42

UNIFORM HORIZONTAL PROJECTILE MOTION

42.1 INTRODUCTION

This contribution complements our preceding papers devoted to the theoretical problem of the uniform motions in the homogeneous and central gravitational fields [7,8]. The mathematical framework of the problem is presented in more details e.g. in [1–6].

From the mechanics it is known that motions of the particles are classified with respect to two aspects. The first aspect is the shape of the trajectory, we distinguish between *rectilinear* motions and *curvilinear* motions. The second aspect of the classification is the speed, we distinguish between *uniform* and *non uniform* motions. In the case of uniform motions the magnitude of the instantaneous velocity $\boldsymbol{v}$ remains constant during the entire motion, $|\boldsymbol{v}| = v_0 = \text{const}$. In the case of non uniform motions the magnitude of the instantaneous velocity is changing during the motion, $|\boldsymbol{v}| = v(t)$.

It is evident that the classical horizontal projectile motion is neither rectilinear nor uniform. To make the classical horizontal projectile motion uniform we consider this motion as a *constrained mechanical system* subjected to a certain subsidiary condition - *constraint*, which ensures desired character of the motion.

In this article we study the modification of the classical horizontal projectile motion onto the uniform one. The problem is solved as the initial value problem for the *reduced motion equation* of the constrained mechanical system arisen from the mechanical system of one particle in the homogeneous gravitational field subjected to one *nonholonomic constraint* nonlinear in the components of the velocity,

$$|\boldsymbol{v}| = \sqrt{v_x^2 + v_y^2} = \sqrt{\dot{x}^2 + \dot{y}^2} = v_0, \tag{42.1}$$

which represents the requirement of the uniformity of the motion. This constraint is called *isotachytonic* constraint.

In the paper we present the comparison of both horizontal projectile motions (classical and uniform) under the same initial conditions, both particles starts from the initial height h_0 with the initial velocity $\boldsymbol{v}_0 = (v_0, 0)$. The air resistance is neglected. The presented problem can serve as a suitable topic for an seminar of theoretical mechanics on the university level.

42.2 CLASSICAL HORIZONTAL PROJECTILE MOTION

We consider a particle m moving in the homogeneous gravitational field ($\boldsymbol{g} = (0, -g)$ gravitational acceleration). The particle starts from the point $[0, h_0]$ with the initial velocity $\boldsymbol{v}_0 = (v_0, 0)$, i.e. it is oriented in the horizontal direction. The Lagrangian of the considered mechanical system is

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy. \tag{42.2}$$

The Newtonian motion equations respective the Euler-Lagrange equations of the Lagrangian (42.2) have the simple form

$$m\ddot{x} = 0, \quad m\ddot{y} = -mg. \tag{42.3}$$

With respect to the initial conditions $x(0) = 0$, $y(0) = h_0$, $\dot{x}(0) = v_0$, and $\dot{y}(0) = 0$ we get the particular solution of (42.3)

$$x(t) = v_0 t, \quad y(t) = h_0 - \tfrac{1}{2}gt^2, \tag{42.4}$$

which represents the parametric expression of the classical horizontal projectile motion. By the elimination of the time parameter t, $t = x/v_0$, we obtain

$$y = h_0 - \tfrac{g}{2v_0^2}x^2, \tag{42.5}$$

which represents the equation of a parabola. However, the trajectory is realized only along the piece of the parabola (42.5) from the starting point $[0, h_0]$ to the impact point $[R, 0]$, where R denotes the distance of the impact from the origin (range).

The time dependence of the Cartesian components $\boldsymbol{v}_x = v_x\boldsymbol{i}$, $\boldsymbol{v}_y = v_y\boldsymbol{j}$ of the velocity $\boldsymbol{v}$ is given by

$$v_x(t) = \dot{x}(t) = v_0, \quad v_y(t) = \dot{y}(t) = -gt, \tag{42.6}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$ are the unit vectors in the direction of x-axis and y-axis, respectively. Hence, the time dependence of the magnitude of the instantaneous velocity is expressed by

$$v(t) = \sqrt{v_x^2(t) + v_y^2(t)} = \sqrt{v_0^2 + g^2t^2}. \tag{42.7}$$

Cartesian components $\boldsymbol{a}_x = a_x\boldsymbol{i}$, $\boldsymbol{a}_y = a_y\boldsymbol{j}$ of the acceleration $\boldsymbol{a}$ are

$$a_x(t) = \ddot{x}(t) = \frac{\mathrm{d}}{\mathrm{d}t}v_x(t) = 0, \quad a_y(t) = \ddot{y}(t) = \frac{\mathrm{d}}{\mathrm{d}t}v_y(t) = -g. \tag{42.8}$$

Hence, the magnitude of the instantaneous acceleration is

$$a(t) = \sqrt{a_x^2(t) + a_y^2(t)} = g. \tag{42.9}$$

The other possible decomposition of the acceleration vector $\boldsymbol{a}$ is on the tangent $\boldsymbol{a}_t$ and normal $\boldsymbol{a}_n$ components. The tangent component is given by

$$\boldsymbol{a}_t = a_t \boldsymbol{e}_{\boldsymbol{v}}, \qquad a_t = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{g^2 t}{\sqrt{v_0^2 + g^2 t^2}}, \tag{42.10}$$

where $\boldsymbol{e}_{\boldsymbol{v}}$ is the unit vector in the direction of the instantaneous velocity $\boldsymbol{v}$. The normal component of the acceleration is

$$\boldsymbol{a}_n = \boldsymbol{a} - \boldsymbol{a}_t, \qquad a_n = \sqrt{a^2 - a_t^2} = \frac{v_0 g}{\sqrt{v_0^2 + g^2 t^2}}. \tag{42.11}$$

On the other hand, for the magnitude of the normal component a_n holds

$$a_n(t) = \frac{v^2(t)}{r(t)} = v^2(t)\varkappa(t), \tag{42.12}$$

where $r(t)$ is the radius of the curvature of the trajectory at the time t and $\varkappa(t) = 1/r(t)$ is the curvature of the trajectory at the time t. If we compare the formulas (42.11) and (42.12) we obtain the expression of the curvature $\varkappa(t)$ of the trajectory of the classical horizontal projectile motion,

$$\varkappa(t) = \frac{v_0 g}{\sqrt{(v_0^2 + g^2 t^2)^3}}. \tag{42.13}$$

The same result can be calculated using the classical geometric formula for the curvature $\varkappa$ of a plain curve under arbitrary parametrisation

$$\varkappa(t) = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{\sqrt{(\dot{x}^2 + \dot{y}^2)^3}} \tag{42.14}$$

after substituting and applying (42.6), (42.7) and (42.8).

The kinematic parameters of the classical horizontal projectile motion are:

- total time $T = \sqrt{\frac{2h_0}{g}}$,
- range $R = v_0\sqrt{\frac{2h_0}{g}}$,
- impact speed $v_R = \sqrt{v_0^2 + 2gh_0}$,
- angle of impact $\cos\alpha = \dfrac{|\boldsymbol{v}_R \cdot \boldsymbol{i}|}{|\boldsymbol{v}_R||\boldsymbol{i}|} = \dfrac{v_0}{\sqrt{v_0^2 + 2gh_0}}$,
  where $\boldsymbol{v}_R = \boldsymbol{v}(T) = (v_0, -\sqrt{2gh_0})$ is the impact velocity vector, $|\boldsymbol{v}_R| = v_R$, $\boldsymbol{i} = (1, 0)$ is the unit vector in the direction of x-axis,

- curvature of the trajectory $\varkappa(t) = \dfrac{v_0 g}{\sqrt{(v_0^2 + g^2t^2)^3}}$.

Finally we computed the length of the trajectory of the classical horizontal projectile motion

$$s = \int_0^T v(t)\mathrm{d}t = \int_0^T \sqrt{v_0^2 + g^2t^2}\mathrm{d}t. \tag{42.15}$$

The integration and the evaluation in the limits from 0 to T leads us to the result

$$s = \sqrt{\tfrac{h_0}{2g}}\sqrt{v_0^2 + 2gh_0} + \tfrac{v_0^2}{2g}\ln\left(\frac{\sqrt{2gh_0} + \sqrt{v_0^2 + 2gh_0}}{v_0}\right). \tag{42.16}$$

42.3 THE DYNAMICS OF THE PARTICLE IN THE PLANE SUBJECT TO ONE NON-HOLONOMIC CONSTRAINT

Consider a mechanical system of one particle in the configuration space $\mathbb{R}^2$ (plane) characterized by the *Lagrangian* $L = T - V$, where T is the *kinetic energy* of the particle and V is its *potential energy.* Denote by q^1, q^2, certain generalized coordinates in the configuration space. The motion of the particle is then governed by the *Newtonian motion equations*

$$\ddot{q}^1 = -\frac{\partial V}{\partial q^1}, \quad \ddot{q}^2 = -\frac{\partial V}{\partial q^2}, \tag{42.17}$$

which arise as the *Euler-Lagrange equations*

$$\frac{\partial L}{\partial q^1} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^1}\right) = 0, \quad \frac{\partial L}{\partial q^2} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^2}\right) = 0, \tag{42.18}$$

of the Lagrangian L.

In the case when the motion of the particle is restricted by one *nonholonomic constraint* depending on time t, generalized coordinates q^1, q^2, and generalized components $\dot{q}^1$, $\dot{q}^2$, of the velocity $\boldsymbol{v}$,

$$f(t, q^1, q^2, \dot{q}^1, \dot{q}^2) = 0, \tag{42.19}$$

which represents certain geometric surface (manifold) in the corresponding evolution space $\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$, the dynamics of the motion is explained by the presence of a additional constraint force

$$\boldsymbol{\Phi} = \mu\left(\frac{\partial f}{\partial \dot{q}^1}, \frac{\partial f}{\partial \dot{q}^2}\right), \tag{42.20}$$

called *Chetaev constraint force*, where μ is the *Lagrange multiplier.* Such expression of the constraint force was treated as suitable and it was attested in many practical applications of the nonholonomic systems.

The motion of the constrained particle is then governed by the nonholonomic Euler-Lagrange equations

$$\frac{\partial L}{\partial q^1} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^1}\right) = \mu\frac{\partial f}{\partial \dot{q}^1}, \qquad \frac{\partial L}{\partial q^2} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^2}\right) = \mu\frac{\partial f}{\partial \dot{q}^2}. \tag{42.21}$$

In nonholonomic terminology the equations (42.21) are called *Chetaev equations* of motion or *deformed* motion equations since they arise by the deformation of the unconstrained Euler-Lagrange equations (42.18) of the original unconstrained mechanical systems by means of the Chetaev constraint force (42.20). The correctness of the deformed equations was verified from the *nonholonomic variational principle* [5].

42.4 UNIFORM HORIZONTAL PROJECTILE MOTION

While in the case of the classical horizontal projectile motion the magnitude of the instantaneous velocity changes due to (42.7), in the case of the uniform horizontal projectile motion we request uniformity of the magnitude of the instantaneous velocity of the particle, i.e.

$$v(t) = \sqrt{\dot{x}^2(t) + \dot{y}^2(t)} = v_0 = \text{const.}, \tag{42.22}$$

where v_0 is the magnitude of the initial velocity $\boldsymbol{v}$. The equation (42.22) represents one nonholonomic constraint which can be rewrite as follows

$$f \equiv \dot{x}^2(t) + \dot{y}^2(t) - v_0^2 = 0. \tag{42.23}$$

This constraint is called *isotachytonic* constraint, and it is evident that it is nonholonomic constraint nonlinear in the components of the velocity $\boldsymbol{v}$.

Without loss of generality we assume that the motion takes place in the half plane, where $\dot{x} > 0$. Therefore the equation (42.23) can be rewrite into explicit form

$$\dot{x} = \sqrt{v_0^2 - \dot{y}^2}. \tag{42.24}$$

The Chetaev constraint force $\boldsymbol{\Phi}$ (42.20) arising in the context of the constraint (42.23) is the vector

$$\boldsymbol{\Phi} = (\Phi_x, \Phi_y) = \mu \left(\frac{\partial f}{\partial \dot{x}}, \frac{\partial f}{\partial \dot{y}} \right) = \mu(2\dot{x}, 2\dot{y}), \tag{42.25}$$

where μ is the Lagrange multiplier.

Deformed equations of motion (42.21) applied on our problem give us

$$m\ddot{x} = 2\mu\dot{x}, \quad m\ddot{y} = 2\mu\dot{y} - mg, \tag{42.26}$$

which together with the differential equation of the constraint (42.24) govern the uniform horizontal projectile motion.

By the elimination of the Lagrange multiplier μ we get

$$\mu = -\frac{m\dot{y}\ddot{y}}{2(v_0^2 - \dot{y}^2)}. \tag{42.27}$$

After the substitution of (42.27) to the second equation of the system (42.26) and after routine manipulations we obtain one motion equation called *reduced* equation of motion for the uniform horizontal projectile motion

$$\ddot{y} = -\tfrac{g}{v_0^2}(v_0^2 - \dot{y}^2), \tag{42.28}$$

which together with the constraint equation (42.24) and with respect to the initial conditions $x(0) = 0$, $y(0) = h_0$, $\dot{x}(0) = v_0$ and $\dot{y}(0) = 0$, enables us to find the parametric equations of the considered motion. For technical details concerning solving of the differential equation (42.28) we refer to [7]. As the original result we present the parametric equations of the uniform horizontal projectile motion,

$$x(t) = -\tfrac{\pi v_0^2}{2g} + \tfrac{2v_0^2}{g} \arctan \mathrm{e}^{\frac{g}{v_0}t}, \quad y(t) = h_0 - \tfrac{v_0^2}{g} \ln \cosh(\tfrac{g}{v_0}t). \tag{42.29}$$

The time dependence of the Cartesian components $\boldsymbol{v}_x$, $\boldsymbol{v}_y$ of the instantaneous velocity $\boldsymbol{v}$ is given by

$$v_x(t) = \dot{x}(t) = \frac{v_0}{\cosh(\frac{g}{v_0}t)}, \quad v_y(t) = \dot{y}(t) = -v_0 \tanh(\tfrac{g}{v_0}t). \tag{42.30}$$

In fact, the reader could verify that the magnitude of the instantaneous velocity remains constant,

$$v(t) = \sqrt{\dot{x}^2(t) + \dot{y}^2(t)} = v_0 = \text{const.} \tag{42.31}$$

The Cartesian components $\boldsymbol{a}_x$, $\boldsymbol{a}_y$ of the acceleration $\boldsymbol{a}$ are

$$a_x(t) = \ddot{x}(t) = -g\frac{\tanh(\frac{g}{v_0}t)}{\cosh(\frac{g}{v_0}t)}, \quad a_y(t) = \ddot{y}(t) = \frac{-g}{\cosh^2(\frac{g}{v_0}t)}. \tag{42.32}$$

Hence, the magnitude of the instantaneous acceleration is

$$a(t) = \sqrt{a_x^2(t) + a_y^2(t)} = \frac{g}{\cosh(\frac{g}{v_0}t)}. \tag{42.33}$$

However, the tangent component $\boldsymbol{a}_t = a_t \boldsymbol{e}_{\boldsymbol{v}}$ of the acceleration is zero, $a_t = \mathrm{d}v/\mathrm{d}t = 0$, since $v(t) = v_0$. Therefore for the normal component $\boldsymbol{a}_n$ of the acceleration we obtain

$$\boldsymbol{a}_n = \boldsymbol{a} - \boldsymbol{a}_t, \qquad a_n = \sqrt{a^2 - a_t^2} = a = \frac{g}{\cosh(\frac{g}{v_0}t)}. \tag{42.34}$$

On the other hand, for the magnitude of the normal component a_n holds

$$a_n(t) = \frac{v^2(t)}{r(t)} = v^2(t)\varkappa(t), \tag{42.35}$$

where $r(t)$ is the radius of the curvature of the trajectory at the time t and $\varkappa(t) = 1/r(t)$ is the curvature of the trajectory at the time t. If we compare the formulas (42.34) and

(42.35) we obtain the expression of the curvature $\varkappa(t)$ of the trajectory of the uniform horizontal projectile motion,

$$\varkappa(t) = \frac{g}{v_0^2 \cosh(\frac{g}{v_0}t)}. \tag{42.36}$$

The same result can be calculated using the classical geometric formula for the curvature (42.14) after applying the formulas (42.30) and (42.32).

The explicit expression of the trajectory (after the elimination of the time parameter t from the first equation of (42.29)) is

$$y(x) = h_0 + \tfrac{v_0^2}{g} \ln \left|\cos\left(\tfrac{g}{v_0^2}x\right)\right|. \tag{42.37}$$

The trajectory of the uniform horizontal projectile motion is realized only along bold part of the positive first period of the function (42.37), see Fig. 42.1.

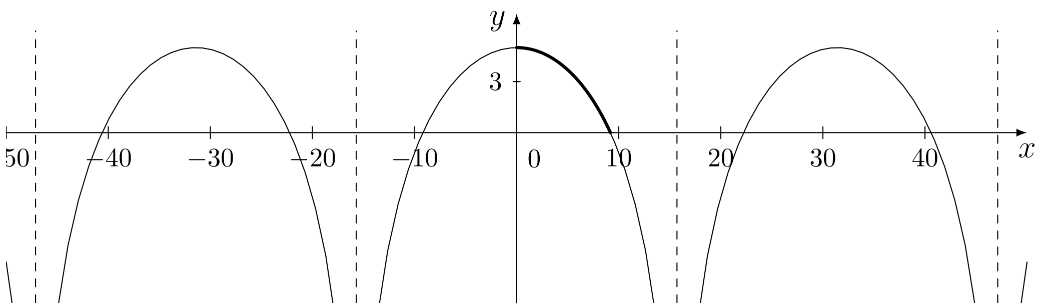

Fig. 42.1 Explicit expression of the trajectory

Source: own elaboration

The kinematic parameters of the uniform horizontal projectile motion are:

- total time $T = \frac{v_0}{g} \operatorname{arccosh} \mathrm{e}^{\frac{gh_0}{v_0^2}}$
- range $R = \frac{v_0^2}{g} \arccos \mathrm{e}^{-\frac{gh_0}{v_0^2}} = \frac{v_0^2}{g}\alpha$, where α is the angle of the impact
- impact speed $v_R = v_0 = \text{const.}$,
- angle of impact $\cos\alpha = \dfrac{|\boldsymbol{v}_R \cdot \boldsymbol{i}|}{|\boldsymbol{v}_R||\boldsymbol{i}|} = \mathrm{e}^{-\frac{gh_0}{v_0^2}} = R\frac{g}{v_0^2}$,

  where $\boldsymbol{v}_R = \boldsymbol{v}(T) = \left(v_0\,\mathrm{e}^{-\frac{gh_0}{v_0^2}}, -v_0\,\mathrm{e}^{-\frac{gh_0}{v_0^2}}\sqrt{\mathrm{e}^{\frac{2gh_0}{v_0^2}}-1}\right)$ is the impact velocity vector, $|\boldsymbol{v}_R| = v_R = v_0$, $\boldsymbol{i} = (1, 0)$ is the unit vector in the direction of x-axis,
- curvature of the trajectory $\varkappa(t) = \dfrac{g}{v_0^2 \cosh(\frac{g}{v_0}t)}$.

Finally, we compute the length of the trajectory of the uniform horizontal projectile motion,

$$s = \int_0^T v(t)\mathrm{d}t = v_0 \int_0^T \mathrm{d}t = v_0 T = \tfrac{v_0^2}{g} \operatorname{arccosh} \mathrm{e}^{\frac{gh_0}{v_0^2}}. \tag{42.38}$$

42.5 COMPARISON OF CLASSICAL AND UNIFORM HORIZONTAL PROJECTILE MOTION

We compare kinematic parameters of the both motions, the horizontal projectile motion on the one side and the uniform horizontal projectile motion on the other side with respect to the same initial conditions; $x(0) = 0$, $y(0) = h_0$ with the initial velocity $\boldsymbol{v}_0 = (v_0, 0)$ for the following numerical values $v_0 = 10\,\mathrm{m}\cdot\mathrm{s}^{-1}$, $g = 10\,\mathrm{m}\cdot\mathrm{s}^{-2}$ and $h_0 = 5\,\mathrm{m}$, cf. Tab. 42.1.

Tab. 42.1 Kinematic parameters - comparison

Horizontal projectile motion	Uniform horizontal projectile motion
$R = v_0\sqrt{\frac{2h_0}{g}} = 10\,\mathrm{m}$	$R = \frac{v_0^2}{g}\arccos \mathrm{e}^{-\frac{gh_0}{v_0^2}} \doteq 9.19\,\mathrm{m}$
$T = \frac{R}{v_0} = \sqrt{\frac{2h_0}{g}} = 1\,\mathrm{s}$	$T = \frac{v_0}{g}\operatorname{arccosh} \mathrm{e}^{\frac{gh_0}{v_0^2}} \doteq 1.09\,\mathrm{s}$
$v_R = \sqrt{v_0^2 + 2gh_0} \doteq 14.14\,\mathrm{m}\cdot\mathrm{s}^{-1}$	$v_R = v_0 = 10\,\mathrm{m}\cdot\mathrm{s}^{-1}$
$\alpha = \arccos\frac{v_0}{\sqrt{v_0^2+2gh_0}} = 45^\circ$	$\alpha = \arccos \mathrm{e}^{-\frac{gh_0}{v_0^2}} \doteq 52^\circ\,39'\,36''$
$s \overset{(42.16)}{\doteq} 11.48\,\mathrm{m}$	$s = \frac{v_0^2}{g}\operatorname{arccosh} \mathrm{e}^{\frac{gh_0}{v_0^2}} \doteq 10.85\,\mathrm{m}$

Source: own elaboration

On the following picture, Fig. 42.2, one can see the direct comparison of trajectories of both motions under the same initial conditions.

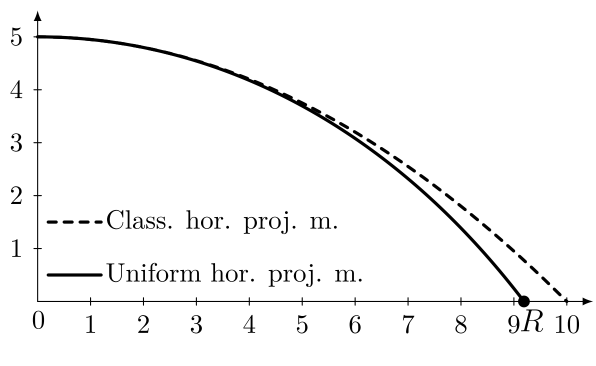

Fig. 42.2 Trajectories of both motions

Source: own elaboration

42.6 DYNAMICS OF UNIFORM HORIZONTAL PROJECTILE MOTION

The classical horizontal projectile motion is caused by the influence of the gravitational force $\boldsymbol{F}_g = (0, -mg)$, which has the constant magnitude $F_g = mg$ and the permanent direction still oriented vertically downward. Since the particle starts with the initial velocity vector oriented in the horizontal direction, its movement consists of the uniform rectilinear motions in the x-axis and simultaneously of the free fall oriented vertically downward and it ends by the impact of the particle on the horizontal plane.

In the case of the uniform horizontal projectile motion it acts on the particle beyond the gravitational force $\boldsymbol{F}_g$ also the Chetaev constraint force $\boldsymbol{\Phi}$, which ensures the requirement of the constant magnitude of the velocity of the particle. Recall that the Chetaev constraint force is expressed by the relation (42.25)

$$\boldsymbol{\Phi} = \mu(t)(2\dot{x}, 2\dot{y}) = 2\mu(t)\boldsymbol{v}(t) = 2\mu(t)v_0\boldsymbol{e}_{\boldsymbol{v}(t)}, \tag{42.39}$$

where $\boldsymbol{v}(t)$ is the instantaneous velocity vector, $\boldsymbol{e}_{\boldsymbol{v}(t)}$ is the unit vector in the direction $\boldsymbol{v}(t)$. It is evident that during the motion the Chetaev force must change the magnitude $|\boldsymbol{\Phi}| = 2v_0|\mu(t)|$ and also its direction $\boldsymbol{e}_{\boldsymbol{v}(t)}$, which is determined by the direction of $\boldsymbol{v}(t)$. Time dependence of the Lagrange multiplier $\mu(t)$ is

$$\mu(t) = -\frac{m\dot{y}\ddot{y}}{2(v_0^2 - \dot{y}^2)} \overset{(42.28)}{=} \frac{mg}{2v_0^2}\dot{y} \overset{(42.30)}{=} -\frac{mg}{2v_0}\tanh\left(\tfrac{g}{v_0}t\right). \tag{42.40}$$

Time dependence of the Chetaev force $\boldsymbol{\Phi}(t)$ is then

$$\boldsymbol{\Phi}(t) = 2\mu(t)v_0\boldsymbol{e}_{\boldsymbol{v}(t)} = \phi(t)\boldsymbol{e}_{\boldsymbol{v}(t)} = -mg\tanh\left(\tfrac{g}{v_0}t\right). \tag{42.41}$$

We conclude that the influence of the Chetaev constraint force can be alternatively substituted by certain external force $\boldsymbol{F}_e$ acting in the vector line of $\boldsymbol{v}(t)$ and its magnitude is regulated by the relation

$$|\boldsymbol{F}_e| = |\boldsymbol{\Phi}(t)| = |\phi(t)| = mg\tanh\left(\tfrac{g}{v_0}t\right). \tag{42.42}$$

Stress that the scalar function $\phi(t) < 0$, i.e. it has opposite direction than $\boldsymbol{v}(t)$ and it compensates the classical accelerated motion of the particle.

The Chetaev force $\boldsymbol{\Phi}$ plays the significant role in the energetic balance of the uniform horizontal projectile motion. Indeed, mechanical work $W_{\boldsymbol{\Phi}}$ of the Chetaev force $\boldsymbol{\Phi}$ along the trajectory γ of the uniform horizontal projectile motion balances the changes of the potential energy instead of the kinetic energy which remains constant during the motion. Details can be found in [7].

CONCLUSION

The modification of the classical horizontal projectile motion onto uniform one brings changes in the kinematics, dynamics and also in the energetic balance of the considered motion. From the point of view of kinematics, the trajectory is not a part of parabola, but it is a small piece of the graph of a certain periodic transcendent function (42.37), Fig. 42.1. There are changes in the kinematic parameters against the classical horizontal projectile motion under the same initial conditions h_0, v_0. The range R of the uniform horizontal motion is at the shorter distance, the length s of the trajectory is shortened and the total time T of the motion becomes slightly longer.

Dynamics of the modified horizontal projectile motion is explained by the presence of the Chetaev constraint force $\mathbf{\Phi}$ which ensures the uniformity of the motion. It has shown that the constraint force must change its magnitude (42.42) during the motion and also its direction $\boldsymbol{e}_{\boldsymbol{v}(t)}$.

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UNIFORM HORIZONTAL PROJECTILE MOTION

Abstract: *In the paper we present the modification of the classical horizontal projectile motion in the homogeneous gravitational field called uniform horizontal projectile motion. The particle passing in the homogeneous gravitational field is treated as a mechanical system subjected to a nonholonomic nonlinear constraint which keeps the magnitude of the instantaneous velocity of the particle constant during the motion. We present the comparison of the kinematic parameters of both motions (classical and uniform horizontal projectile motions) under the same initial conditions. The presented problem can serve as a suitable topic for an elementary course of theoretical mechanics on the university level.*

Keywords: *nonholonomic mechanical systems, reduced and deformed equations of motion, Chetaev constraint force, horizontal projectile motion, kinematic parameters.*

ROVNOMĚRNÝ VODOROVNÝ VRH

Abstrakt: *V článku prezentujeme modifikaci klasického vodorovného vrhu v homogenním gravitačním poli na rovnoměrný vodorovný vrh. Částice pohybující se v homogenním gravitačním poli je chápána jako mechanický systém podrobený neholonomní nelineární vazbě, která udržuje konstantní velikost okamžité rychlosti částice během pohybu. Prezentujeme srovnání kinematických parametrů obou pohybů (klasického i rovnoměrného vodorovného vrhu) vzhledem ke stejným počátečním podmínkám. Prezentovaný problém může sloužit jako vhodné téma k elementárnímu kurzu teoretické mechaniky na vysokoškolské úrovni.*

Klíčová slova: *neholonomní mechanický systém, redukované a deformované pohybové rovnice, Chetaeova vazebná síla, vodorovný vrh, kinematické parametry.*

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