Karbasian Mahdi
Malek-e Ashtar University of Technology, Isfahan, Iran

Mahdavi Mojtaba
Islamic Azad University of NajafAbad (IAUN), Isfahan, Iran

A new method for computing the reliability of composite systems

Keywords

reliability, extreme value, parallel, series, life distribution

Abstract

This paper tries to represent a new method for computing the reliability of a system, which is arranged in series or parallel model. In this method we estimate life distribution function of whole system using the asymptotic Extreme Value (EV) distribution of Type I, or Gumbel theory. We use EV distribution in minimal mode, for estimate the life distribution function of series system and maximal mode for parallel system. All parameters also are estimated by Moments method. Reliability function and failure (hazard) rate and p-th percentile point of each function are determined. Other important indexes such as Mean Time to Failure (MTTF), Mean Time to repair (MTTR), for non-repairable and renewal systems in both of series and parallel structures will be computed.

1. Introduction

There are some different methods to computing the reliability of a system. We can use two general ways for these computations:

1.By using the reliability of components.

2.By using the life distribution function of system.

In the large systems, which have a lot of different components with unknown reliability, it will be very important to find the life distribution function of system. In this paper we're going to represent a new method for estimate the life distribution of these systems, when their components become a lot and difference. We trying to use desire properties of Extreme Value (EV) theorem which has been offered by Gumbel in 1954 (Gumbel, E. J.1954). He demonstrated asymptotic probability distributions for minimal and maximal value in a random sample, under the special conditions. We conform these PDFs to computing the reliability of series and parallel-based systems by estimating the life distribution of each system.

According the obtained functions all-important indexes of reliability and dependability of a system with any arrange will be determined for renewal and non-renewable mode. The Mean Time to Failure (MTTF), Mean Time to Replacement (MTTR), Hazard function (*h(t)*), are some of these parameters. We acquire another useful relations for reliability calculations and analysis system.

2. Extreme Value's distributions of a sample

In this section two types of extreme values and their PDFs with probability properties are introduced.

2.1. Maximal value distribution function in a random sample

Suppose the random sample $x_1, x_2, ..., x_n$ and define $X(1)$ and $X(n)$ as minimum and maximum of x_i in sample and call them the Extreme Values:

$$X(1) = Min\{x_1, x_2, ... x_n\}$$
$$X(n) = Max\{x_1, x_2, ..., x_n\} \quad (1)$$

In the specific conditions these will have the special behaviour independent the initial distribution.

At the 1954, Gumbel could present the $X(n)$, when the n closest infinitive, has a fix and independent probability model with CDF as following:

$$F_{EV}(x,\gamma,\delta) = P(X \le x) = \exp\left(-\exp\left(-\tfrac{x-\gamma}{\delta}\right)\right) \tag{2}$$

$$-\infty < x < +\infty \quad , \quad x > \gamma$$

where γ and δ are Location and Scale parameters. This function is called Gumbel dis., or EV dis [1]. type I, too. After him Benjamin at the 1970 and Ang at the 1984 developed his concept on EV type II and III [4]. So in this paper we just discos about type I. The PDF of Gumbel dist. that is denoted by $f_{EV}^{x(n)}$ can be found after differentiating:

$$f_{EV}^{x(n)}(x;\gamma,\delta) = \frac{d}{dx} F_{EV}$$

$$= \tfrac{1}{\delta}\exp\left(-\left(\tfrac{x-\gamma}{\delta}\right) - \exp\left(-\tfrac{x-\gamma}{\delta}\right)\right) \tag{3}$$

$$-\infty < x < +\infty \quad , \quad \delta > 0$$

If $\gamma = 0, \delta = 1$, we have the Standard Gumbel dis. with the curve same as *Figure 1*:

$$f(x) = e^{-x}e^{-e^{-x}} \tag{4}$$

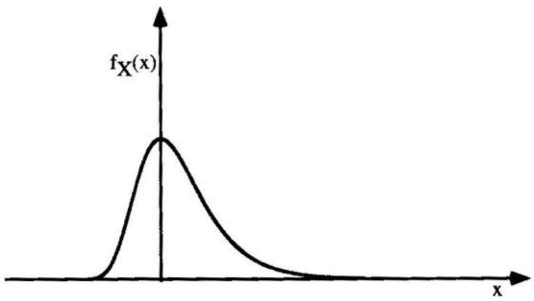

Figure 1. A general curve of EV dis. for maximal

Other statistical attributes of this function are:

$$\mu = \gamma + 0.5772\,\delta \tag{5}$$

$$\sigma = \tfrac{\pi}{\sqrt{6}}\delta = 1.283\,\delta \tag{6}$$

$$Me = \gamma - \delta\ln(\ln(2)) = \gamma - 0.3665\,\delta \tag{7}$$

$$t_p = \gamma - \delta\ln\left(\ln\left(\tfrac{1}{p}\right)\right) \tag{8}$$

$$S.K = 5.4 \tag{9}$$

$$K = 1.14 \tag{10}$$

where μ is expected value, σ is standard deviation, Me is median, t_p is p-th percentile point, $S.K$ is skewness, and K is kurtosis of this dis. function [3].

2.2. Minimal value distribution function in a random sample

Second form of EV distribution relates to minimal value in a random sample. In the other hand can demonstrate when the sample size *n*, increase enough, the *X(1)*, has an independent probability model with the following asymptotic CDF:

$$F_{EV}^{x(1)}(x;\lambda,\delta) = P(X \le x) = 1 - \exp\left(-\exp\left(\tfrac{x-\gamma}{\delta}\right)\right) \tag{11}$$

$$-\infty < x < +\infty \quad , \quad x > \gamma$$

By differentiating of formula (11), the PDF of *X(1)* which is denoted by $f_{EV}^{x(1)}$ define:

$$f_{EV}^{x(1)} = \tfrac{1}{\delta}\exp\left(\tfrac{x-\gamma}{\delta} - \exp\left(\tfrac{x-\gamma}{\delta}\right)\right) \tag{12}$$

$$-\infty < x < +\infty \quad , \quad x > \gamma$$

If $\gamma = 0$, $\delta = 1$, we have the Standard form of EV dis.(Min) as bellow:

$$f(x) = e^{x}e^{-e^{x}} \tag{13}$$

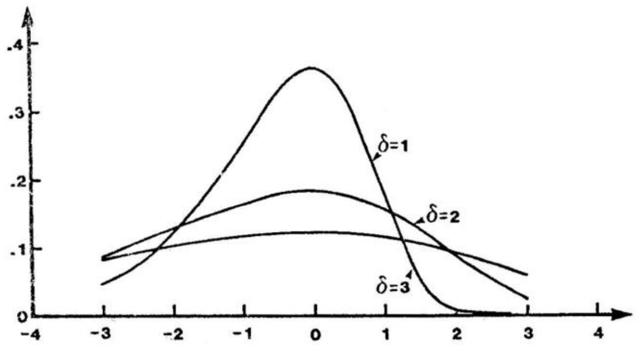

Figure 2. A general curve of EV dis. for minimal

Other statistical attributes of this function, are:

$$\mu = \gamma - 0.5772\,\delta \tag{14}$$

$$\sigma = \tfrac{\pi}{\sqrt{6}}\delta = 1.283\,\delta \tag{15}$$

$$Me = \gamma + \delta\ln(\ln(2)) = \gamma + 0.3665\,\delta \tag{16}$$

$$t_p = \gamma + \delta\ln\left(\ln\left(\tfrac{1}{1-p}\right)\right) \tag{17}$$

$$S.K = -5.4 \tag{18}$$

$$K = 1.14 \tag{19}$$

According the relations of the above, Gumbel distribution (EV dis.) curves is skewed to right in maximal mode, and skewed to left in minimal mode. Both of these forms are more kurtosis than Normal distribution.

3. Reliability of composite systems

Two single combinations of systems are series and parallel. *Figure 3* and *Figure 4* show them. Suppose C_i is the symbol of *i*-th Component. As we know the reliability of a series system is:

$$R_{sys}(1) = \prod_{i=1}^{n} R_i \tag{20}$$

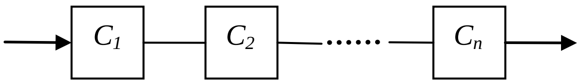

Figure 4. A simple view of series system

where R_i is reliability of *i*-th component. This relation for a parallel system is:

$$R_{sys}(2) = 1 - \prod_{i=1}^{n} (1 - R_i) \tag{21}$$

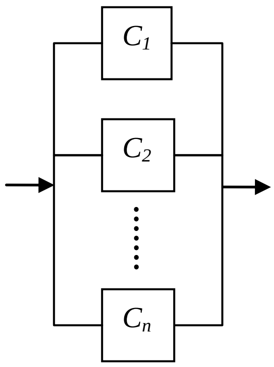

Figure 5. A single view of parallel system

When we don't have any information on the reliability of subsystems, or components, R_i, we can not use formula (20) and (21). So we must determine the life distribution function of whole system [5].
Suppose *C(i)* is the i-th part of system that fails. Thus in series system, duration life of whole system equals to the first part which fails, as *C(1)*. So life duration of a parallel system equals to last failed part, *C(n)*. Thus by given PDF of *C(1)* and *C(2)* we can determine the life distribution of series and parallel system. We want to apply the results of section 2.1 and 2.2 for estimation the PDF of *C(1)* and *C(2)*.
Consider the parallel structure in *Figure 4*. Suppose the quantity of subsystems or components, *n*, is large. The life duration of this system is equal to life duration of C(1) and meet the PDF same EV dis. in minimum mode, $f_{EV}^{x(1)}$. At the same way, the structure in *Figure 5* has life duration same the C(n) and meet the PDF of EV dis. in maximum mode, $f_{EV}^{x(n)}$. According these concepts we're going to determine the reliability indexes of both of the systems.

4. Computing the reliability of large composite system using the EV distribution

4.1. Series systems

Consider the given system in *Figure 4*, again. According to results of previous section, the reliability of this system can be computed by relation $f_{EV}^{x(1)}$:

$$\begin{aligned} R(t) = P(T \geq t) &= \int_t^\infty f(x)dx \\ &= \int_t^\infty f_{EV}^{x(1)}(x;\gamma,\delta)dx \\ &= 1 - F_{EV}^{x(1)}(\mathrm{t};\gamma,\delta) \\ &= \exp\left(-\exp\left(\tfrac{t-\gamma}{\delta}\right)\right) \end{aligned} \tag{22}$$

For determining the failure rate or hazard rate, we should try for *h(t)*:

$$\begin{aligned} H(t) &= -\ln\left(1 - F_{EV}^{\mathrm{x}(1)}(t)\right) \\ &= -\ln\left(\exp\left(-\exp\left(\tfrac{t-\gamma}{\delta}\right)\right)\right) \\ &= \exp\left(\tfrac{t-\gamma}{\delta}\right) \end{aligned} \tag{23}$$

$$\begin{aligned} h(t) &= \frac{d}{dt} H(t) \\ &= \tfrac{1}{\delta} \exp\left(\tfrac{t-\gamma}{\delta}\right) \end{aligned} \tag{24}$$

In the Preventive Maintenance (PM) for non-repairable parts, which should be replaced with any failure, we need to predict the time of the first failure. So determining the Mean Time to Failure (MTTF) will be necessary. By using this Index determine the scheduled inspections for on time replacements that can always keep the system in an available condition. However this index is very useful for non-repairable systems and we try to compute for described systems in the condition of this paper given in *Figure 4*:

$$MTTF = E(T)$$

$$= \int_0^\infty tf(t)dt$$

$$= \int_0^\infty R(t)dt$$

$$= \int_0^\infty \exp\left(-\exp\left(\tfrac{t-\gamma}{\delta}\right)\right)dt$$

$$= -\delta \exp\left(-\tfrac{t-\gamma}{\delta} - \exp\left(\tfrac{t-\gamma}{\delta}\right)\right)\Big|_0^\infty$$

$$= \delta \exp\left(\tfrac{\gamma}{\delta} - \exp\left(-\tfrac{\gamma}{\delta}\right)\right) \qquad (25)$$

For renewal systems we need to predict the during times between failures for periodic service before breaking down. So it was necessary to determine the Mean Time Between Failure (MTBF), which can be computed by statistics data of performance during time in the normal or intensive condition [1], [5].

4.2. Parallel systems

Consider given structure in *Figure 5* and suppose the same condition in section 4.1. According the results we discussed in section 2.1 and 3, life distribution of this system is equal to *C(n)* PDF. So we can determine reliability as bellow:

$$R(t) = P(T \geq t) = \int_0^\infty f(x)\,\mathrm{dx}$$

$$= 1 - F_{EV}^{x(n)}(t\,;\gamma\,,\delta)$$

$$= 1 - \exp\left(-\exp\left(-\tfrac{t-\gamma}{\delta}\right)\right) \qquad (26)$$

Other indexes of reliability can be computed same the series system [3]. For example the Failure rate or hazard function is:

$$H(t) = -\ln(1 - F_{EV}^{x(n)}(t))$$

$$= -\ln\left(1 - \exp\left(-\exp\left(-\tfrac{t-\gamma}{\delta}\right)\right)\right)$$

$$h(t) = \frac{d}{dt}H(t)$$

$$= \frac{\frac{1}{\delta}\exp\left(-\frac{t-\gamma}{\delta} - \exp\left(-\frac{t-\gamma}{\delta}\right)\right)}{1 - \exp\left(-\exp\left(-\frac{t-\gamma}{\delta}\right)\right)} \qquad (27)$$

5. Estimating parameters

In this step we must present the statistical estimators for each parameter in determined formulas. The main parameters of Gumbel distribution are δ (scale parameter), and γ (location parameter).

Using the Moments method for estimate the scale parameter of the above we obtain:

$$\hat{\delta} = \frac{\sqrt{6}}{\pi} S \qquad (28)$$

Where *S* is the standard deviation of sample. The below formulas will be obtained for location parameters:

$$\hat{\gamma}_{C(1)} = \bar{x} + 0.5772\,\hat{\delta} \qquad (29)$$

$$\hat{\gamma}_{C(n)} = \bar{x} - 0.5772\,\hat{\delta} \qquad (30)$$

where $\bar{x}$ is mean of the random sample and $\hat{\gamma}_{C(i)}$ is momentum estimator of location parameter for minimum (*C(1)*) or maximum (*C(n)*) mode of EV distribution. Formula (29) is for minimal mode and (30) for maximal mode.

6. Conclusion

During this paper we tried to present a new method for determining the reliability of composite systems with unknown reliable components. in this letter some useful relations and formulas were exhibited for reliability indexes computations. this new method is recommended strongly for large structures with various components. Although in this paper we discussed about single series or parallel systems, but it can de developed for complex systems.

However we offer to apply this presented theorem for large complex and applicable systems.

References

[1] Gumbel, E. J.(1954). Statistical Theory of Extreme values and Some Practical Applications. *Applied Mathematics Series Publication* No. 33.

[2] Gumbel, E. J. (1962). *Statistics of Extremes*. New York.

[3] Jar din A. K. S. (1943). Maintenance, Replacement, and Reliability.

[4] Nowak, A. S. & Collinas, K. R. (2000). *Reliability of Structures*. Singapore. Mc Graw Hill.

[5] Zacks, Shelemyuhu. (1992). *Introduction to Reliability Analysis.* New York. Springer-Verlag.