

## MINIMUM ENERGY CONTROL OF FRACTIONAL DESCRIPTOR POSITIVE DISCRETE-TIME LINEAR SYSTEMS

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Necessary and sufficient conditions for the positivity and reachability of fractional descriptor positive discrete-time linear systems are established. The minimum energy control problem for descriptor positive systems is formulated and solved. Sufficient conditions for the existence of a solution to the minimum energy control problem are given. A procedure for computation of optimal input sequences and a minimal value of the performance index is proposed and illustrated by a numerical example.

**Keywords:** fractional system, descriptor system, positive system, minimum energy control.

### 1. Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. Overviews of the state of the art in positive systems theory were made by Farina and Rinaldi (2000) as well as Kaczorek (2001). A variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

Mathematical fundamentals of fractional calculus are given by Oldham and Spanier (1974), Ostalczyk (2008) as well as Podlubny (1999). Positive fractional linear systems were investigated by Kaczorek (2008a; 2011b; 2011c; 2012). The stability of fractional linear 1D discrete-time and continuous-time systems was investigated by Busłowicz (2008), Dzieliński and Sierociuk (2008) as well as Kaczorek (2012), who also characterized the stability of 2D fractional positive linear systems (Kaczorek, 2009) and introduced the notion of practical stability of positive fractional discrete-time linear systems (Kaczorek, 2008b). Some recent interesting results in fractional systems theory and its applications can be found in the works of Dzieliński *et al.* (2009), Kaczorek (2011a; 2008c), Radwan *et al.* (2009), Tenreiro Machado and Ramiro Barbosa (2006) or Vinagre *et al.* (2002). The minimum energy control problem for standard linear systems was formulated and solved by Klamka (1991; 1983; 1976) and for 2D linear systems

with variable coefficients by Kaczorek and Klamka (1986). The controllability and minimum energy control problems of fractional discrete-time linear systems were investigated by Klamka (2010), while minimum energy control of fractional positive continuous-time linear systems was addressed by Kaczorek (2013a), who also discussed its counterpart for descriptor positive discrete-time linear systems (Kaczorek, 2013b).

In this paper, necessary and sufficient conditions for the positivity and reachability of fractional descriptor systems will be established, and the minimum energy control problem will be formulated and solved.

The paper is organized as follows. In Section 2, the reduction of fractional descriptor linear systems to an equivalent standard system by the use of the shuffle algorithm is addressed. In Section 3, the solution to the standard equivalent fractional system is given, and conditions for the positivity of the fractional descriptor system are established. Necessary and sufficient conditions for the reachability of positive fractional descriptor systems are given in Section 4. The minimum energy control problem is formulated and solved in Section 5. Concluding remarks are given in Section 6.

The following notation will be used:  $\mathbb{R}$ , the set of real numbers;  $\mathbb{R}^{n \times m}$ , the set of  $n \times m$  real matrices;  $\mathbb{R}_+^{n \times m}$ , the set of  $n \times m$  matrices with nonnegative entries and  $\mathbb{R}_+^n = \mathbb{R}_+^{n \times 1}$ ;  $x^T$ , the transpose of the vector  $x$ ;  $I_n$ , the

$n \times n$  identity matrix.

## 2. Reduction of fractional descriptor systems to standard systems

Consider the descriptor discrete-time linear system

$$E\Delta^\alpha x_{i+1} = Ax_i + Bu_i, \quad 0 < \alpha < 1, \quad i \in \mathbb{Z}_+ = \{0, 1, \dots\}, \quad (1)$$

where

$$\Delta^\alpha x_i = \sum_{j=0}^i c_j x_{i-j}, \quad c_j = (-1)^j \binom{\alpha}{j},$$

$$\binom{\alpha}{j} = \begin{cases} 1 & \text{for } j = 0, \\ \frac{\alpha(\alpha-1)\dots(\alpha-j+1)}{j!} & \text{for } j = 1, 2, \dots, \end{cases} \quad (2)$$

is the  $\alpha$ -th order fractional difference of the state vector,  $x_i \in \mathbb{R}^n$  and  $u_i \in \mathbb{R}^m$  are respectively the state and input vectors,  $E, A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ .

It is assumed that  $\det E = 0$  and the pencil  $Ez - A$  is regular, i.e.,

$$\det[Ez - A] \neq 0 \quad \text{for some } z \in \mathbb{C}. \quad (3)$$

Substitution of (2) into (1) yields

$$Ex_{i+1} = A_\alpha x_i - Ec_2 x_{i-1} - Ec_3 x_{i-2} - \dots - Ec_i x_1 - Ec_{i+1} x_0 + Bu_i, \quad i \in \mathbb{Z}_+, \quad (4)$$

where  $A_\alpha = A + E\alpha$ .

**Theorem 1.** *If the pencil of the fractional descriptor system (1) is regular (i.e., (3) holds), then the system can be reduced to the standard equivalent form with the use of the shuffle algorithm:*

$$x_{i+1} = \bar{A}_i x_i + \bar{A}_{i-1} x_{i-1} + \dots + \bar{A}_0 x_0 + \bar{B}_0 u_i + \bar{B}_1 u_{i+1} + \dots + \bar{B}_q u_{i+q}, \quad (5a)$$

where

$$\bar{A}_k \in \mathbb{R}^{n \times n}, \quad k = 0, 1, \dots, i, \\ \bar{B}_j \in \mathbb{R}^{n \times m}, \quad k = 0, 1, \dots, q, \quad (5b)$$

and  $q$  is the number of the shuffles.

*Proof.* The following elementary row operations will be used (Kaczorek, 1992; 2011):

1. Multiplication of the  $i$ -th row by a real number  $c$ . This operation will be denoted by  $L[i \times c]$ .

2. Addition to the  $i$ -th row of the  $j$ -th row multiplied by a real number  $c$ . This operation will be denoted by  $L[i + j \times c]$ .
3. Interchange of the  $i$ -th and  $j$ -th rows. This operation will be denoted by  $L[i, j]$ .

Performing elementary row operations on the array

$$\begin{bmatrix} E & A_\alpha & -Ec_2 & \dots & -Ec_{i+1} & B \end{bmatrix} \quad (6)$$

or, equivalently, on (4), we get

$$\begin{bmatrix} E_1 & A_{\alpha 1} & -E_1 c_2 & \dots & -E_1 c_{i+1} & B_1 \\ 0 & A_{\alpha 2} & 0 & \dots & 0 & B_2 \end{bmatrix} \quad (7)$$

and

$$E_1 x_{i+1} = A_{\alpha 1} x_i - E_1 c_2 x_{i-1} - E_1 c_3 x_{i-2} - \dots - E_1 c_{i+1} x_0 + B_1 u_i, \quad (8a)$$

$$0 = A_{\alpha 2} x_i + B_2 u_i. \quad (8b)$$

Substituting  $i$  in (8b) by  $i + 1$ , we obtain

$$-A_{\alpha 2} x_{i+1} = B_2 u_{i+1}. \quad (9)$$

Equations (8a) and (9) can be written in the form

$$\begin{bmatrix} E_1 \\ -A_{\alpha 2} \end{bmatrix} x_{i+1} = \begin{bmatrix} A_{\alpha 1} \\ 0 \end{bmatrix} x_i + \begin{bmatrix} -E_1 c_2 \\ 0 \end{bmatrix} x_{i-1} + \dots + \begin{bmatrix} -E_1 c_{i+1} \\ 0 \end{bmatrix} x_0 + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u_i + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u_{i+1}. \quad (10)$$

The array

$$\begin{bmatrix} E_1 & A_{\alpha 1} & -E_1 c_2 & \dots & -E_1 c_{i+1} & B_1 & 0 \\ -A_{\alpha 2} & 0 & 0 & \dots & 0 & 0 & B_2 \end{bmatrix} \quad (11)$$

can be obtained from (7) by performing a shuffle.

If the matrix

$$\begin{bmatrix} E_1 \\ -A_{\alpha 2} \end{bmatrix}$$

is nonsingular, then solving (10) we obtain the standard system

$$x_{i+1} = \begin{bmatrix} E_1 \\ -A_{\alpha 2} \end{bmatrix}^{-1} \left( \begin{bmatrix} A_{\alpha 1} \\ 0 \end{bmatrix} x_i + \begin{bmatrix} -E_1 c_2 \\ 0 \end{bmatrix} x_{i-1} + \dots + \begin{bmatrix} -E_1 c_{i+1} \\ 0 \end{bmatrix} x_0 + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u_i + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u_{i+1} \right). \quad (12)$$

If the matrix is singular, then performing elementary row operations on (11) we obtain

$$\begin{bmatrix} E_2 & A_{\alpha 2} & -E_2 c_2 & \dots & -E_2 c_{i+1} & B_2 & B_4 \\ 0 & A_{\alpha 3} & 0 & \dots & 0 & B_3 & B_5 \end{bmatrix}, \tag{13}$$

where  $E_2$  has full row rank and  $\text{rank } E_2 \geq \text{rank } E_1$ .

Substituting  $i$  in  $0 = A_{\alpha 3} x_i + B_3 u_i + B_5 u_{i+1}$  by  $i + 1$ , we obtain

$$-A_{\alpha 3} x_{i+1} = B_3 u_{i+1} + B_5 u_{i+2}. \tag{14}$$

The equations

$$E_2 x_{i+1} = A_{\alpha 2} x_i - E_2 c_2 - \dots - E_2 c_{i+1} + B_2 u_i + B_4 u_{i+1}$$

and (14) can be written down as

$$\begin{aligned} & \begin{bmatrix} E_2 \\ -A_{\alpha 3} \end{bmatrix} x_{i+1} \\ &= \begin{bmatrix} A_{\alpha 2} \\ 0 \end{bmatrix} x_i + \begin{bmatrix} -E_2 c_2 \\ 0 \end{bmatrix} x_{i-1} \\ &+ \dots + \begin{bmatrix} -E_2 c_{i+1} \\ 0 \end{bmatrix} x_0 + \begin{bmatrix} B_2 \\ 0 \end{bmatrix} u_i \\ &+ \begin{bmatrix} B_4 \\ B_3 \end{bmatrix} u_{i+1} + \begin{bmatrix} 0 \\ B_5 \end{bmatrix} u_{i+2}. \end{aligned} \tag{15}$$

The array

$$\begin{bmatrix} E_2 & A_{\alpha 2} & -E_2 c_2 & \dots & -E_2 c_{i+1} & B_2 & B_4 & 0 \\ -A_{\alpha 3} & 0 & 0 & \dots & 0 & 0 & B_3 & B_5 \end{bmatrix} \tag{16}$$

can be obtained from (13) by performing a shuffle.

If

$$\det \begin{bmatrix} E_2 \\ -A_{\alpha 3} \end{bmatrix} \neq 0,$$

we can find  $x_{i+1}$  from (15). Otherwise, we repeat the procedure for (16). If the pencil is regular, then after  $q$  steps we obtain a nonsingular matrix

$$\begin{bmatrix} E_q \\ -A_{\alpha q+1} \end{bmatrix}$$

and the desired equation (5). ■

**Example 1.** Consider the fractional descriptor system (1) with  $\alpha = 0.5$  and the matrices

$$\begin{aligned} E &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \\ B &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}. \end{aligned} \tag{17}$$

To reduce the descriptor system to an equivalent standard system using elementary row operations, we reduce the array

$$\begin{aligned} & [ E \quad A_{\alpha} \quad -E c_2 \quad \dots \quad -E c_{i+1} \quad B ] \\ &= \begin{bmatrix} 1 & 0 & 0 & 1.5 & 0 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -c_2 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & \dots & -c_{i+1} & 0 & 0 & 0 & 1 & 0 \\ 0 & \dots & 0 & -c_{i+1} & 0 & 1 & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{aligned} \tag{18}$$

to the form (7). In this case, the array (18) has already the desired form (7) with

$$\begin{aligned} E_1 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \\ A_{\alpha 1} &= \begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \\ E_1 c_2 &= \begin{bmatrix} c_2 & 0 & 0 \\ 0 & c_2 & 0 \end{bmatrix}, \\ &\vdots \\ E_1 c_{i+1} &= \begin{bmatrix} c_{i+1} & 0 & 0 \\ 0 & c_{i+1} & 0 \end{bmatrix}, \\ A_{\alpha 2} &= [ 0 \quad 0 \quad -1 ], \\ B_1 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \\ B_2 &= [ 1 \quad 1 ]. \end{aligned} \tag{19}$$

Performing the shuffle, we obtain

$$\begin{aligned} & \begin{bmatrix} E_1 & A_{\alpha 1} & -E_1 c_2 & \dots & -E_1 c_{i+1} & B_1 & 0 \\ -A_{\alpha 2} & 0 & 0 & \dots & 0 & 0 & B_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 1.5 & 0 & 0 & -c_2 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -c_2 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ -c_{i+1} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -c_{i+1} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{aligned} \tag{20}$$

and the equivalent standard system

$$\begin{aligned} x_{i+1} &= \bar{A}_i x_i + \bar{A}_{i-1} x_{i-1} + \dots + \bar{A}_0 x_0 \\ &+ \bar{B}_0 u_i + \bar{B}_1 u_{i+1}, \quad i \in \mathbb{Z}_+, \end{aligned} \tag{21a}$$

where

$$\bar{A}_i = \begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{aligned} \bar{A}_{i-1} &= \begin{bmatrix} -c_2 & 0 & 0 \\ 0 & -c_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ &\vdots \\ \bar{A}_0 &= \begin{bmatrix} -c_{i+1} & 0 & 0 \\ 0 & -c_{i+1} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ \bar{B}_0 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \\ \bar{B}_1 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}. \end{aligned} \tag{21b}$$

In this case, the number of shuffle is  $q = 1$ . ◆

### 3. Solution for the standard equivalent system and the positivity of fractional descriptor systems

To find a solution  $x_i$  of the standard discrete-time linear system (5), we shall apply the  $\mathcal{Z}$ -transform method.

Let  $X(z)$  be the  $\mathcal{Z}$ -transform of  $x_i$  defined by

$$\mathcal{Z}[x_i] = \sum_{i=0}^{\infty} x_i z^{-i}. \tag{22}$$

Taking into account that

$$\mathcal{Z}[x_{i-p}] = z^{-p}X(z) + z^{-p} \sum_{j=-1}^{-p} x_j z^{-j}, \tag{23a}$$

$$\mathcal{Z}[x_{i+p}] = z^p X(z) - \sum_{l=0}^{p-1} x_l z^{p-l}, \tag{23b}$$

$p = 1, 2, \dots$  and applying the  $\mathcal{Z}$ -transform to Eqn. (5), we obtain

$$\mathcal{Z}[x_{i+1}] = \sum_{k=0}^i \bar{A}_k \mathcal{Z}[x_{i-k}] + \sum_{j=0}^q \bar{B}_j \mathcal{Z}[u_{i+j}] \tag{24a}$$

and

$$\begin{aligned} zX(z) - zx_0 &= \sum_{k=0}^i \bar{A}_k z^{-k} X(z) \\ &+ \sum_{j=0}^q \bar{B}_j z^q \left[ U(z) - \sum_{l=0}^{q-1} u_l z^{-l} \right], \end{aligned} \tag{24b}$$

where  $U(z) = \mathcal{Z}[u_i]$  and  $x_j = 0, j = -1, \dots, -k$ .

Multiplying (24b) by  $z^{-1}$  and solving with respect to  $X(z)$ , we obtain

$$\begin{aligned} X(z) &= \left[ I_n - \sum_{k=0}^i \bar{A}_k z^{-(k+1)} \right]^{-1} \\ &\times \left[ x_0 + \sum_{j=0}^q \bar{B}_j z^{q-1} \left[ U(z) - \sum_{l=0}^{q-1} u_l z^{-l} \right] \right]. \end{aligned} \tag{25}$$

Substitution of the expansion

$$\left[ I_n - \sum_{k=0}^i \bar{A}_k z^{-(k+1)} \right]^{-1} = \sum_{j=0}^{\infty} \Phi_j z^{-j} \tag{26}$$

into (25) yields

$$\begin{aligned} X(z) &= \sum_{j=0}^{\infty} \Phi_j z^{-j} \\ &\times \left[ x_0 + \sum_{j=0}^q \bar{B}_j z^{q-1} \left[ U(z) - \sum_{l=0}^{q-1} u_l z^{-l} \right] \right]. \end{aligned} \tag{27}$$

From definition of the inverse matrix, we have

$$\begin{aligned} \left[ I_n - \sum_{k=0}^i \bar{A}_k z^{-(k+1)} \right] \left[ \sum_{j=0}^{\infty} \Phi_j z^{-j} \right] \\ = \left[ \sum_{j=0}^{\infty} \Phi_j z^{-j} \right] \left[ I_n - \sum_{k=0}^i \bar{A}_k z^{-(k+1)} \right] = I_n. \end{aligned} \tag{28}$$

Comparison of the coefficients at the same powers of  $z^{-k}, k = 0, 1, \dots$  from (28) yields

$$\begin{aligned} \Phi_0 &= I_n, & \Phi_1 &= \bar{A}_0, \\ \Phi_2 &= \bar{A}_0 \Phi_1 + \bar{A}_1, & \Phi_3 &= \bar{A}_0 \Phi_2 + \bar{A}_1 \Phi_1 + \bar{A}_2 \end{aligned}$$

and, in general,

$$\begin{aligned} \Phi_k &= \bar{A}_0 \Phi_{k-1} + \bar{A}_1 \Phi_{k-2} + \dots + \bar{A}_{k-1} \Phi_0 \\ &= \Phi_{k-1} \bar{A}_0 + \Phi_{k-2} \bar{A}_1 + \dots + \Phi_0 \bar{A}_{k-1}, \end{aligned} \tag{29}$$

Applying the inverse  $\mathcal{Z}$ -transform and the convolution theorem (Kaczorek, 2012) to (27), we obtain the desired solution

$$\begin{aligned} x_i &= \Phi_i x_0 \\ &+ \sum_{k=0}^i \Phi_{i-k-1} \left( \sum_{j=0}^q \bar{B}_j u_{j+k} \right), \quad i \in \mathbb{Z}_+. \end{aligned} \tag{30}$$

Thus, the following theorem has been proved.

**Theorem 2.** The solution of Eqn. (5) has the form (30), where the matrices  $\Phi_k$  are given by (29).

**Definition 1.** The fractional discrete system (1) is called (internally) positive if  $x_i \in \mathbb{R}_+^n$  for every consistent  $x_0 \in \mathbb{R}_+^n$  and all inputs  $u_i \in \mathbb{R}_+^m, i \in \mathbb{Z}_+$ .

**Theorem 3.** The fractional descriptor system (1) is positive if and only if the matrices of the equivalent standard system (5) satisfy the conditions

$$\bar{A}_k \in \mathbb{R}_+^{n \times n}, \quad k = 0, 1, \dots, i$$

and

$$\bar{B}_j \in \mathbb{R}_+^{n \times m}, \quad j = 0, 1, \dots, q. \quad (31)$$

*Proof.* It is well known that the state vectors of the descriptor system (1) and the equivalent standard system (5) are the same. By Definition 1, the descriptor system (1) is positive if and only if the standard system (5) is positive. From (30) it follows that  $x_i \in \mathbb{R}_+^n, i \in \mathbb{Z}_+$  if the conditions (31) are met and  $x_0 \in \mathbb{R}_+^n, u_i \in \mathbb{R}_+^m$ . The necessity of the condition (31) follows from the fact that  $x_i \in \mathbb{R}_+^n$  for every consistent  $x_0 \in \mathbb{R}_+^n$  and arbitrary  $u_i \in \mathbb{R}_+^m, i \in \mathbb{Z}_+$  (Kaczorek, 2012). ■

**Example 2.** (Continuation of Example 1) Check the positivity of the fractional descriptor system (1) with  $\alpha = 0.5$  and the matrices (17). In Example 1 it was shown that the equivalent standard system has the matrices given by (21a) and (21b). The matrices have nonnegative entries since

$$c_2 = \frac{\alpha(\alpha - 1)}{2!} < 0 \quad \text{for } \alpha = 0.5$$

and

$$\begin{aligned} c_{k+1} &= (-1)^{k+1} \frac{\alpha(\alpha - 1) \dots (\alpha - k + 1)(\alpha - k)}{(k + 1)!} \\ &= c_k \frac{k - \alpha}{k + 1} < 0, \quad k = 2, 3, \dots \end{aligned} \quad (32)$$

Therefore, the fractional descriptor system (1) with  $\alpha = 0.5$  and (17) is positive. ◆

#### 4. Reachability of positive fractional descriptor systems

Consider the positive fractional descriptor discrete-time system (1). It is called reachable in  $n$  steps if and only if the equivalent standard system (5) is reachable in  $n$  steps.

**Definition 2.** The positive system (5) is called *reachable in  $n$  steps* if for any given  $x_f \in \mathbb{R}_+^n$  there exists an input sequence  $u_k \in \mathbb{R}_+^m$  for  $k = 0, 1, \dots, h - 1, h = n + q$ , which steers the state of the system from  $x_0 = 0$  to  $x_f \in \mathbb{R}_+^n$ , i.e.,  $x_n = x_f$ .

**Theorem 4.** The positive system (5) is reachable in  $n$  steps if and only if the reachability matrix

$$R_n = [ \Phi_{n-1} \bar{B}_0 \quad \Phi_{n-2} \bar{B}_1 \quad \dots \quad \Phi_{n-q-2} \bar{B}_{q-1} \quad \Phi_{n-q-1} \bar{B}_q ] \quad (33)$$

contains  $n$  linearly independent monomial columns.

*Proof.* Using (30) for  $i = n$  and  $x_0 = 0$ , we obtain

$$\begin{aligned} x_f &= x_n \\ &= [ \Phi_{n-q-1} \bar{B}_q \quad \Phi_{n-q-2} \bar{B}_{q-1} \quad \dots \quad \Phi_{n-2} \bar{B}_1 \quad \Phi_{n-1} \bar{B}_0 ] \begin{bmatrix} u_{n+q} \\ u_{n+q-1} \\ \vdots \\ u_0 \end{bmatrix} \\ &= R_n \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{h+q} \end{bmatrix}, \end{aligned} \quad (34)$$

where  $R_n$  is defined by (33).

From (34) it follows that there exists an input sequence  $u_k \in \mathbb{R}_+^m$  for  $k = 0, 1, \dots, i + q$  if and only if the matrix (33) contains  $n$  linearly independent monomial columns. ■

**Remark 1.** Assuming that the components of the input sequence which do not correspond to the chosen linear independent monomial columns are zero, we obtain a different input sequence which steers the state vector from  $x_0 = 0$  to  $x_n = x_f$ .

**Example 3.** (Continuation of Examples 1 and 2) Check the reachability of the positive fractional descriptor system (1) with  $\alpha = 0.5$  and the matrices (17), and compute an input sequence that steers in two steps the state vector of the system from  $x_0 = [ 0 \ 0 \ 0 ]^T$  to  $x_f = [ 1 \ 1 \ 1 ]^T$ . In this case, the reachability matrix (33) has the form

$$\begin{aligned} R_4 &= [ \bar{B}_1 \quad \bar{B}_0 + \bar{A}_1 \bar{B}_1 \quad \bar{A}_1 \bar{B}_0 ] \\ &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned} \quad (35)$$

The matrix (35) contains three linearly independent monomial columns, and by Theorem 4 the positive fractional descriptor system is reachable in two steps. Choosing as linearly independent monomial columns the fifth, fourth and first column of the matrix (35), we obtain

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{21} \\ u_{12} \\ u_{01} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} u_{21} \\ u_{12} \\ u_{01} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} u_2 \\ u_1 \\ u_0 \end{bmatrix} = \begin{bmatrix} u_{21} \\ u_{22} \\ u_{11} \\ u_{12} \\ u_{01} \\ u_{02} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}. \quad (36)$$

If we choose as linearly independent monomial columns the second, third and last ones of the matrix (35), then we obtain

$$\begin{bmatrix} u_{21} \\ u_{12} \\ u_{01} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1.5 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \frac{10}{15} \end{bmatrix}$$

and

$$\begin{bmatrix} u_2 \\ u_1 \\ u_0 \end{bmatrix} = \begin{bmatrix} u_{21} \\ u_{22} \\ u_{11} \\ u_{12} \\ u_{01} \\ u_{02} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ \frac{10}{15} \end{bmatrix}. \quad (37)$$



### 5. Minimum energy control problem

Consider the fractional descriptor positive system (1) reduced to the form (2). In Section 4 it was shown that if the positive system is reachable then there exist many input sequences which steer the state of the system from  $x_0 = 0$  to the given final state  $x_f \in \mathbb{R}_+^n$ . Among these input sequences we are looking for a sequence  $u_k \in \mathbb{R}_+^m$  for  $k = 0, 1, \dots, n + q - 1$  that minimizes the performance index

$$I(u) = \sum_{i=0}^{h-1} u_i^T Q u_i, \quad (38)$$

where  $Q \in \mathbb{R}_+^{m \times m}$  is a symmetric positive definite matrix such that

$$Q^{-1} \in \mathbb{R}_+^{m \times m}, \quad (39)$$

and  $h = n + q$  is the number of steps in which the state of the system is transferred from  $x_0 = 0$  to the given final state  $x_f \in \mathbb{R}_+^n$ .

The minimum energy control problem for the fractional descriptor positive discrete-time linear system (1) can be stated as follows.

Given the matrices  $E, A, B$  of the descriptor positive system (1), the number of steps  $h$ , the final state  $x_f \in \mathbb{R}_+^n$  and the matrix  $Q \in \mathbb{R}_+^{m \times m}$  of the performance index (38) satisfying the condition (39), find a sequence of inputs

$u_k \in \mathbb{R}_+^m$  for  $k = 0, 1, \dots, h - 1$  that steers the state of the system from  $x_0 = 0$  to  $x_f \in \mathbb{R}_+^n$  and minimizes the performance index (38).

To solve the problem, we define the matrix

$$W_h = R_h Q_h^{-1} R_h^T \in \mathbb{R}_+^{n \times n}, \quad (40)$$

where  $R_h \in \mathbb{R}_+^{n \times hm}$  is given by (33) and

$$Q_h^{-1} = \text{block diag} [Q^{-1}, \dots, Q^{-1}] \in \mathbb{R}_+^{hm \times hm}. \quad (41)$$

The matrix (40) is non-singular if the positive system is reachable in  $h$  steps.

For a given  $x_f \in \mathbb{R}_+^n$ , we may define the input sequence

$$\hat{u} = \begin{bmatrix} \hat{u}_{h-1} \\ \hat{u}_{h-2} \\ \vdots \\ \hat{u}_0 \end{bmatrix} = Q_h^{-1} R_h^T W_h^{-1} x_f, \quad (42)$$

where  $Q_h^{-1}, W_h$  and  $R_h$  are defined by (41), (40) and (33), respectively.

Note that  $\hat{u} \in \mathbb{R}_+^{hm}$  if

$$W_h^{-1} \in \mathbb{R}_+^{n \times n}, \quad (43)$$

and this holds if the condition (39) is met.

**Theorem 5.** Let the fractional descriptor positive system (1) be reachable in  $h$  steps and the conditions (39) and (43) be satisfied. Moreover, let

$$\bar{u} = \begin{bmatrix} \bar{u}_{h-1} \\ \bar{u}_{h-2} \\ \vdots \\ \bar{u}_0 \end{bmatrix} \in \mathbb{R}_+^{hm}, \quad (44)$$

be an input sequence that steers the state of the system from  $x_0 = 0$  to  $x_f \in \mathbb{R}_+^n$ . Then the input sequence (42) also steers the state of the system from  $x_0 = 0$  to  $x_f \in \mathbb{R}_+^n$  and minimizes the performance index (38), i.e.,

$$I(\hat{u}) \leq I(\bar{u}). \quad (45)$$

The minimal value of the performance index (38) is given by

$$I(\hat{u}) = x_f^T W_h^{-1} x_f. \quad (46)$$

*Proof.* If the conditions (39) and (43) are met and the system is reachable in  $h$  steps, then the input sequence (42) is well defined and  $\hat{u} \in \mathbb{R}_+^{hm}$ . We shall show that the input sequence (42) steers the state of the system from  $x_0 = 0$  to  $x_f \in \mathbb{R}_+^n$ . Using (35) and (42), we obtain

$$x_h = R_h \hat{u} = R_h Q_h^{-1} R_h^T W_h^{-1} x_f = x_f, \quad (47)$$

since, by (40),  $R_h Q_h^{-1} R_h^T = W_h$ . Hence,  $x_f = R_h \hat{u} = R_h \bar{u}$  or

$$R_h[\hat{u} - \bar{u}] = 0. \tag{48}$$

The transposition of (48) yields

$$[\hat{u} - \bar{u}]^T R_h^T = 0. \tag{49}$$

Postmultiplying the equality (49) by  $W_h^{-1} x_f$ , we obtain

$$[\hat{u} - \bar{u}]^T R_h^T W_h^{-1} x_f = 0. \tag{50}$$

From (42), we have  $Q_h \hat{u} = R_h^T W_h^{-1} x_f$ . Substitution of this equality into (50) yields

$$[\hat{u} - \bar{u}]^T Q_h \hat{u} = 0, \tag{51}$$

where  $Q_h = \text{block diag} [Q, \dots, Q] \in \mathbb{R}_+^{hm \times hm}$ .

From (51), it follows that

$$\bar{u}^T Q_h \bar{u} = \hat{u}^T Q_h \hat{u} + [\bar{u} - \hat{u}]^T Q_h [\bar{u} - \hat{u}], \tag{52}$$

since, by (51),  $\bar{u}^T Q_h \hat{u} = \hat{u}^T Q_h \hat{u} = \hat{u}^T Q_h \bar{u}$ .

From (52) it follows that (45) holds since  $[\bar{u} - \hat{u}]^T Q_h [\bar{u} - \hat{u}] \geq 0$ . To find the minimal value of the performance index (38), we substitute (42) into (38) and obtain

$$\begin{aligned} I(\hat{u}) &= \sum_{i=0}^{h-1} \hat{u}_i^T Q \hat{u}_i = \hat{u}^T Q_h \hat{u} \\ &= [Q_h^{-1} R_h^T W_h^{-1} x_f]^T Q_h [Q_h^{-1} R_h^T W_h^{-1} x_f] \\ &= x_f^T W_h^{-1} R_h Q_h^{-1} R_h^T W_h^{-1} x_f \\ &= x_f^T W_h^{-1} x_f, \end{aligned} \tag{53}$$

since, by, (40)  $W_h^{-1} R_h Q_h^{-1} R_h^T = I_n$ . ■

From the above discussion we have the following procedure for computation of the optimal input sequence and the minimal value of the performance index (42).

**Procedure 1.**

*Step 1.* Knowing  $E, A, B$ , compute  $\bar{A}_k$  for  $k = 0, 1, \dots, i$  and  $\bar{B}_l$  for  $l = 0, 1, \dots, q$ .

*Step 2.* Using (33), compute  $R_h$ .

*Step 3.* For given  $Q$ , using (41), compute  $Q_h^{-1}$ .

*Step 4.* Using (40), compute  $W_h$  and  $W_h^{-1}$ .

*Step 5.* For given  $x_f \in \mathbb{R}_+^n$ , using (42), compute  $\hat{u}$ .

*Step 6.* Using (46), compute the minimal value of the performance index  $I(\hat{u})$ .

**Example 4.** (Continuation of Examples 1–3) For the fractional descriptor system (1) with  $\alpha = 0.5$  and (17), compute the optimal input sequence that steers the system

from  $x_0 = [0 \ 0 \ 0]^T$  to  $x_f = [1 \ 1 \ 1]^T$ , and minimize the performance index (38) for

$$Q_h = \text{diag} [2, 2, 2, 2]. \tag{54}$$

Using Procedure 1 and results of Example 3, we obtain the following.

*Step 1.* The desired matrices  $\bar{A}_k$  for  $k = 0, 1, \dots, i$  and  $\bar{B}_l$  for  $l = 0, 1$  are given by (21).

*Step 2.* The matrix  $R_4$  is given by (35).

*Step 3.* Using (41) and (54), we obtain

$$Q_4^{-1} = \text{diag} [0.5, 0.5, 0.5, 0.5, 0.5, 0.5]. \tag{55}$$

*Step 4.* Using (40) and (55), we obtain

$$\begin{aligned} W_4 &= R_4 Q_4^{-1} R_4^T \\ &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &\quad \times \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 \end{bmatrix} \\ &\quad \times \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1.5 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1.625 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{56a}$$

and

$$W_4^{-1} = \begin{bmatrix} 0.6154 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{56b}$$

*Step 5.* Using (42), (55) and (56b), we obtain

$$\begin{aligned} \hat{u} &= \begin{bmatrix} \hat{u}_2 \\ \hat{u}_1 \\ \hat{u}_0 \end{bmatrix} = Q_4^{-1} R_4^T W_4^{-1} x_f \\ &= \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 \end{bmatrix} \end{aligned}$$



$$\begin{aligned} & \times \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1.5 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.6154 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ & = \begin{bmatrix} 0.5 \\ 0.4615 \\ 0.5 \\ 0.3077 \\ 0.5 \\ 0.5 \end{bmatrix}. \end{aligned} \quad (57)$$

Step 6. Using (46) and (56b), we obtain

$$\begin{aligned} I(\hat{u}) &= x_f^T W_4^{-1} x_f \\ &= [1 \quad 1 \quad 1] \begin{bmatrix} 0.6154 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= 2.6154. \end{aligned} \quad (58)$$

Note that the optimal input sequence (58) is different from the input sequences (36) and (37). ♦

## 6. Concluding remarks

Necessary and sufficient conditions for the positivity and reachability of fractional descriptor positive discrete-time linear systems have been established (Theorems 1 and Theorem 3). The transformation of the fractional descriptor system to an equivalent standard one with the use of the shuffle algorithm has been addressed. The minimum energy control problem for fractional descriptor positive systems has been formulated and solved (Theorem 5). A procedure for computation of optimal input sequences and a minimal value of the performance index has been proposed (Procedure 1). The procedure has been demonstrated on a numerical example.

An open problem is the extension of these deliberations to fractional positive descriptor 2D continuous-discrete linear systems.

## Acknowledgment

This work was supported by the National Science Centre in Poland under the work G/WE/1/11.

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Received: 21 November 2013

Revised: 22 January 2014