspaces and their original traces are mapped, are non-projected lines and planes in the transformation R from the subspace Σ onto the plane Π.

3.2. Division with respect to characteristic properties of invertible images

With respect to characteristic properties of invertible images of non-projected subspaces, we distinguish three types of subspaces.

1. Subspaces whose invertible images can be obtained with the help of primary traces.
2. Subspaces whose invertible images can be obtained with the help of secondary traces.
3. Subspaces, whose invertible images cannot be formed by only one type of traces.

3.2.1. Invertible images obtained with the help of primary traces of non-projecting subspaces

Non-projected lines belong to subspaces whose invertible images can be obtained with the help of primary traces (the first type). An example of an image of such a line contained in the five-dimensional projective space P_5 is given in Fig. 1.

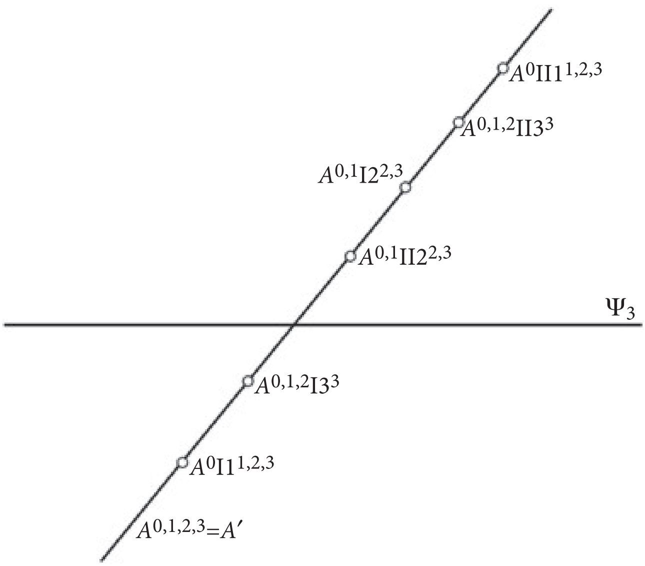

Fig. 1

In the apparatus of mapping, the traces $A_{I3}{}^{0,1,2} = R_3{}^{-1}\,(A_{I3}{}^{0,1,2,3})$ and $A_{II3}{}^{0,1,2} = R_3{}^{-1}(A_{II\,3}{}^{0,1,2,3})$ which belong to the system planes Φ_{I3} and Φ_{II3}, correspondingly, and they determine the straight line $A^{0,1,2}$ contained in the three-dimensional projective plane Π_2. The transformation $R_3{}^{-1}$ is an inverse transformation to the stage projection R_3. The transformation $R_3{}^{-1}$ adopts pairs of different points $\{A^{0,1}{}_{I2}{}^{2}, A^{0,1}{}_{II2}{}^{2}\}$ and $\{A^{0}{}_{I1}{}^{1,2}, A^{0}{}_{II1}{}^{1,2}\}$ in the straight line $A^{0,1,2}$. The transformation $R_2{}^{-1}$ determines traces $A^{0,1}{}_{I2} \in \Phi_{I2}$ and $A^{0,1}{}_{II2} \in \Phi_{II2}$. Those traces determine the straight line $A^{0,1}$ which is a stage projection of the straight line A^{o} four-dimensional plane Π_1. Moreover, the transformation $R_2{}^{-1}$ attribute to the projections $A^{0}{}_{I1}{}^{1,2}$ and $A^{0}{}_{II1}{}^{1,2}$ the projections $A^{0}{}_{I1}{}^{1}$ and $A^{0}{}_{II1}{}^{1} \in A^{0,1}$, correspondingly. The next transformation $R_1{}^{-1}$ attributes to the projections $A^{0}{}_{I1}{}^{1}$ and $A^{0}{}_{II1}{}^{1}$ the traces $A^{0}{}_{I1}$ and $A^{0}{}_{II1}$, which determine in the non-projected straight line $A^{o} = A$.

3.2.2. Invertible images obtained with the help of secondary traces of non-projecting subspaces

Non–projected hyperplanes belong to subspaces whose invertible images can be obtained with the help of secondary traces (the second distinguished type). In this case, all stage projection subspaces are identical to the projection fields corresponding to them. This situation completely excludes the possibility of using primary traces and we must determine hyperplanes with the help of secondary traces. This is illustrated in figure 2.

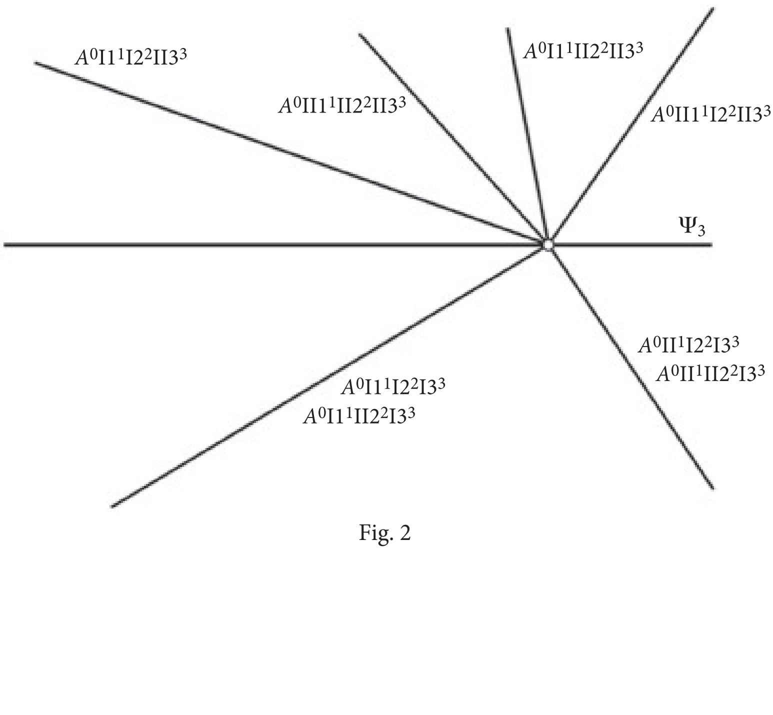

Fig. 2

In the presented mapping apparatus S (Fig. 2), anti-images of projection pairs $\{A^{0}{}_{\mathrm{I1}}{}^{1}{}_{\mathrm{I2}}{}^{2}{}_{\mathrm{I3}}{}^{3}, A^{0}{}_{\mathrm{I1}}{}^{1}{}_{\mathrm{I2}}{}^{2}{}_{\mathrm{III3}}{}^{3}\}$ and $\{A^{0}{}_{\mathrm{I1}}{}^{1}{}_{\mathrm{II2}}{}^{2}{}_{\mathrm{I3}}{}^{3}, A^{0}{}_{\mathrm{I1}}{}^{1}{}_{\mathrm{II2}}{}^{2}{}_{\mathrm{II3}}{}^{3}\}$ in the system planes Φ_{I3} and Φ_{II3} are pairs of the traces $\{A^{0}{}_{\mathrm{I1}}{}^{1}{}_{\mathrm{I2}}{}^{2}{}_{\mathrm{I3}}, A^{0}{}_{\mathrm{I1}}{}^{1}{}_{\mathrm{I2}}{}^{2}{}_{\mathrm{III3}}{}^{3}\}$ and $\{A^{0}{}_{\mathrm{I1}}{}^{1}{}_{\mathrm{II2}}{}^{2}{}_{\mathrm{I3}}, A^{0}{}_{\mathrm{I1}}{}^{1}{}_{\mathrm{II2}}{}^{2}{}_{\mathrm{II3}}\}$. In three--dimensional projected plane Π_2, junctions of these pairs of traces determine the planes $A^{0}{}_{\mathrm{I1}}{}^{1}{}_{\mathrm{I2}}{}^{2}$ and $A^{0}{}_{\mathrm{I1}}{}^{1}{}_{\mathrm{II2}}{}^{2}$, correspondingly. The product of the planes is the straight line $A^{0}{}_{\mathrm{I1}}{}^{1}{}_{2}{}^{2}$. The traces $A^{0}{}_{\mathrm{I1}}{}^{1}{}_{\mathrm{I2}}$, $A^{0}{}_{\mathrm{I1}}{}^{1}{}_{\mathrm{II2}}$ and the node $A^{0}{}_{\mathrm{I1}}{}^{1}{}_{2}$ are anti-images of the planes $A^{0}{}_{\mathrm{I1}}{}^{1}{}_{\mathrm{I2}}{}^{2}$ and $A^{0}{}_{\mathrm{I1}}{}^{1}{}_{\mathrm{II2}}{}^{2}$ in three-dimensional system subspaces φ_{I2}, φ_{II2} and in the node plane Ψ_2, correspondingly.

The traces $A^{0}{}_{\mathrm{I1}}{}^{1}{}_{\mathrm{I2}}$, $A^{0}{}_{\mathrm{I1}}{}^{1}{}_{\mathrm{II2}}$ are different planes with a common straight line. The junction of these subspaces is the three-dimensional subspace $A^{0}{}_{\mathrm{I1}}{}^{1}$ contained in the four-dimensional projected plane Π_1. In the four-dimensional system subspace Φ_{I1}, the anti-image of the projection $A^{0}{}_{\mathrm{I1}}{}^{1}$ is three-dimensional subspace $A^{0}{}_{\mathrm{I1}}$.

Analogically, the remaining two projection pairs $\{A^{0}{}_{\mathrm{III1}}{}^{1}{}_{\mathrm{I2}}{}^{2}{}_{\mathrm{I3}}{}^{3}, A^{0}{}_{\mathrm{III1}}{}^{1}{}_{\mathrm{I2}}{}^{2}{}_{\mathrm{III3}}{}^{3}\}$ and $\{A^{0}{}_{\mathrm{III1}}{}^{1}{}_{\mathrm{II2}}{}^{2}{}_{\mathrm{I3}}{}^{3}, A^{0}{}_{\mathrm{III1}}{}^{1}{}_{\mathrm{II2}}{}^{2}{}_{\mathrm{II3}}{}^{3}\}$ determine the three-dimensional subspace $A^{0}{}_{\mathrm{III1}}$ in the four-dimensional system subspace. The obtained three-dimensional subspaces $A^{0}{}_{\mathrm{I1}}$ and $A^{0}{}_{\mathrm{III1}}$ have the common plane $A_1{}^0$ in the three-dimensional node subspaces Ψ_1. The subspaces $A^{0}{}_{\mathrm{I1}}$ and $A^{0}{}_{\mathrm{III1}}$ are traces of the four-dimensional subspace $A^{0}{}_{\mathrm{I1}}$ $A^{0}{}_{\mathrm{III1}} = A^0$, which is the hyperplane $A = A^0$ in the field $\Delta = P_5$.

3.2.3. Invertible images obtained with the help of primary and secondary traces of non-projecting subspaces

In the case, when the considered non-projected subspace $A \subset P_n\,(n \geq 4)$ is not a straight line or a hyperplane, it is not possible to obtain its invertible image only on the basis of primary or secondary traces (Fig. 1, Fig. 2).

For example, it is impossible to get an invertible image of a non-projected plane A as a subspace contained in P_4 and not contained in the projected plane $\Pi_1 = \Delta_2$ only with the help of one type of traces (Fig. 3).

In the example illustrated in Fig. 3, the projections $A^{0,1}{}_{\mathrm{I2}}{}^{2}$ and $A^{0,1}{}_{\mathrm{II2}}{}^{2}$ determine the plane $A^{0,1}$ in the trace system U_2. The straight lines $A^{0}{}_{\mathrm{I1}}{}^{1}$ and $A^{0}{}_{\mathrm{III1}}{}^{1}$ are anti-images of projections $A^{0}{}_{\mathrm{I1}}{}^{1,2}$ and $A^{0}{}_{\mathrm{III1}}{}^{1,2}$ in this plane. They have the common point $A^{0}{}_{1}{}^{1}$. The straight lines $A^{0}{}_{\mathrm{I1}} \subset \Phi_{\mathrm{I1}}$ and $A^{0}{}_{\mathrm{III1}} \subset \Phi_{\mathrm{III1}}$ with the common point $A_1{}^0$ are anti-images of the projections $A^{0}{}_{\mathrm{I1}}{}^{1}$ and $A^{0}{}_{\mathrm{III1}}{}^{1}$. The straight lines $A^{0}{}_{\mathrm{I1}}$, $A^{0}{}_{\mathrm{III1}}$ are traces and the point $A_1{}^0$ is a node and they determine the plane $A^{o} = A$ in the system U_1. The plane A is contained in the space P_4 and A is not contained in the projection field Δ_2. In the presented image of the subspace A, the pairs of the straight lines $\{A^{0,1}{}_{\mathrm{I2}}, A^{0,1}{}_{\mathrm{II2}}\}$ and $\{A^{0}{}_{\mathrm{I1}}, A^{0}{}_{\mathrm{III1}}\}$ are primary traces and the pairs of the points $\{A^{0}{}_{\mathrm{I1}}{}^{1}{}_{\mathrm{I2}}$, $A^{0}{}_{\mathrm{I1}}{}^{1}{}_{\mathrm{II2}}\}$ and $\{A^{0}{}_{\mathrm{III1}}{}^{1}{}_{\mathrm{I2}}, A^{0}{}_{\mathrm{III1}}{}^{1}{}_{\mathrm{II2}}\}$ are secondary traces.

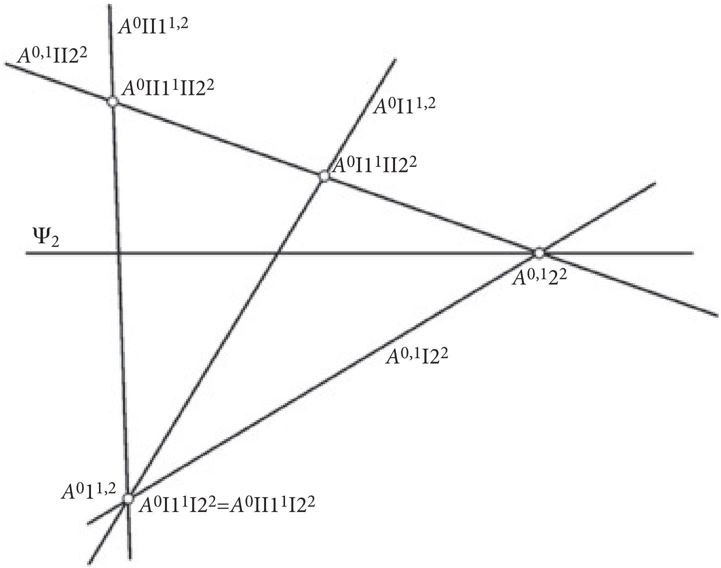

Fig. 3

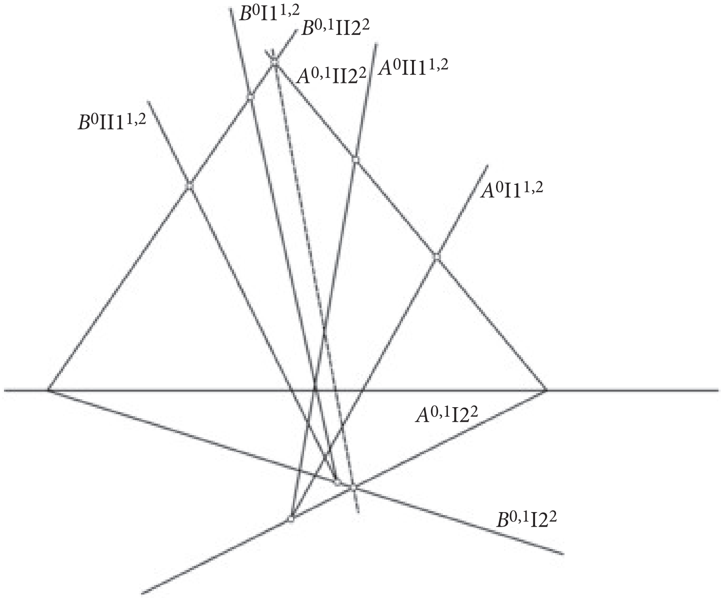

Fig. 4a

3.2.4. Implicit images of any subspaces

The presented multistage bundle projections with secondary non-projected trace node subspaces form the basis for the preparation of inverse images of subspaces irrespectively of their being projected or unprojected, or of their being tracely determined or undetermined. An anti-image of each subspace can be given as an implicit image. An implicit image of subspace is then a multi-unit set of invertible images of subspaces. This set's anti-image is a subspace which is the product or the junction of those subspaces. In the example illustrated in Fig. 4a, the pair of invertible images of the planes A and B in P_4 is an invertible image of two subspaces $C = A \cap B$ and $D = A \oplus B$.

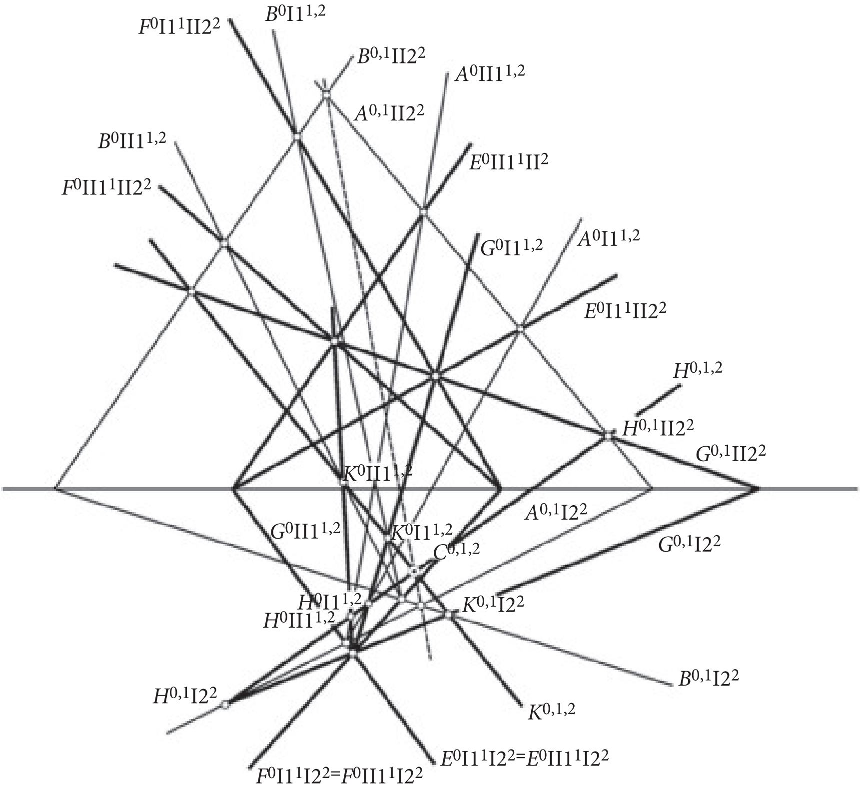

Fig. 4b

To determine whether the subspace C is a point or a straight line and respectively whether the subspace D is the space P_4 or a hyperplane different from P_4, the hyperplanes E and F, where $E \supset A$, $F \supset B$ have been adopted (Fig. 4b). The plane $G = E \cap F$ is the common part of these hyperplanes. The plane G intersects the planes A and B in the different straight lines H and K. The subspace $C = A \cap B$ is the point $H \cap K$, so $D = P_4$. Thus, the pair of images of planes is an invertible image of the point $C = A \cap B$. Its invertible image is also the image of the intersection $C \cap H = C$ and the intersection $C \cap K = C$. The invertible images of the point C (Fig. 4b), obtained in this way, are the result of transformation of a given implicit image (Fig. 4a). The phase projection $C^{0,1}$ of the point C belongs to a common straight line of the planes $A^{0,1}$ and $B^{0,1}$. It is drawn as a dash line in figure 4b.

4. Conclusions

In this article we presented the kind of the multistage bundle projection which was named *multistage bundle projection with secondary non-projected tracely node subspaces (MBP II).* Assumptions and chosen properties of this projection are another important contribution in the theory of one-project mappings of the projective space P_n onto a plane. Presented mapping extends significantly constructive possibilities in the field of images of subspaces, independently of their types.

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Wieloetapowy rzut wiązkowy o wtórnie nierzutujących śladowych podprzestrzeniach węzłowych

Streszczenie. Przedstawione w niniejszym artykule założenia i wybrane właściwości odmiany wieloetapowego rzutu wiązkowego o wtórnie nierzutujących śladowych podprzestrzeniach węzłowych są kolejnym uzupełnieniem pola jedno-rzutowych odwzorowań n-wymiarowej przestrzeni rzutowej P_n na płaszczyznę. Prezentowane odwzorowanie realizowane jest etapowo: w poszczególnych krokach tego rzutowania jako rzutnie przyjmujemy podprzestrzenie należące do wiązkowego układu śladowego. Ponadto, istotnym jest, iż rzuty wtórne podprzestrzeni węzłowych są podprzestrzeniami nierzutującymi. Przedstawione odwzorowanie znacząco poszerza możliwości konstrukcyjne w zakresie obrazów podprzestrzeni n-wymiarowych, niezależnie od ich rodzaju.

Słowa kluczowe: matematyka, geometria wykreślna, geometria n-wymiarowa, przestrzeń rzutowa, wieloetapowy rzut wiązkowy