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SOLUTION OF THE STATE EQUATIONS OF DESCRIPTOR FRACTIONAL DISCRETE-TIME LINEAR SYSTEMS WITH REGULAR PENCILS

Abstract

A method for finding of the solutions of the state equations of descriptor fractional discrete-time linear systems with regular pencils is proposed. The derivation of the solution formula is based on the application of the $\mathcal{Z}$ transform and the convolution theorem. A procedure for computation of the transition matrix is proposed. The effectiveness of the proposed method is demonstrated on a simple numerical example.

1. INTRODUCTION

Descriptor (singular) linear systems with regular pencils have been considered in many papers and books [1-4, 10-12, 15, 17, 18, 20]. The eigenvalues and invariants assignment by state and output feedbacks have been investigated in [10, 11] and the realization problem for singular positive continuous-time systems with delays in [15]. The computation of Kronecker's canonical form of a singular pencil has been analyzed in [20]. A delay dependent criterion for a class of descriptor systems with delays varying in intervals has been proposed in [2].

Fractional positive continuous-time linear systems have been addressed in [9] and positive linear systems with different fractional orders in [8]. A new concept of the practical stability of the positive fractional 2D systems has been proposed in [14]. The reachability of the positive fractional linear systems has been considered in [9] and some selected problems in theory of fractional linear systems in the monograph [16].

A new class of descriptor fractional linear systems and electrical circuits has been introduced, their solution of state equations has been derived and a method for decomposition of the descriptor fractional linear systems with regular pencils into dynamic and static parts has been proposed in [6]. Positive fractional continuous-time linear systems with singular pencils has been considered in [7].

In this paper a method for finding of the solutions of the state equations of descriptor fractional discrete-time linear systems with regular pencils will be proposed.

The paper is organized as follows. In section 2 the solution to the state equation of the descriptor system is derived using the method based on the $\mathcal{Z}$ transform and the convolution theorem. A method for computation of the transition matrix is proposed in section 3. In section 4 the proposed method is illustrated on a simple numerical example. Concluding remarks are given in section 5.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices and $\Re^n = \Re^{n\times 1}$, Z_+ - the set of $n\times n$ nonnegative matrices, I_n - the $n\times n$ identity matrix

2. SOLUTION OF THE STATE EQUATION

Consider the descriptor fractional discrete-time linear system

$$E\Delta^\alpha x_{i+1} = Ax_i + Bu_i, \; i\in Z_+ = \{0,1,2,...\}, \; 0<\alpha<1 \tag{1}$$

where α is fractional order, $x_i \in \Re^n$ is the state vector $u_i \in \Re^m$ is the input vector and $E, A\in \Re^{n\times n}$, $B\in \Re^{n\times m}$. It is assumed that det E = 0 but the pencil (E, A) is regular, i.e.

$$\det[Ez - A] \neq 0 \text{ for some } z\in \mathrm{C} \text{ (the field of complex numbers).} \tag{2}$$

Without lost of generality we may assume

$$E = \begin{bmatrix} E_1 & 0 \\ 0 & 0 \end{bmatrix} \in \Re^{n\times n}, \quad E_1 \in \Re^{r\times r} \text{ and rank } E_1 = \text{rank } E = r < n. \tag{3}$$

Consistent boundary conditions for (1) are given by Ex_0. The fractional difference of the order $\alpha \in [0,1)$ is defined by

$$\Delta^\alpha x_i = \sum_{k=0}^{i} c_k x_{i-k} \tag{4a}$$

where

$$c_k = (-1)^k \binom{\alpha}{k}, \quad k = 0,1,... \tag{4b}$$

and

$$\binom{\alpha}{k} = \begin{cases} 1 & \text{for} \quad k=0 \\ \dfrac{\alpha(\alpha-1)...(\alpha-k+1)}{k!} & \text{for} \quad k=1,2,... \end{cases} \tag{4c}$$

Substitution of (4a) into (1) yields

$$Ex_{i+1} = Fx_i - \sum_{k=2}^{i+1} Ec_k x_{i-k+1} + Bu_i, \; i\in Z_+ \tag{5}$$

where $F = A - Ec_1 = A + E\alpha$.
Applying to (5) the $\mathcal{Z}$ transform and taking into account that [11]

$$\mathcal{Z}[x_{i-p}] = z^{-p}X(z) + z^{-p}\sum_{j=-1}^{-p} x_j z^{-j}, \quad p = 1,2,... \tag{6}$$

we obtain

$$X(z) = [Ez - F]^{-1}\{Ex_0 z - H(z) + BU(z)\} \tag{7a}$$

where

$$X(z)=\mathcal{Z}[x_i]=\sum_{i=0}^{\infty}x_i z^{-i}\ ,\ U(z)=\mathcal{Z}[u_i]=\sum_{i=0}^{\infty}u_i z^{-i}\ ,\ H(z)=\mathcal{Z}[h_i]\ ,\ h_i=\sum_{k=2}^{i+1}Ec_k x_{i-k+1}\ . \quad (7b)$$

Let

$$[Ez-F]^{-1}=\sum_{j=-\mu}^{\infty}\psi_j z^{-(j+1)} \quad (8)$$

where μ is positive integer defined by the pair (E, A) [11, 20]. Comparison of the coefficients at the same powers of z of the equation

$$[Ez-F]\left(\sum_{j=-\mu}^{\infty}\psi_j z^{-(j+1)}\right)=\left(\sum_{j=-\mu}^{\infty}\psi_j z^{-(j+1)}\right)[Ez-F]=I_n \quad (9a)$$

yields

$$E\psi_{-\mu}=\psi_{-\mu}E=0 \quad (9b)$$

and

$$E\psi_k-F\psi_{k-1}=\psi_k E-\psi_{k-1}F=\begin{cases}I_n & \text{for} \quad k=0\\ 0 & \text{for} \quad k=1-\mu,2-\mu,\dots,-1,1,2,\dots\end{cases} \quad (9c)$$

From (9b) and (9c) we have the matrix equation

$$G\begin{bmatrix}\psi_{0\mu}\\ \psi_{1N}\end{bmatrix}=\begin{bmatrix}V\\ 0\end{bmatrix} \quad (10a)$$

where

$$G=\begin{bmatrix}G_1 & 0\\ G_{21} & G_2\end{bmatrix}\in\Re^{(N+\mu+1)n\times(N+\mu+1)n},\ G_{21}=\begin{bmatrix}0 & \dots & 0 & -F\\ 0 & \dots & 0 & 0\\ \vdots & \dots & \vdots & \vdots\\ 0 & \dots & 0 & 0\end{bmatrix}\in\Re^{Nn\times(\mu+1)n},$$

$$G_1=\begin{bmatrix}E & 0 & 0 & \dots & 0 & 0 & 0\\ -F & E & 0 & \dots & 0 & 0 & 0\\ 0 & -F & E & \dots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & -F & E & 0\\ 0 & 0 & 0 & \dots & 0 & -F & E\end{bmatrix}\in\Re^{(\mu+1)n\times(\mu+1)n},\ G_2=\begin{bmatrix}E & 0 & 0 & \dots & 0 & 0 & 0\\ -F & E & 0 & \dots & 0 & 0 & 0\\ 0 & -F & E & \dots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & -F & E & 0\\ 0 & 0 & 0 & \dots & 0 & -F & E\end{bmatrix}\in\Re^{Nn\times Nn},$$

$$\psi_{0\mu}=\begin{bmatrix}\psi_{-\mu}\\ \psi_{1-\mu}\\ \vdots\\ \psi_0\end{bmatrix}\in\Re^{(\mu+1)n\times n},\ \psi_{1N}=\begin{bmatrix}\psi_1\\ \psi_2\\ \vdots\\ \psi_N\end{bmatrix}\in\Re^{Nn\times n},\ V=\begin{bmatrix}0\\ \vdots\\ 0\\ I_n\end{bmatrix}\in\Re^{(\mu+1)n\times n} \quad (10b)$$

The equation (10a) has the solution $\begin{bmatrix}\psi_{0\mu}\\ \psi_{1N}\end{bmatrix}$ for given G and V if and only if

$$\operatorname{rank}\left\{G, \begin{bmatrix} V \\ 0 \end{bmatrix}\right\} = \operatorname{rank} G \,. \tag{11}$$

It is easy to show that the condition (11) is satisfied if the condition (2) is met.
Substituting (8) into (7a) we obtain

$$X(z) = \left(\sum_{j=-\mu}^{\infty} \psi_j z^{-(j+1)} \right) [Ex_0 z - H(z) + BU(z)] \,. \tag{12}$$

Applying the inverse transform $\mathcal{Z}^{1}$ and the convolution theorem to (12) we obtain

$$x_i = \psi_i E x_0 - \sum_{k=0}^{i+\mu-1} \psi_{i-k-1} \sum_{j=2}^{k+1} c_j x_{k-j+1} + \sum_{k=0}^{i+\mu-1} \psi_{i-k-1} B u_k \,. \tag{13}$$

To find the solution to the equation (1) first we compute the transition matrices ψ_j for $j = -\mu, 1-\mu, ..., 1, 2, ...$ and next using (13) the desired solution.

3. COMPUTATION OF TRANSITION MATRICES

To compute the transition matrices ψ_k for $k = -\mu, 1-\mu, ..., N, ...$ the following procedure is recommended.

Procedure 1.

Step 1. Find a solution $\psi_{0\mu}$ of the equation

$$G_1 \psi_{0\mu} = V \tag{14}$$

where G_1, $\psi_{0\mu}$ and V are defined by (10b). Note that if the matrix E has the form (3) then the first r rows of the matrix $\psi_{0\mu}$ are zero and its last $n - r$ rows are arbitrary.

Step 2. Choose $n - r$ arbitrary rows of the matrix ψ_0 so that the equation

$$\begin{bmatrix} E & 0 \\ -F & E \end{bmatrix} \begin{bmatrix} \psi_0 \\ \psi_1 \end{bmatrix} = \begin{bmatrix} I_n + F\psi_{-1} \\ 0 \end{bmatrix} \tag{15}$$

has a solution with arbitrary last $n - r$ rows of the matrix ψ_1.

Step 3. Knowing $\psi_{0\mu}$ choose the last $n - r$ rows of the matrix ψ_1 so that the equation

$$\begin{bmatrix} E & 0 \\ -F & E \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix} \psi_0 \tag{16}$$

has a solution with arbitrary last $n - r$ rows of the matrix ψ_2. Repeating the last step for $\begin{bmatrix} \psi_2 \\ \psi_3 \end{bmatrix}, \begin{bmatrix} \psi_3 \\ \psi_4 \end{bmatrix}, \ldots$ we may compute the desired matrices ψ_k for $k = -\mu, 1-\mu, ...$.

The details of the procedure will be shown on the following example.

4. EXAMPLE

Find the solution to the equation (1) for α = 0.5 with the matrices

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{17}$$

and the initial condition $Ex_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

In this case the pencil (2) of (17) is regular since

$$\det[Ez - A] = \begin{vmatrix} z & 0 \\ -1 & 2 \end{vmatrix} = 2z \tag{18}$$

μ = 1 and

$$F = [A + E\alpha] = \begin{bmatrix} \alpha & 0 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 1 & -2 \end{bmatrix}. \tag{19}$$

Using Procedure 1 we obtain the following.

Step 1. In this case the equation (14) has the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\alpha & 0 & 1 & 0 \\ -1 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi_{-1} \\ \psi_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{20}$$

and its solution with the arbitrary second row $[\psi_{21}^0 \quad \psi_{22}^0]$ of ψ_0 is given by

$$\begin{bmatrix} \psi_{-1} \\ \psi_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \\ 1 & 0 \\ \psi_{21}^0 & \psi_{22}^0 \end{bmatrix}. \tag{21}$$

Step 2. We choose the row $[\psi_{21}^0 \quad \psi_{22}^0]$ of ψ_0 so that the equation

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\alpha & 0 & 1 & 0 \\ -1 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi_0 \\ \psi_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{22}$$

has the solution

$$\begin{bmatrix} \psi_0 \\ \psi_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0.5 & 0 \\ \alpha & 0 \\ \psi_{21}^1 & \psi_{22}^1 \end{bmatrix} \tag{23}$$

with the second arbitrary row $[\psi_{21}^1 \quad \psi_{22}^1]$ of ψ_1.

Step 3. We choose $[\psi_{21}^1 \quad \psi_{22}^1]$ so that the equation

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & 0 & 1 & 0 \\ -1 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ \psi_{21}^1 & \psi_{22}^1 \\ -\alpha^2 & 0 \\ \psi_{21}^2 & \psi_{22}^2 \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{24}$$

has the solution

$$\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0.5\alpha & 0 \\ \alpha^2 & 0 \\ \psi_{21}^2 & \psi_{22}^2 \end{bmatrix} \tag{25}$$

with arbitrary $[\psi_{21}^2 \quad \psi_{22}^2]$.

Continuing the procedure we obtain

$$\psi_{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad \psi_k = \begin{bmatrix} \alpha^k & 0 \\ 0.5\alpha^k & 0 \end{bmatrix} \text{ for } k = 0,1,\ldots \tag{26}$$

Using (13), (17) and (19) we obtain the desired solution of the form

$$x_i = \psi_i \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sum_{k=0}^{i} \psi_{i-k-1} \sum_{j=2}^{k+1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} c_j x_{k-j+1} + \sum_{k=0}^{i} \psi_{i-k-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} u_k \tag{29}$$

where c_j are defined by (4b).

5. CONCLUDING REMARKS

A new method for finding of the solution of the state equation of descriptor fractional discrete-time linear systems with regular pencils has been proposed. Derivation of the solution formula has been based on the application of the $\mathcal{Z}$ transform and the convolution theorem. A procedure for computation of the transition matrices has been proposed and its application has been demonstrated on a simple numerical example. The presented method can be easily extended to continuous-time descriptor fractional linear system with regular pencils. An open problem is an extension of the method for 2D descriptor fractional discrete and continuous-discrete linear systems.

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ROZWIĄZANIE RÓWNAŃ STANU DESKRYPTOWYCH UKŁADÓW DYSKRETNYCH RZĘDÓW NIECAŁKOWITYCH O PĘKACH REGULARNYCH

Streszczenie

Podano metodę wyznaczania rozwiązań równań stanu deskryptowych układów dyskretnych rzędów niecałkowitych o pękach regularnych. Rozwiązanie to zostało wyprowadzone korzystając z przekształcenia zet i twierdzenia o transformacie splotu. Zaproponowano procedurę wyznaczania macierzy tranzycji tych układów. Proponowaną metodę zilustrowano przykładem numerycznym.

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