Fuzzy Modal Operators and Their Applications

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Abstract:
In this paper we present some fuzzy modal operators and show their two possible applications. These operators are fuzzy generalizations of modal operators well-known in modal logics. We present an application of some compositions of these operators in approximations of fuzzy sets. In particular, it is shown how skills of candidates can be matched for selecting research projects. The underlying idea is based on the observation that fuzzy sets approximations can be viewed as intuitionistic fuzzy sets introduced by Atanassov. Distances between intuitionistic fuzzy sets, proposed by Szmidt and Kacprzyk, support the reasoning process. Also, we point out how modal operators are useful for representing linguistic hedges, that is terms like "very", "definitely", "rather", or "more or less".

Keywords: *Modal operators, Fuzzy sets, Approximation operators, Intuitionistic fuzzy sets, Linguistic hedges*

1. Introduction

The term *modal operators* usually refers to logical connectives for modal logics which are characterized by expressing a modal attitude (necessity, possibility, belief, knowledge) about propositions they are applied to. Semantically, these operators are interpreted as mappings defined on a universe of binary relations (in a nonempty domain in discourse, say X) and subsets of a domain X, which return another subset of X. Such mappings are also called modal operators. The operators of possibility and necessity are typical examples of modal operators. Another pair of modal operators, usually referred to as *sufficiency* (or *negative necessity*) and *dual sufficiency* (or *impossibility*) were introduced in order to represent expressions like "*necessary false*" and "*possibly false*", respectively (cf. Humberstone [12], Gargov [10], and Goranko [11]).

Modal operators found many interesting applications. Probably the most famous ones are *rough sets* introduced by Pawlak [16, 17] where necessity and possibility operators are used for set approximations.

In more general settings modal operators are mappings of the form $R(X,Y) \times \wp(Y) \to \wp(X)$, where $R(X,Y)$ stands for the family of all binary relations on two nonempty domains X and Y, and $\wp(Z)$ is the power set of Z. In formal concept analysis (FCA) the sufficiency operator is known as *derivation operator* (cf. Wille [31]). Düntsch and Gediga [8] discussed compositions of modal operators in qualitative data analysis and in [9] they considered these operators in the context of knowledge and skills structures.

Modal operators, as traditionally investigated and applied, are based on classical structures like sets and relations. From the standpoint of practical applications this approach is sufficient when we deal with precise data. However, when the available information is imprecise, or vague, more general structures are needed. Fuzzy set theory introduced by Zadeh [32] offers numerous tools and techniques for representing, processing, and analyzing information which is imprecise in its nature. For example, assume that we are to evaluate student's skills during some course. Clearly, it is essential to know to what extend student's knowledge and abilities match our requirements and the yes-no information is practically meaningless. Also, it is important to state to what extent some skill is required for realizing a particular research project and the useful/usefulness answers may highly restrict correctness of decision process concerning selection of proper candidates. Therefore, fuzzy sets and fuzzy relations are natural tools for representing this kind of data. Fuzzy generalizations of modal operators seem to be adequate for drawing conclusions from fuzzy information.

In this paper we present fuzzy generalization of modal operators and show two their applications. In the first application it is shown how fuzzy modal operators can be applied for supporting process of selecting candidates for research projects. The underlying information is candidates' skills and projects' requirements. We point out that the compositions of fuzzy sufficiency and fuzzy dual sufficiency operators form fuzzy approximation operators. As observed, these approximations lead to Atanassov's intuitionistic fuzzy sets. Distances between intuitionistic fuzzy sets determine decisions on selecting candidates for projects.

The second application is focused on the representation of linguistic hedges. Traditional, and still very popular representation, originally proposed by Zadeh [33], is a powering technique. Precisely, if F is a fuzzy set representing some property P (e.g., *good*, *high*, *old*), then for $\alpha > 1$, F^α stands for *very P* (or *definitely P*, *extremely P*), while $\sqrt[\alpha]{P}$ represents *rather P* (or *more or less P*, *quite P*). This approach is purely technical and, in our opinion, passes over the fact that linguistic hedges can be treated as specific modal expressions. Following this observation we present representations of linguistic hedges using fuzzy necessity and fuzzy possibility operators. In particular, given some property of objects (typically an adjective in natural language) represented by a fuzzy set P in the set X of objects, we say that an object $x \in X$ is *very P* (e.g., *very young*) to the degree to which all objects from X resembling x posses the property P. Similarly, $x \in X$ is *rather P* (e.g., *rather young*) to the degree to which some object from X, which resembles x, has the property P. This way linguistic hedges provide characterizations of objects relatively to other objects.

The paper is organized as follows. In Section 2 we recall basic notions and terminology which will be used in the paper. Fuzzy modal operators are presented in Section 3. We define four basic operators and consider two their compositions. It is pointed out that, given a fuzzy set F, these operators constitute fuzzy approximations of F and lead to an intuitionistic fuzzy set. In Section 4 we propose how these operators may be useful for matching research projects for potential candidates taking into account requirements imposed on particular projects and skills shown by candidates. The selection criterion is based on a distance between intuitionistic fuzzy sets. The next section is focused on modeling linguistic modifiers by means of fuzzy possibility and fuzzy necessity operators. Several schemes are presented and the corresponding representation is discussed. Concluding remarks complete the paper.

2. Preliminaries

In this section we present basic notions and some of their properties which clarify our discussion in the present paper.

2.1. Fuzzy Sets

Let X be a non-empty domain. A *fuzzy set in X* is any mapping $F : X \to [0,1]$. For any $x \in X$, $F(x)$ is the degree to which x belongs to F. Given two fuzzy sets in X, A and B, we say that

- A in (totally) included in B, written $A \subseteq B$, if $A(x) \leqslant B(x)$ for every $x \in X$,
- A is (totally) equal to B, written $A = B$, if $A(x) = B(x)$ for every $x \in X$.

A *kernel* of a fuzzy set A in X is defined as

$$ker(A) = \{x \in X : A(x) = 1\}$$

while the *support* of a fuzzy set A in X is the set

$$supp(A) = \{x \in X : A(x) > 0\}.$$

The family of all fuzzy sets in X will be denoted by $\mathcal{F}(X)$.

A *binary fuzzy relation in X and Y* (or just a *fuzzy relation in X* and *Y*) is a fuzzy set in $X \times Y$. For every $x \in X$ and for every $y \in Y$, $R(x,y)$ is the degree to which x is R-related with y. We will write $\mathcal{R}(X,Y)$ to denote the family of all fuzzy relations in X and Y. For $R \in \mathcal{R}(X,Y)$, the converse relation $R^{-1} \in \mathcal{R}(Y,X)$ is defined as $R^{-1}(y,x) = R(x,y)$, $x \in X$ and $y \in Y$. If $X = Y$, then we have a binary fuzzy relation on X (*fuzzy relation on X*, for short). a fuzzy relation R on X is called

- *reflexive* if $R(x,x) = 1$ for every $x \in X$,
- *symmetric* if $R(x,y) = R(y,x)$ for all $x,y \in X$,
- *sup-min transitive* if
$$\sup_{y \in X} \min(R(x,y), R(y,z) \leqslant R(x,z)$$
for all $x,y,z \in X$.

2.2. Intuitionistic Fuzzy Sets

Intuitionistic fuzzy sets, originally proposed by Atanassov [1, 2], is an interesting generalization of fuzzy sets where both degrees of membership and non-membership are involved. More specifically, an intuitionistic fuzzy set in X is given by $A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\}$ where $\mu_A, \nu_A \in \mathcal{F}(X)$ with $\mu_A(x) + \nu_A(x) \leq 1$ for every $x \in X$, are called a membership and a non-membership function, respectively. For $x \in X$, $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is a hesitation margin reflecting the lack of knowledge of whether x belongs to A or not. Clearly, every fuzzy set $A \in \mathcal{F}(X)$ is a specific intuitionistic fuzzy set with $\nu_A(x) = 1 - \mu_A(x)$, i.e., $\pi_A(x) = 0$ for every $x \in X$.

As argued by Szmidt and Kacprzyk [30], distance measures between intuitionistic fuzzy sets should involve all three parameters. Namely, for two intuitionistic fuzzy sets, A and B, in a finite universe $X = \{x_1, \ldots, x_n\}$, the *normalized Hamming distance* is defined as

$$\delta_H(A,B) = \frac{1}{2n} \sum_{i=1}^{n} |\mu_A(x) - \mu_B(x)| + |\nu_A(x) - \nu_B(x)| + |\pi_A(x) - \pi_B(x)|.$$

2.3. Fuzzy Logical Connectives

Fuzzy logical connectives generalize logical connectives of classical logic. The most popular generalization of classical conjunction are *triangular norms* (*t-norms*, for short). Specifically, a triangular norm is a function $\otimes : [0,1]^2 \to [0,1]$ satisfying the following conditions

(T1) $x \otimes y = y \otimes x$ for all $x, y \in [0,1]$ (commutativity),

(T2) $x \otimes (y \otimes z) = (x \otimes y) \otimes z$ for all $x,y,z \in [0,1]$ (associativity),

(T3) increasing in both arguments, i.e., $x \leqslant z$ implies $x \otimes y \leqslant z \otimes y$ and $y \otimes x \leqslant y \otimes z$ for all $x,y,z \in [0,1]$,

(T4) $x \otimes 1 = x$ for every $x \in [0,1]$ (boundary condition).

Typical examples of t-norms are:

- the standard t-norm (the largest t-norm)
$$x \otimes_Z y = \min(x,y)$$
- the product
$$x \otimes_P y = xy$$
- the Łukasiewicz t-norm
$$x \otimes_L y = \max(0, x + y - 1).$$

A t-norm is called *left-continuous* whenever it is left-continuous on both arguments. Note that all three t-norms mentioned above are left-continuous. For the extensive studies on t-norm we refer to Klement, Mesiar, and Pap [13].

A *fuzzy implication* $\rightarrow$ is a $[0,1]^2 - [0,1]$ map with increasing first and decreasing second partial mappings and satisfying $1 \rightarrow 1 = 0 \rightarrow 0 = 0 \rightarrow 1 = 1$, and $1 \rightarrow 0 = 0$. The well-known fuzzy implications are

- the Kleene-Dienes implication

$$x \rightarrow_{KD} y = \max(1-x, y),$$

- the Reichenbach implicator

$$x \rightarrow_R y = 1 - x + xy,$$

- the Łukasiewicz implication

$$x \rightarrow_L y = \min(1, 1 - x + y),$$

- the Gödel implication

$$x \rightarrow_G y = \begin{cases} 1 & \text{for } x \leqslant y \\ y & \text{elsewhere.} \end{cases}$$

A special class of fuzzy implications are *residual implications*: given a left-continuous t-norm $\otimes$, its residual implication is defined for all $x, y \in [0,1]$,

$$x \rightarrow y = \sup\{z \in [0,1] : x \otimes z \leqslant y\}.$$

The Łukasiewicz the the Gaines implications are examples of residual implications based on $\otimes_L$ and $\otimes_M$, respectively. For an extended survey on fuzzy implications we refer to Baczyński and Jayaram [3].

A *fuzzy negation* is a mapping $\neg : [0,1] \rightarrow [0,1]$, non-increasing and satisfying $\neg 0 = 1$ and $\neg 1 = 0$. It is involutive iff $\neg\neg x = x$ for every $x \in [0,1]$. Residual implications induce fuzzy negations of the form $\neg x = (x \rightarrow 0)$. Since residual implications are defined on the basis of left-continuous t-norms $\otimes$, these type of fuzzy negations will be referred to as fuzzy negations induced by $\otimes$. The standard fuzzy negation $\neg x = 1 - x$, $x \in [0,1]$, is the involutive negation induced by the Łukasiewicz t-norm. It is well-known that it is the only involutive negation induced by left-continuous t-norms.

Using fuzzy logical connectives basic operations on fuzzy sets are defined. In particular, for a t-norm $\otimes$ and for a fuzzy negation $\neg$, the $\otimes$-intersection of two fuzzy sets, $A, B \in \mathcal{F}(X)$, is defined for every $x \in X$, $(A \cap_{\otimes} B)(x) = A(x) \otimes B(x)$, and a fuzzy $\neg$-complementation of $A \in \mathcal{F}(X)$ is given by $(\neg A)(x) = \neg A(x)$ for every $x \in X$.

3. Fuzzy Modal Operators

Let X be a set of objects and let Y be a set of properties. Any fuzzy set $A \in \mathcal{F}(X)$ may be viewed as a fuzzy attribute (property): for every object $x \in X$, $A(x)$ is the degree to which A characterizes x. Similarly, any fuzzy set $B \in \mathcal{F}(Y)$ may be interpreted as a description of an individual: for any property $y \in Y$, $B(y)$ is the degree to which y characterizes B.

Let $R \in \mathcal{R}(X,Y)$ be a fuzzy relation which represents characterizations of objects from X: for every object $x \in X$ and for every property $y \in Y$, $R(x,y)$ is the degree to which x posses y. Given a graded information about objects represented by a relation $R \in \mathcal{R}(X,Y)$, we can derive new information about two types of relationships, namely a relationship between objects from X determined by their properties and a relationship between properties from Y basing on objects having these properties. These relationships are represented by *fuzzy information relations* which are generalizations of information relations widely studied within the framework of the rough set-style data analysis (see, e.g., Orłowska [14], Demri and Orlowska [7]). Fuzzy information relations were investigated by Radzikowska and Kerre [23,27,29], logical systems capable to reason about such relations were considered by Radzikowska [20].

For two objects x_1 and x_2 from X, we say that

- x_1 is *relevant to* x_2 to the degree to which all properties of x_1 are also properties of x_2;
- x_1 and x_2 are *compatible* to the degree to which they both share some common property;
- x_1 and x_2 are *coherent* to the degree to which they both do not have some property.

In order to derive such relationships *fuzzy modal operators* are used. These operators, being generalizations of operators well-known in modal logics, were extensively studied by Radzikowska and Kerre [18, 19, 22, 24, 25, 25, 26, 28, 29]. Algebraic and logical aspects were presented by Orłowska, Radzikowska, and Rewitzky [15]. Recall that these operators are $\mathcal{F}(Y) \rightarrow \mathcal{F}(X)$ mappings defined as follows. Let $\otimes$ be a t-norm, let $\rightarrow$ be a fuzzy implication, and let $\neg$ be a fuzzy negation. Given a fuzzy relation $R \in \mathcal{R}(X,Y)$, for every fuzzy set $B \in \mathcal{F}(Y)$ and for every $x \in X$,

$$[R]_{\rightarrow} B(x) = \inf_{y \in Y} (R(x,y) \rightarrow B(y)) \quad (1)$$

$$\langle R \rangle_{\otimes} B(x) = \sup_{y \in Y} (R(x,y) \otimes B(y)) \quad (2)$$

$$[\![R]\!]_{\rightarrow} B(x) = \inf_{y \in Y} (B(y) \rightarrow R(x,y)) \quad (3)$$

$$\langle\!\langle R \rangle\!\rangle_{\otimes,\neg} B(x) = \sup_{y \in Y} (\neg R(x,y) \otimes \neg B(y)). \quad (4)$$

The above operators are called *fuzzy necessity*, *fuzzy possibility*, *fuzzy sufficiency*, and *fuzzy dual sufficiency*, respectively. They have the following natural interpretation in data analysis: for any individual $B \in \mathcal{F}(Y)$ and for any $x \in X$,

- $[R]_{\rightarrow} B(x)$ is the degree to which the object x is relevant to the individual B;
- $\langle R \rangle_{\otimes} B(x)$ is the degree to which the individual B and the object x are compatible;

- $[\![R]\!]_{\rightarrow}B(x)$ is the degree to which the individual B is relevant to the object x;
- $\langle\!\langle R\rangle\!\rangle_{\otimes,\neg}B(x)$ is the degree to which the individual B and the object x are coherent.

Taking R^{-1} and an attribute $A \in \mathcal{F}(X)$, in the similar way we obtain fuzzy relevance (resp. compatibility, coherence) between properties. E.g., for any property $y \in Y$, $[R^{-1}]A(y)$ is the degree to which y is relevant to A.

Let us recall some basic properties of operators (1)–(4).

Property 3.1 *For all fuzzy sets $A, B \in \mathcal{F}(Y)$ and for any fuzzy relation $R \in \mathcal{R}(X,Y)$,*

(a) *$A \subseteq B$ implies $[R]_{\otimes}A \subseteq [R]_{\otimes}B$, $\langle R\rangle_{\otimes}A \subseteq \langle R\rangle_{\otimes}B$, $[\![R]\!]_{\otimes}B \subseteq [\![R]\!]_{\otimes}A$, and $\langle\!\langle R\rangle\!\rangle_{\otimes,\neg}B \subseteq \langle\!\langle R\rangle\!\rangle_{\otimes,\neg}A$;*

(b) *If $\rightarrow$ and $\Rightarrow$ are two fuzzy implications such that $\rightarrow \leqslant \Rightarrow$ (i.e., $x \rightarrow y \leqslant x \Rightarrow y$ for all $x, y \in [0,1]$), then $[R]_{\rightarrow}A \subseteq [R]_{\Rightarrow}A$ and $[\![R]\!]_{\rightarrow}A \subseteq [\![R]\!]_{\Rightarrow}A$;*

(c) *If $\otimes$ and $\odot$ are two t-norms such that $\otimes \leqslant \odot$, then $\langle R\rangle_{\otimes}A \subseteq \langle R\rangle_{\odot}A$ and $\langle\!\langle R\rangle\!\rangle_{\otimes,\neg}A \subseteq \langle\!\langle R\rangle\!\rangle_{\odot,\neg}A$ for any fuzzy negation $\neg$.*

Now, take a left-continuous t-norm $\otimes$, the residual implication $\rightarrow$ based on $\otimes$, and the fuzzy negation $\neg$ induced by $\otimes$. Then all four operators (1)–(4) can be indexed by $\otimes$ only.

Now, let us define the following two $\mathcal{F}(Y) - \mathcal{F}(Y)$ operations for every $B \in \mathcal{F}(Y)$ and for every $y \in Y$:

$$\Delta^R_{\otimes}B = \langle\!\langle R^{-1}\rangle\!\rangle_{\otimes}\langle\!\langle R\rangle\!\rangle_{\otimes}B \tag{5}$$

$$\nabla^R_{\otimes}B = [\![R^{-1}]\!]_{\otimes}[\![R]\!]_{\otimes}B. \tag{6}$$

These operators have the following important *approximation property*:

Property 3.2 *Let $\otimes$ be a left-continuous t-norm such that its residual implication induces an involutive fuzzy negation. Then for every $R \in \mathcal{R}(X,Y)$ and for every $B \in \mathcal{F}(Y)$,*

$$\Delta^R_{\otimes}B \subseteq B \subseteq \nabla^R_{\otimes}B.$$

Note that the approximation property holds only for the Łukasiewicz t-norm $\otimes_L$ since it is the only left-continuous t-norm which induces an involutive fuzzy negation. The operators (5) and (6) determined by $\otimes_L$ will be written Δ_L and ∇_L, respectively. For any fuzzy relation $R \in \mathcal{R}(X,Y)$ and for any fuzzy set $B \in \mathcal{F}(Y)$, $\Delta^R_L B$ is a *lower bound of* B whereas $\nabla^R_L B$ is an *upper bound of* B with respect to R, respectively. The pair $(\Delta^R_L B, \nabla^R_L B)$ is called an (Δ^R_L, ∇^R_L)–approximation of B with respect to R. For any $y \in Y$, $\Delta_L B(y)$ may be viewed as the degree to which y *at least* belongs to B and $\nabla_L B(y)$ can be read as the degree to which y *at most* belongs to B. Accordingly, for every $y \in Y$, the value $\nabla_L B(y) - \Delta_L B(y)$ is a *hesitation region*, that is, the degree to which it is unknown whether y belongs to B or not. This leads us to the following observation.

Observation 3.1 *Let $R \in \mathcal{R}(X,Y)$ and let $A \in \mathcal{F}(Y)$. Define the following two mappings $\mu_A, \nu_A : X \rightarrow [0,1]$ for every $x \in X$,*

$$\mu_A(x) = \Delta^R_L A(x)$$
$$\nu_A(x) = 1 - \nabla^R_L A(x).$$

Then $\{(x, \mu_A(x), \nu_A(x)) : x \in X\}$ is an intuitionistic fuzzy set.

An application of these operators will be shown in Section 4.

Now, let us consider a fuzzy relation R on X, i.e., $R \in \mathcal{F}(X,X)$. Radzikowska and Kerre [26, 28, 29] showed the following property.

Property 3.3 *Let $\otimes$ be a left-continuous t-norm, let $\rightarrow$ be its residual implication, and let $\neg$ be the fuzzy negation induced by $\otimes$. Then for every $A \in \mathcal{F}(X)$,*

(a) *R is reflexive*
iff $\forall A \in \mathcal{F}(X)$, $A \subseteq \langle R\rangle_{\otimes}A$
iff $\forall A \in \mathcal{F}(X)$, $[R]_{\rightarrow}A \subseteq A$;

(b) *R is symmetric*
iff $\forall A \in \mathcal{F}(X)$, $\langle R\rangle_{\otimes}[R]_{\rightarrow}A \subseteq A$
iff $\forall A \in \mathcal{F}(X)$, $A \subseteq [R]_{\rightarrow}\langle R\rangle_{\otimes}A$.

(c) *$\neg\langle R\rangle_{\otimes}A \subseteq [R]_{\rightarrow}(\neg A)$ and $\langle R\rangle_{\otimes}(\neg A) \subseteq \neg[R]_{\rightarrow}A$; if $\neg$ is involutive, then both inclusions are equalities.*

Then the following corollary follows.

Corollary 3.1 *Let $\otimes$ be a left-continuous t-norm and let $\rightarrow$ be its residual implication. Then for every reflexive and symmetric fuzzy relation R on X and for every $A \in \mathcal{F}(X)$,*

$$[R]_{\rightarrow}A \subseteq \langle R\rangle_{\otimes}[R]_{\rightarrow}A \subseteq A \subseteq [R]_{\rightarrow}\langle R\rangle_{\otimes}A \subseteq \langle R\rangle_{\otimes}A.$$

4. Skills Assessment and Projects Matching

In this section we show how fuzzy operators (5) and (6) can be applied to a proper selection of candidates to research projects that are to be carried out at some department.

Assume that a set P of projects is given. Each one requires some skills guaranteed its accomplishment. Let S be a set of these skills. Researchers responsible for projects present their requirements by determining to what extend particular skills are demanded for their projects. A natural way for representation of such descriptions is to use a fuzzy relation $R \in \mathcal{R}(P,S)$, where $R(p,s)$ is the degree to which a skill $s \in S$ is required for a project p. Next, let a group C of candidates (students or researchers) apply for these projects. They passed some tests which show their abilities in required skills: for each candidate $c \in C$ it was evaluated to what extent he/she posses particular skills from S. Again, fuzzy structures are useful for representation of candidates' abilities. In consequence, we have another fuzzy relation $Q \in \mathcal{R}(C,S)$ such that for every candidate $c \in C$ and for every skill $s \in S$, $R(c,s)$ is the degree to which c's abilities coincide with the skill s.

Tab. 1. Projects' requirements

R	*Java*	*DBases*	*DMining*	*Statistics*	*Algorithmics*
p_1	0.7	0.9	1.0	0.6	0.5
p_2	0.2	0.8	0.9	0.8	0.6
p_3	0.9	0.6	0.4	1.0	0.8

Tab. 2. Candidates' skills

Q	*Java*	*DBases*	*DMining*	*Statistics*	*Algorithmics*
Tom	0.2	0.5	1.0	1.0	0.4
Susan	0.4	1.0	1.0	0.4	0.6
Jane	1.0	0.6	0.6	0.8	0.7
Bill	0.6	0.3	0.9	1.0	0.0
Mary	0.2	0.8	0.7	0.5	0.1
Ted	1.0	0.5	0.7	0.6	0.9

The task is to choose the most adequate candidates for each project.

First, observe that for any project $p \in P$, pR is its description in terms of skills it requires, and for every candidate $c \in C$, cQ is his/her description in terms of his/her abilities. Then the simplest solution of our problem seems to take distances between fuzzy sets cQ and pR – the proper choice of a candidate for a project p is pointed out by the shortest distance.

However, this method has a substantial drawback. Observe that the relation R explicitly shows requirements for particular projects, but implicitly R gives information about relationships between projects and between skills. This implicit information should be taken into account when the candidate selection is to be made adequately.

Following the interpretation presented in Section 3, any fuzzy set $A \in \mathcal{F}(S)$ can be viewed a problem and $A(s)$ is the degree to which a skill $s \in S$ is required to solve A. Analogously, any fuzzy set $B \in \mathcal{F}(P)$ may represent some feature and $B(p)$ is the degree to which a project $p \in P$, if carried out, requires B. Taking a fuzzy implication $\rightarrow$, a t-norm $\otimes$, and a fuzzy negation $\neg$, for any problem $A \in \mathcal{F}(S)$ and for any project $p \in P$,

- $[\![R]\!]_{\rightarrow}A(p)$ is the degree to which the problem A is relevant to the project p;
- $\langle\!\langle R\rangle\!\rangle_{\otimes,\neg}A(p)$ is the degree to which the project p and the problem A are coherent.

Similarly, for an attribute $B \in \mathcal{F}(P)$ and for any skill $s \in S$, $[\![R^{-1}]\!]_{\rightarrow}B(s)$ is the degree to which the attribute B is relevant to the skill s, whereas $\langle\!\langle R^{-1}\rangle\!\rangle_{\otimes,\neg}B(s)$ is the degree to which the attribute B and the skill s are coherent.

Now, take a left-continuous t-norm. For any problem $A \in \mathcal{F}(S)$ and for any skill $s \in S$,

- $\triangle_{\otimes}A(s)$ is the degree to which the skill s is coherent with the attribute $\langle\!\langle R\rangle\!\rangle_{\otimes}A$, or equivalently, the degree to which some project incoherent with the problem A does not require the skill s;
- $\nabla_{\otimes}A(s)$ is the degree to which the attribute $[R]_{\otimes}A$ is relevant to s; in other words, the degree to which all projects to which the problem A is relevant to, require the skill s.

By Property 3.2, any problem $A \in \mathcal{F}(S)$ can be approximated using a relation R and any left-continuous t-norm $\otimes$. In particular, for a candidate $c \in C$ and a description cQ of his/her abilities, we have

$$\triangle_{\otimes}cQ \subseteq cQ \subseteq \nabla_{\otimes}cQ.$$

Hence, for all candidates $c \in C$, we obtain lower and upper bounds of their abilities with respect to particular skills $s \in S$. These approximated evaluations take into account both abilities of candidates resulting from tests they passed, and requirements for projects they applied for. Note that these requirements are twofold: on one hand they follow from researchers' needs (described directly in the relation R) and, in addition, those ones which result from relationships between both projects and skills (implicitly follow from the relation R). Consequently, the proposed approximation uses both explicit and implicit knowledge of all projects' coordinators. Clearly, such an information is required for selecting proper candidates.

Example 4.1 Let P be a set of three projects p_1, p_2, p_3 and let S be a set of five skills required: Programming in Java (*Java*), Data Bases (*DBases*), Data Mining (*DMining*), Statistics (*Statistics*), and Algorithmics (*Algorithmics*). A fuzzy relation $R \in \mathcal{R}(P,S)$ given in Tab. 1 represents requirements for projects $p \in P$ in terms of skills $s \in S$. Next, let $C = \{Tom, Alan, Jim, andBill\}$ be a group of four candidates for projects P. A fuzzy relation $Q \in \mathcal{F}(C,S)$ presented in Tab. 2 represents candidates' skills, Now, taking the Łukasiewicz logical connectives: t-norm $\otimes_L$, implications $\rightarrow_L$, and the negation $\neg_L$ (in fact, the standard negation $\neg_S$), $(\triangle^R_L, \nabla^R_L)$–approximations of particular candidates' abilities are presented in Tab. 3. □

However, there is still a question as to which candidate is the proper one for particular projects. To cope with this problem we adopt the methodology as in [21]. Namely,

Step 1: Determine $(\triangle^R_L, \nabla^R_L)$–approximation of descriptions pR of each project $p \in P$;

Tab. 3. Candidates' assessments

Q	*Java*	*DBases*	*DMining*	*Statistics*	*Algorithmics*
Tom	(0.2,0.4)	(0.4,1.0)	(0.6,1.0)	(0.3,1.0)	(0.4,0.8)
Susan	(0.4,0.4)	(0.4,1.0)	(0.6,1.0)	(0.4,0.7)	(0.5,0.6)
Jane	(0.8,1.0)	(0.6,0.8)	(0.6.0.6)	(0.6,0.9)	(0.6,0.8)
Bill	(0.4,0.6)	(0.2,1.0)	(0.4,0.9)	(0.0,1.0)	(0.0,0.9)
Mary	(0.2,0.2)	(0.3,0.8)	(0.5,0.7)	(0.1,0.6)	(0.1,0.5)
Ted	(0.8,1.0.)	(0.5,0.9)	(0.6,0.7)	(0.5,1.0)	(0.5,0.9)

Tab. 4. Projects' assessments

R	*Java*	*DBases*	*DMining*	*Statistics*	*Algorithmics*
p_1	(0.7,0.7)	(0.4,0.9)	(0.6,1.0)	(0.4,0.6)	(0.5,0.5)
p_2	(0.2,0.2)	(0.4,0.8)	(0.6,0.9)	(0.3,0.8)	(0.4,0.6)
p_3	(0.8,0.9)	(0.2,0.6)	(0.4,0.4)	(0.4,1.0)	(0.5,0.8)

Tab. 5. Intuitionistic fuzzy relation R' (projects-skills)

R'	*Java*	*DBases*	*DMining*	*Statistics*	*Algorithmics*
p_1	(0.7,0.3,0.0)	(0.4,0.1,0.5)	(0.6.0.0,0.4)	(0.4,0.4,0.2)	(0.5,0.5,0.0)
p_2	(0.2,0.8,0.0)	(0.4,0.2,0.4)	(0.6,0.1,0.3)	(0.3,0.2,0.5)	(0.4,0.4,0.2)
p_3	(0.8,0.1,0.1)	(0.2,0.4,0.4)	(0.4,0.6,0.0)	(0.4,0.0,0.6)	(0.5,0.2,0.3)

Tab. 6. Intuitionistic fuzzy relation Q' (candidates-skills)

Q'	*Java*	*DBases*	*DMining*	*Statistics*	*Algorithmics*
Tom	(0.2,0.6,0.2)	(0.4,0.0,0.6)	(0.6,0.0,0.4)	(0.3,0.0,0.7)	(0.4,0.2,0.4)
Susan	(0.4,0.6,0.0)	(0.4,0.0,0.6)	(0.6,0.0,0.4)	(0.4,0.3,0.3)	(0.5,0.4,0.1)
Jane	(0.8,0.0,0.2)	(0.6,0.2,0.2)	(0.6,0.4,0.0)	(0.6,0.1,0.3)	(0.6,0.2,0.2)
Bill	(0.4,0.4,0.2)	(0.2,0.0,0.8)	(0.4,0.1,0.5)	(0.0,0.0,1.0)	(0.0,0.1,0.9)
Mary	(0.2,0.8,0.0)	(0.3,0.2,0.5)	(0.5,0.3,0.2)	(0.1,0.4,0.5)	(0.1,0.5,0.4)
Ted	(0.8,0.0,0.2)	(0.5,0.1,0.4)	(0.6,0.3,0.1)	(0.5,0.0,0.5)	(0.5,0.1,0.4)

Step 2: Calculate intuitionistic fuzzy relations, R' and Q' determined by R and Q, respectively, and the approximation operators (5) and (6): $R' = \{(\mu_{R'}(p,s), \nu_{R'}(p,s), \chi_{R'}(p,s)) : p \in P, s \in S\}$, $Q' = \{(\mu_{Q'}(c,s), \nu_{Q'}(p,s), \chi_{Q'}(p,s)) : c \in C, s \in S\}$ are given by:

$$\mu_{R'}(p,s) = \Delta_L^R(pR)$$
$$\nu_{R'}(p,s) = 1 - \nabla_L^R(pR)(s)$$
$$\chi_{R'}(p,s) = \nabla_L^R(pR)(s) - \Delta_L^R(pR)(s)$$

and

$$\mu_{Q'}(c,s) = \Delta_L^R(cQ)$$
$$\nu_{Q'}(c,s) = 1 - \nabla_L^R(cQ)(s)$$
$$\chi_{Q'}(c,s) = \nabla_L^R(cQ)(s) - \Delta_L^R(cQ)(s),$$

respectively.

Step 3: Distances between intuitionistic fuzzy sets point out the proper candidate selection: a candidate c is chosen for a project p whenever the distance $dist(cQ', pR')$ is the shortest for all $c \in C$.

Example 4.1 (cont.) Tab. 4 shows the results of (Δ_L^R, Δ_L^R)–approximations of cQ for every $c \in C$. Intuitionistic fuzzy relations R' and Q', derived from these approximations, are given in Tab. 5 and Tab. 6, respectively. Normalized Hamming distances between respective intuitionistic fuzzy sets are presented in Tab. 7. Therefore, *Susan* should be chosen for the project p_1, she and *Mary* for p_2, and *Ted* for p_3. □

Tab. 7. Distances between candidates' and projects' descriptions

	p_1	p_2	p_3
Tom	0.3	0.18	0.36
Susan	**0.12**	**0.16**	0.4
Jane	0.32	0.36	0.22
Bill	0.5	0.44	0.46
Mary	0.32	**0.16**	0.4
Ted	0.3	0.32	**0.18**

The approximation operators (5) and (6) are not the only ones that enables to estimate information given by relations. Radzikowska [21] presented another pair of operators also constructed from modal-like operators which allow us for similar approximations. Also, following rough set-style data analysis one can derive fuzzy information relations and on their basis approximate fuzzy sets using fuzzy necessity and fuzzy possibility.

5. Modeling Linguistic Hedges

In this section we show how fuzzy necessity and fuzzy possibility operators can be applied for modeling linguistic hedges. This approach was presented by De Cock, Radzikowska, and Kerre [5, 6] and then developed by De Cock and Kerre [4].

In natural language many properties of objects are normally expressed by adjectives, for example *good*, *young*, *warm*. Using fuzzy-set theoretical approach, they are represented by fuzzy sets. Linguistic modifiers (also referred to as *linguistic hedges*) are specific type of linguistic expressions like *very*, *extremely*, *more or less*, *quite*. While applied to adjectives, linguistic hedges allow us to express an emphasis we impose on the corresponding properties. In general, there are two types of linguistic hedges: *intensifying* and *weakening*. While the former strengthen the emphasis imposed on the term they are applied to (e.g., *very good*, *extremely warm*, *definitely high*), the latter weaken this emphasis (e.g., *quite good*, *more or less warm*, *rather high*).

In the literature two types of interpretation of linguistic hedges are use: inclusive and non-inclusive. Roughly speaking, for a given property P, in the inclusive interpretation any object qualified as "*very P*" is also viewed as having the property P and an object which posses the property P is also referred to as *quite P*. For instance, if someone is called *very tall*, then he/she is also viewed as *tall* and *quite tall*. Therefore, when representing linguistic terms by fuzzy sets, the following semantic entailment holds:

$$extremely\ P \subseteq very\ P \subseteq \mathrm{P} \subseteq more\ or\ less\ P \subseteq rater\ P.$$

On the other hand, in the non-inclusive interpretation objects qualified as *very P* are not considered P (e.g., people of 90 years old or more, called *very old*, are not viewed as just *old*). In this section only the inclusive interpretation is considered and modified terms are represented by supersets or subsets of the original term.

In the literature there are many approaches for modeling linguistic hedges. Probably the most popular representation, proposed by Zadeh [33], is a powering technique: given a fuzzy predicate P (stated a property of objects and represented by a fuzzy set), the modified term is represented by P^α with $\alpha > 1$ for intensifying hedges and $\alpha \in (0,1)$ for weakening ones. One disadvantage of this approach is that both a kernel and a support of P are preserved: $ker(P) = ker(P^\alpha)$ and $supp(P) = supp(P^\alpha)$. However, it seems counterintuitive: if John is 25 years old, he is obviously viewed as *young* to the degree 1, yet intuitively he is *very young* up to the lower degree, say 0.9. Moreover, this method is based on technical operations only and does not take into account any inherited meaning from modified terms. It is worth noting that linguistic hedges add a special emphasis to adjectives they are applied to. For example, while saying that "*George is a very good doctor*" one wants to emphasise George's medical qualifications. This conviction may be viewed as an implicit reference to medical qualifications of other doctors. In this sense linguistic hedges have a relative flavor and, as such, are in fact modal expressions, like *certainly*, *sometimes*, or *presumably*. Consequently, they are to be modeled in the similar way as modalities, that is using relational methods.

Following this idea we present another representation of linguistic hedges basing on the notion of *resemblance*. This approach was proposed by De Cock, Radzikowska, and Kerre [5, 6], and by De Cock and Kerre [4]. Intuitively, having a universe X of objects, any $x \in X$ is threated as resembling itself and, if $x \in X$ resembles $y \in X$, then also y resembles x. This relationship is represented by reflexive and symmetric fuzzy relation (originally defined on the basis of pseudo-metric spaces, yet in practice it is often assumed that the underlying pseudo-metric is the identity). Transitivity is not required since it may lead to counterintuitive results. For example, a temperature of 0°C resembles 1°C up to the degree 1, also 1°C totally resembles 2°C , the same with 10°C and 11°C, yet 0°C resembles 11°C to the degree definitely less than 1. Having established a resemblance relation R on a universe in discourse, linguistic hedges are modelled by fuzzy necessity (1) and fuzzy possibility (2) operators. Namely, given a fuzzy predicate P, an intensifying modifier $iMod$, and a weakening modifier $wMod$, we use the following general schemas:

$$iMod(P) = [R]_{\rightarrow}P$$
$$wMod(P) = \langle R\rangle_{\otimes}P.$$

For example, if $P \in \mathcal{F}(X)$ stands for *good*, then *very good* is represented by $[R]_{\rightarrow}P$, while *quite good* is represented by $\langle R\rangle_{\otimes}P$. This representation reflects our underlying intuition: x is called *very good* up to the degree to which all objects resembling x are qualified as *good*.

Various intensifying (resp. weakening) modifiers reflect different emphasis on terms they are applied to. Assume that two intensifying modifiers, $iMod_1$ and $iMod_2$, are such that $iMod_1$ reflects a stronger emphasis than $iMod_2$. Then, for any fuzzy predicate P,

$$iMod_1(P) \subseteq iMod_2(P). \tag{7}$$

The following three schemes are proposed for the representation of the resulting modified terms:

Scheme 1.i: Take two fuzzy implications, $\rightarrow$ and $\Rightarrow$, such that $\rightarrow \leqslant \Rightarrow$. Then

$$iMod_1(P) = [R]_{\rightarrow}P$$
$$iMod_2(P) = [R]_{\Rightarrow}P.$$

Here the inclusion (7) is guaranteed by Property 3.1(c).

Scheme 2.i: Take a t-norm $\otimes$ and a fuzzy implication $\rightarrow$. Then

$$iMod_1(p) = [R]_{\rightarrow}P$$
$$iMod_2(P) = \langle R\rangle_{\otimes}[R]_{\rightarrow}P.$$

Now, (7) holds due to Corollary 3.1.

Scheme 3.i: Take two fuzzy implications $\rightarrow$ and $\Rightarrow$. Then

$$iMod_1(P) = [R]_{\Rightarrow}[R]_{\rightarrow}P$$

The inclusion (7) also holds by reflexivity of R and Property 3.3(a).

For example, the intuition dictates that from among intensifying modifiers *extremely*, *definitely*, and *very*, the first one is the strongest modifier, while the last one is the the weakest one. Hence, for any fuzzy predicate P, we expect

$$\textit{extremely } P \subseteq \textit{definitely } P \subseteq \textit{very } P.$$

Using the above schemes we represent these terms as

$$\textit{extremely } P = [R]_{\rightarrow}[R] \Rightarrow R$$
$$\textit{definitely } P = [R]_{\Rightarrow}P$$
$$\textit{very } P = [R]_{\rightarrow}P$$

where fuzzy implications $\rightarrow$ and $\Rightarrow$ satisfy $\Rightarrow \leqslant \rightarrow$.

Similarly, for weakening modifiers we have three schemes. Assume that $wMod_1$ and $wMod_2$ are two weakening modifiers such that the former reflects weaker emphasis than the latter one. Consequently, for any fuzzy predicate P,

$$wMod_2(P) \subseteq wMod_1(P). \quad (8)$$

Scheme 1.w: Take two t-norms, $\otimes$ and $\odot$, satisfying $\otimes \leqslant \odot$. Then

$$wMod_1(P) = \langle R\rangle_{\odot}P$$
$$wMod_2(P) = \langle R\rangle_{\otimes}P.$$

(8) is guaranteed by Property 3.1(d).

Scheme 2.w: Take a fuzzy implication $\rightarrow$ and a t-norm $\otimes$. The

$$wMod_1(P) = \langle R\rangle_{\otimes}P$$
$$wMod_2(P) = [R]_{\rightarrow}\langle R\rangle_{\otimes}P.$$

(8) holds by Corollary 3.1.

Scheme 3.w: Take two t-norms, $\otimes$ and $\odot$. Then

$$wMod_1(P) = \langle R\rangle_{\odot}\langle R\rangle_{\otimes}P$$
$$wMod_2(P) = \langle R\rangle_{\otimes}P.$$

Again, (8) is satisfied by reflexivity of R and Property 3.3(a).

Assume that for three weakening modifiers: *rather*, *quite*, and *more or less* semantic entailment is such that

$$\textit{rather } P \subseteq \textit{quite } P \subseteq \textit{more or less } P.$$

Then

$$\textit{rather } P = \langle R\rangle_{\otimes}P$$
$$\textit{quite } P = \langle R\rangle_{\odot}P$$
$$\textit{more or less } P = \langle R\rangle_{\otimes}\langle R\rangle_{\odot}P$$

where $\otimes$ and $\odot$ are t-norms such that $\otimes \leqslant \odot$.

Example 5.1 Let $X = [0, +\infty)$ be a universe of temperatures and let a resemblance relation on X be given by

$$R(x, y) = \min\left(1, \max\left(0\,, 2 - \frac{|x-y|}{2}\right)\right)$$

Note that R is reflexive and symmetric, but not sup-$\otimes$-transitive for any t-norm $\otimes$. Indeed, we have

$$\begin{aligned} R(10, 13\tfrac{1}{4}) &\otimes R(13\tfrac{1}{4}, 15\tfrac{1}{2}) \\ &\leqslant \min(R(10, 13\tfrac{1}{4}), R(13\tfrac{1}{4}, 15\tfrac{1}{2})) \\ &= \min(\tfrac{3}{8}, \tfrac{7}{8}) \\ &= \tfrac{3}{8} \\ &\not\leqslant 0 \\ &= R(10, 15\tfrac{1}{2}). \end{aligned}$$

Let a fuzzy set $W \in \mathcal{F}(X)$, representing a term *warm*, be given by

$$W(x) = \begin{cases} 0 & x \leq 20 \\ \frac{1}{5}x - 4 & 20 < y < 25 \\ 1 & x \geq 25 \end{cases}$$

Now, fuzzy sets $([R]_{KD}W$ and $[R]_{KD}[R]_{KD}W$ given below, represent terms *very warm* and *definitely warm*, respectively.

$$[R]_{KD}W(x) = \begin{cases} 0 & x \leq 22 \\ \frac{1}{7}x - \frac{22}{7} & 22 < x < 29 \\ 1 & x \geq 29 \end{cases}$$

$$[R]_{KD}[R]_{KD}W(x) = \begin{cases} 0 & x \leq 24 \\ \frac{1}{9}x - \frac{8}{3} & 24 < x < 33 \\ 1 & x \geq 33 \end{cases}$$

On the other hand, $\langle R\rangle_L W$ and $\langle R\rangle_M W$ represent *more or less warm* and *rather warm*, respectively.

$$\langle R\rangle_L W(x) = \begin{cases} 0 & x \leq 18 \\ \frac{1}{5}x - \frac{18}{5} & 18 < x < 23 \\ 1 & x \geq 23 \end{cases}$$

$$\langle R\rangle_M W(x) = \begin{cases} 0 & x \leq 16 \\ \frac{1}{7}x - \frac{16}{7} & 16 \leq x \leq 23 \\ 1 & x \geq 23 \end{cases}$$

Note that

$$[R]_{KD}[R]_{KD}W \subseteq [R]_{KD}W \subseteq W \subseteq \langle R\rangle_L W \subseteq \langle R\rangle_M W.$$

These membership functions are depicted in Fig. 1. □

Consider now the following three expressions:

(E1) *x is rather very P*;

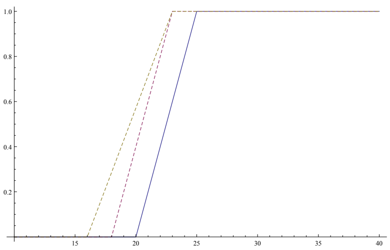
(E2) *x is P*;

(E3) *x is definitely quite P*.

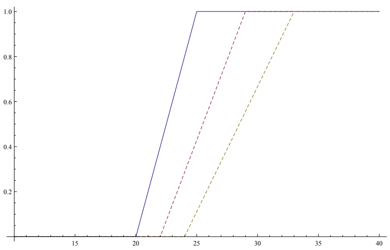
Note that in (E1) a weaker emphasis is put on the fact represented by *very P*, but stronger than in the statement (E2) – consequently, the expressivity in somewhere in between. Similarly, (E3) is a stronger expression that *x is quite P*, but weaker than (E2), so

Fig. 1. Membership functions for linguistic terms

(a) Membership functions for linguistic terms (from left to right): $\langle R\rangle_M W$ ***(****rather warm****),*** $\langle R\rangle_L W$ ***(****more or less warm****),*** W ***(****warm****)***



(b) Membership functions for linguistic terms (from left to right): W ***(****warm****),*** $[R]_{KD}W$ ***(****very warm****),*** $[R]_{KD}[R]_{KD}W$ ***(****definitely warm****)***



its expressive power is intermediate. In our framework the statements (E1) and (E3) can be represented by $\langle R\rangle_{\otimes}[R]_{\rightarrow}P$ and $[R]_{\Rightarrow}\langle R\rangle_{\odot}P$, respectively, where $\otimes$ (resp. $\odot$) is a left-continuous t-norm and $\rightarrow$ (resp. $\Rightarrow$) is its residual implication. By Corollary 3.1, we have

$$\langle R\rangle_{\otimes}[R]_{\rightarrow}P \subseteq P \subseteq [R]_{\Rightarrow}\langle R\rangle_{\odot}P,$$

which coincides with our intuition.

There is a kind of dualism between some linguistic hedges. Namely, let the following expressions be given:

(E4a) *rather not P*;

(E5a) *not very P*.

In particular, if we say that a temperature outside is *rather not warm*, it obviously cannot be treated as *extremely warm*, whence *rather not P* ⊆ *not very P*. By Property 3.3(c) this can be modeled by operators (2) for (E4a) and by (1) for (E5a) using a left-continuous t-norm $\otimes$, its residual implication $\rightarrow$, and the negation $\neg$ induced by $\otimes$. However, if in some cases *rather not P=not very P* is required, one can apply Łukasiewicz connectives.

Similarly, consider the following expressions:

(E4b) *definitely not P*;

(E5b) *not quite P*.

For example, if one says that outside is not even *quite warm*, the intuition dictates that it is *definitely not warm*, thus *not quite P* ⊆ *definitely not P*. As before, by Property 3.3(c) this case may be supported by choosing a left-continuous t-norm $\otimes$, its residual implication, and the negation induced by $\otimes$. Łukasiewicz connectives are to be applied whenever equality is desired.

6. Concluding Remarks

In this paper we have presented two applications of fuzzy modal operators. First, we have shown how these operators may be used for fuzzy set approximations. Basing on the observation that fuzzy set approximations maybe viewed as intuitionistic fuzzy sets, we have presented the application of these operators in the problem of skills matching for selecting research projects. Also, we have pointed out how fuzzy possibility and fuzzy necessity operators can be used for modeling linguistic hedges. This representation is based on the observation that linguistic hedges may be viewed as specific kind of modal expressions. The presented approach reflects the contextual meaning of these modifiers which is, in our opinion, intuitively justified.

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