ISSN 1232-3829
tts 10/2013
Technika Transportu Szynowego koleje ▪ tramwaje ▪ metro

OSMOLOVSKII Nikolai, WÓJTOWICZ Marek, JANISZEWSKI Szymon

TIME OPTIMAL CONTROL FOR A TWO-DIMENSIONAL LINEAR SYSTEM WITH A FIRST ORDER STATE CONSTRAINT

Abstract

This paper provides an analysis of time optimal control problem of motion of a material point along a horizontal axis, without friction. The point is controlled by a power directed along this axis. An absolute value of the power is limited by one. The velocity in the reverse direction is also limited. In the analysis of this problem, the maximum principle is applied.

INTRODUCTION

We consider the following optimization problem. A material point (a trolley) of the mass equal to one moves along the horizontal axis without friction. The point is controlled by a power directed along the same axis. An absolute value of the power is limited by one. Let the position of the point at time t be $x(t)$ and its velocity $y(t)$. Let the value of the power at time t be $u(t)$. A movement in the reverse direction with the velocity exceeding $a>0$ is forbidden. Al the initial time $t=0$, the initial position $x(0)=x_0$ and the initial velocity $y(0)=y_0$ are given. Also, at the final time $t=T$, the final position $x(T)=x_T$ and the final velocity $y(T)=y_T$ are given. It is necessary to minimize the time of the process *T*. Since $m=1$, by the Newton law we have

$$u(t)=\ddot{x}(t)=\dot{y}(t).$$

Thus, the problem has the form:

$$T\to\min, \tag{1}$$

subject to the constraints

$$\dot{x}(t)=y(t),\ \ \dot{y}(t)=u(t), \tag{2}$$

$$x(0)=x_0,\ \ y(0)=y_0 \tag{3}$$

$$x(T)=x_T,\ \ y(T)=y_T \tag{4}$$

$$|u(t)|\le 1,\ \ y(t)\ge -a \tag{5}$$

Here $x,y,u\in\mathbb{R}$. We assume that the control variable $u(t)$ is measurable and bounded and the state variables $x(t)$ and $y(t)$ are absolutely continuous. Since $\dot{y}=u$, the state constraint $y\ge -a$ has the order one.

The first version of this problem appeared in the book of l. S. Pontryagin et al. [6], but, in this book, it was considered only the case, where $x_T=y_T=0$, and there was no state constraint of the form $y(t)\ge -a$. The problem with the state constraint $y(t)\ge -a$ was briefly analyzed in the book of V. M. Tikhomirov and A. D. Ioffe [2], but again in the case, where

$x_T = y_T = 0$. The problem with arbitrary x_T and y_T and without state constraints was analyzed in the book of A. A. Milyutin and N. P. Osmolovskii [5]. We will give a detailed analysis of the above problem.

1. MAXIMUM PRINCIPLE

Let a process

$$\mathcal{T} := \left(x(t), y(t), u(t) | t \in [0, T] \right)$$

be a solution to the problem. We assume that

$$y(0) > -a, \quad y(T) > -a \tag{6}$$

Let us formulate the first order necessary optimality conditions for the process P, or the maximum principle in the Dubovitskii-Milyutin form, cf. [1], [3], [4]. We introduce the *Pontryagin function* (or the *Hamiltonian*):

$$H = \psi_x y + \psi_y u \tag{7}$$

And the augmented Pontryagin function (or the augmented Hamiltonian):

$$\bar{H} = \psi_x y + \psi_y u + \frac{d\mu}{dt}(y + a) \tag{8}$$

(where $\frac{d\mu}{dt}$ is generalized derivative). The maximum principle conditions are the following: on the interval $[0, T]$, there exist functions of bounded variation $\psi_x(t), \psi_y(t)$ and a Lebesgue – Stieltjes measure $d\mu$ (defined by a function of a bounded variation μ) such that the following conditions hold:

a) *non-triviality condition*: the triple $\left(\psi_x(t), \psi_y(t), d\mu(t)\right)$ is nontrivial;

b) *non-positivity condition*:

$$d\mu(t) \geq 0 \tag{9}$$

c) *complimentary slackness condition*:

$$d\mu(t)\left(y(t) + a\right) = 0, \tag{10}$$

d) *adjoint equalitions*:

$$-\dot{\psi}_x(t) = \bar{H}_x \iff -\dot{\psi}_x(t) = 0, \tag{11}$$

$$-d\psi_y(t) = \bar{H}_y dt \iff -d\psi_y(t) = \psi_x(t)dt + d\mu(t), \tag{12}$$

e) *condition of the maximum of the Pontryagin function*:

$$\max_{v \in [-1,1]} \psi_y(t) v = \psi_y(t) u(t),$$

or in the equivalent form

$$u(t) \in \operatorname{Sign} \psi_y(t) := \begin{cases} \{+1\}, & \psi_y(t) > 0 \\ \{-1\}, & \psi_y(t) < 0, \\ [-1, 1], & \psi_y(t) = 0, \end{cases} \tag{13}$$

f) *constancy and non-negativeness of the Pontryagin function* for the optimal process:

$$\psi_x(t) y(t) + \psi_y(t) u(t) = const \geq 0. \tag{14}$$

Let us study properties of extremals of the problem.

2. ANALYSIS OF THE CONDITIONS OF THE MAXIMUM PRINCIPLE: THE MOVEMENT INSIDE THE STATE CONSTRAINT

Let a process

$$\mathcal{T} := \left(x(t), y(t), u(t) | t \in [0,T] \right)$$

be an extremal of the problem, i.e., the conditions of the maximum principle hold. Set

$$\mathcal{M}_0 = \{ t \in [0,T] \mid y(t) = -a \}.$$

It is clear that $\mathcal{M}_0$ is a closed and bounded set. Consequently, $\mathcal{M}_0$ is a compact (possibly empty) subset of the interval $[0,T]$. We call $\mathcal{M}_0$ the *contact set with the boundary of the state constraint*.

Let us consider an interval $(t_0,t_1) \subset [0,T], (t_0 < t_1)$ such that $y(t) > -a$ on this interval. According to assumption (6), such interval exists. By complementary slackness condition (10) we have: $d\mu(t) = 0$ for all $t \in (t_0,t_1)$. Consequently, on the interval (t_0,t_1), the extremal satisfies the well-known system of conditions:

Let us study the expended system on an interval $\Delta \subset [0,T]$:

$$-\dot{\psi}_x(t) = 0, \qquad -\dot{\psi}_y(t) = \psi_x(t),$$
$$\dot{x}(t) = y(t), \ \ \dot{y}(t) = u(t) \in \operatorname{Sign} \psi_y(t).$$

If $\psi_y(t) \equiv 0$ on the interval (t_0,t_1), then the control $u(t)$ can be chosen arbitrary on this interval. In this case, we say that (t_0,t_1) is an interval of *singular regime*.

LEMMA 3.1 *There is no interval* $(t_0,t_1) \subset [0,T] \setminus \mathcal{M}_0$ *of singular regime in the problem*.

Proof. Assume the contrary. Let $\psi_y(t) \equiv 0$ on the interval $(t_0,t_1) \subset [0,T] \setminus \mathcal{M}_0$. Then the equation $-\dot{\psi}_x(t) = 0$ on $[0,T]$ implies that $\psi_x = \text{const}$ on $[0,T]$ and the equation $-\dot{\psi}_y(t) = \psi_x(t)$, satisfied on (t_0,t_1), implies that $\psi_x(t) \equiv 0$ on $[0,T]$. Then from adjoint equation (12) it follows that

$$-d\psi_y = d\mu \geq 0 \tag{15}$$

on $[0,T]$. Consequently, the function $\psi_y(t)$ is non-increasing on the whole interval $[0,T]$, and moreover, $\psi_y(t) \equiv 0$ on (t_0,t_1). If $d\mu$ is the zero measure, then clearly $\psi_y(t)$ is zero function, and hence the triple $(\psi_x, \psi_y, d\mu)$ is trivial)i.e., equal to zero), which contradicts the non-triviality condition.

Consequently,

$$d\mu \neq 0 \tag{16}$$

and then the set $\mathcal{M}_0$ is nonempty.

Let

$$t' = \min\{ t | t \in \mathcal{M}_0 \}, \quad t'' = \max\{ t | t \in \mathcal{M}_0 \}.$$

Since $\mathcal{M}_0$ is compact set, the minimum and the maximum in these formulas are attained. Conditions (15) and (16), together with complementary slackness condition (10 imply that

$$\psi_y(t'-0) > \psi_y(t''+0) \tag{17}$$

Since $\psi_y(t) \equiv 0$ on (t_0,t_1), the inclusion $(t',t'') \subset (t_0,t_1)$ is impossible; consequently, either (i) $t' \leq t_0$, or (ii) $t_1 \leq t''$. Consider each of the two possible cases.

(i) Suppose that $t' \leq t_0$. Then $\psi_y(t'-0) > \psi_y(t_0+0) = 0$. Let us show that the strict inequality $\psi_y(t'-0) > 0$ does not hold. Indeed, if $\psi_y(t'-0) > 0$, then $\psi_y(t) > 0$ in some left half-neighborhood of the point t' and then $\dot{y}(t) = u(t) = 1$ in the same half-neighborhood, but the latter means that $y(t) < y(t') = -a$ for all $t < t'$ close enough to t'

which is impossible. Thus we have $\psi_y(t'-0)=0$. But then according (17) we obtain $\psi_y(t''-0)>0$. It follows that $\psi_y(t)>0$ in some right half-neighborhood of the point t'' and then $\dot{y}(t)=u(t)=-1$ in the same half-neighborhood. The latter means that $y(t)<y(t'')=-a$ for all $t>t''$ close enough to t'' which is also impossible.

(ii) Suppose that $t_1 \le t''$. Then $0=\psi_y(t_1-0)\ge\psi_y(t''+0)$. The strict inequality $\psi_y(t''-0)<0$ does not hold. Indeed, in this case $\psi_y(t)<0$ in some right half-neighborhood of the point t'', and then $\dot{y}(t)=u(t)=-1$ in the some half-neighborhood, but the latter means that $y(t)<y(t'')=-a$ for all $t>t''$ close enough to t'', which is impossible. Therefore, $\psi_y(t''-0)=0$. Then, according to (17), we get $\psi_y(t'-0)>0$. The latter means that $\psi_y(t)>0$ in some left half-neighborhood of the point t', and then $\dot{y}(t)=u(t)=1$ in the some half-neighborhood. The latter implies that $y(t)<y(t')=-a$ for all $t<t'$ close enough to t', which is also impossible.

Thus, assuming that $\psi_y(t)\equiv 0$ on an interval (t_0,t_1), we come to a contradiction. This proves the lemma.

So, let $(t_0,t_1)\subset[0,T]\setminus\mathcal{M}_0$ be an arbitrary interval. Then, the function $\psi_y(t)$ cannot vanish on this interval and hence $\psi_y(t)$ is a nonzero linear function on this interval. Consequently, $\psi_y(t)$ changes its sign on (t_0,t_1) taking values +1 or -1 (*bang-bang control*) and changing its sign on (t_0,t_1) not more than once.

Now consider subinterval $(\tau_0,\tau_1)\subset(t_0,t_1)$ such that $u(t)=\text{const}$ on (τ_0,τ_1), i. e., $u(t)=1$ on (τ_0,τ_1). Then integrating equations (2) on (τ_0,τ_1) we get

$$y(t)=ut+C_1, \quad x(t)=\frac{u}{2}t^2+C_1t+C_2 \tag{18}$$

Let us find a relation between x and y on (τ_0,τ_1). Conditions

$$\dot{x}=y, \quad \dot{y}=u, \quad u=\text{const}\ne 0, \quad u^2=1$$

imply

$$\frac{dx}{dy}=\frac{y}{u}=uy\,.$$

Consequently,

$$dx=uydy\,,$$

whence

$$x=\frac{u}{2}y^2+C\,. \tag{19}$$

Thus on the phase plane xOy we have two families of parabolas. The first family corresponds to the control $u(t)=1$ and has the form

$$x=\frac{1}{2}y^2+C\,. \tag{20}$$

The direction of movement along parabolas of the first family is defined by the condition: if $t\to+\infty$, then $x\to+\infty$ and $y\to+\infty$ (cf. (18) with $u=1$). The second family corresponds to the control $u(t)=-1$ and has the form

$$x=-\frac{1}{2}y^2+C\,. \tag{21}$$

The direction of movement along parabolas of the first family is defined by the condition: if $t \to +\infty$, then $x \to +\infty$ and $y \to +\infty$ (cf. (18) with $u = -1$).

So, if $t \in (t_0, t_1) \subset [0,T] \setminus \mathcal{M}_0$, then the point (x, y) moves along a parabola of one of the two families (20) or (21))on the phase plane xOy in the corresponding direction), and then it can switch to a parabola of another family (but not more than once), and continue its motion along this parabola in the corresponding direction. In what follows we will see that if $\mathcal{M}_0$ is nonempty, then the switching on an interval $(t_0, t_1) \subset [0,T] \setminus \mathcal{M}_0$ is impossible

3. THE CONTACT SET WITH THE BOUNDARY OF THE STATE CONSTRAINT

Denote by $[\psi_y](t)$ the jump of the function ψ_y at the point t, i.e.,

$$[\psi_y](t) = \psi_y(t+0) - \psi_y(t-0).$$

The adjoint equation (12) implies

COROLLARY 3.1 *At any point* $t \in (0,T)$ *we have*

$$[\psi_y](t) = -[\mu](t) \le 0. \qquad (22)$$

The following lemma holds.

LEMMA 3.1 *The set* $\mathcal{M}_0$ *is connected.*

Proof. Assume the contrary: the compact set $\mathcal{M}_0$ is not connected. Then there exist points t' and t'', and τ on $[0,T]$, $t' < \tau < t''$, such that $y(t') = y(t'') = -a$ and $y(\tau) > -a$. Since $[0,T] \setminus \mathcal{M}_0$ is an open set (in the induced topology of the interval $[0,T]$), without loss of generality we can assume that $y(t) > -a$ for all $t \in (t', t'')$. Then, as we know, the control $u(t)$ is piecewise constant function on (t', t'') taking values 1 or -1 with at most one switching point, and hence $y(t)$ is a (continuous) piecewise linear function with at most one break point. Let τ be a switching point of the control $u(t)$ and hence τ is a break point of the function $y(t)$. The conditions $\dot{y} = u,\ y(t') = y(t'') = -a$, $y(t) > -a$ for all $t \in (t', t'')$ imply that $y(t)$ increases on (t', τ) and decreases on (τ, t''), i.e., $u(t) = 1$ on (t', τ) and $u(t) = -1$ on (τ, t''). Moreover, it is clear that $\tau = (t' + t'')/2$. Furthermore that the function $\psi_y(t)$ is linear on the interval (t', t'') and has the following signs on this interval:

$$\psi_y(t) > 0 \ \text{ on } \ (t', \tau); \quad \psi_y(t) < 0 \ \text{ on } \ (\tau, t'').$$

Consequently,

$$\psi_y(t'+0) > 0, \qquad \psi_y(t''-0) < 0.$$

The conditions

$$\psi_y(t'+0) > 0, \quad [\psi_y](t') = \psi_y(t'+0) - \psi_y(t'-0) \le 0$$

Imply that $\psi_y(t''-0) < 0$, and hence $\psi_y(t) > 0$ in a left half-neighborhood of the point t'. Consequently, $\dot{y}(t) = u(t) = 1$ in the same half-neighborhood, i.e., the function $y(t)$ is strictly increasing in this half-neighborhood. The latter means that, for $t < t'$ and t close enough to t', we have $y(t) < y(t') = -a$, which is impossible. We come to a contradiction. This proves the lemma.

Since $\mathcal{M}_0$ is a connected compact set, we get the following assertion, $y(t)$.

COROLLARY 3.2 *The set* $\mathcal{M}_0$ *is a closed interval, or a singleton, or an empty set.*

If $\mathcal{M}_0$ is an empty set, then, as we know, $u(t)$ is a bang-bang control taking values ± 1 with at most one switching point. In fact this case has been considered in the preceding section. Now let us consider the case where $\mathcal{M}_0$ is a singleton.

4. THE CASE OF A SINGLE CONTACT WITH THE BOUNDARY OF STATE CONSTRAINT

LEMMA 4.1 *Let* $\mathcal{M}_0 = \{t'\}$ *be a singleton. Then* $d\mu = 0$.

Proof. Suppose that $\mathcal{M}_0$ is a singleton $\{t'\}$. Then, it is clear that t' is a minimum point of the function $y(t)$, and moreover, $y(t)$ decreases in a left half-neighborhood of the point t' with the derivative $\dot{y} = u = -1$, and $y(t)$ increases in a right half-neighborhood of this point with the derivative $\dot{y} = u = -1$. It follows that $\psi_y(t) < 0$ in the left half-neighborhood of the point t' and $\psi_y(t) < 0$ in the right half-neighborhood of this point. Consequently,

$$\psi_y(t'-0) \le 0, \qquad \psi_y(t'+0) \ge 0.$$

This implies that

$$\left[\psi_y\right](t') = \psi_y(t'+0) - \psi_y(t'-0) \ge 0.$$

From the other hand, according to Corollary 3.1, we have $\left[\psi_y\right](t') \le 0$. It follows that $\left[\psi_y\right](t') = 0$. Then, again using Corollary 3.1, we obtain that $\left[\psi_y\right](t') = 0$. By the complementary slackness condition, the measure $d\mu$ is concentrated on a singleton $\{t'\}$. Hence $d\mu = 0$. The lemma is proved.

Thus in the case, where $\mathcal{M}_0$ is a singleton, we have that $d\mu = 0$ and hence the nonnegativity condition and the complementary slackness condition are fulfilled automatically, while the adjoint system takes the form:

$$\dot{\psi}_x = 0, \quad \dot{\psi}_y = -\psi_x.$$

Thus, the components of the extremal $\left(x(t), y(t), u(t), \psi_x(t), \psi_y(t)\right)$ satisfying the state constraint $y(t) \ge -a$, are defined by the same system of conditions, as in the case, where the state constraint is absent. The soltions to this system are well-known.

5. THE CASE, WHERE THE CONTACT SET WITH THE BOUNDARY OF STATE CONSTRAINT IS AN INTERVAL

LEMMA 5.1 *Assume that* $\mathcal{M}_0$ *is an interval* $\mathcal{M}_0 = [t_0, t_1] \subset [0, T]$, $t_0 < t_1$. *Then the measure* $d\mu$ *has no atoms, i.e., the function* μ *has no jumps, and hence* μ *is a continuous function. Moreover, the measure* $d\mu$ *is absolutely continuous and has a constant density on* $[t_0, t_1]$*:* $d\mu = \dot{\mu}\, dt$, *where* $\dot{\mu} = \text{const} \ge 0$ on $[t_0, t_1]$.

Proof. Let $\mathcal{M}_0=[t_0,t_1]$, $0<t_0<t_1<T$. Then $y(t)=a$ on (t_0,t_1) hence $\dot{y}(t)=0$, and therefore $u(t)=0$ on (t_0,t_1). In virtue of the condition $u(t)\in \mathrm{Sign}\,\psi_y(t)$ this implies that

$$\forall_{t\in(t_0,t_1)} \ \psi_y(t)=0 \tag{23}$$

Taking into account the adjoint equation $-\dot{\psi}_y=\psi_x+\dot{\mu}$ we conclude that

$$\forall_{t\in(t_0,t_1)} \ \dot{\mu}=-\psi_x=\text{const}\geq 0 .$$

Thus the measure $d\mu$ has a constant density on the open interval (t_0,t_1). But what about the ends of the interval?

Consider the point t_0. Since $y(t_0)=-a$ and $y(t)>-a$ in a left half-neighborhood of the point t_0, the linear function $y(t)$ decrease in a left half-neighborhood of this point, hence $\dot{y}(t)=u(t)=-1$ in this left half-neighborhood. Then in virtue of the condition $u(t)\in \mathrm{Sign}\,\psi_y(t)$ we have: $\psi_y(t)<0$ in the same half-neighborhood. It follows that $\psi_y(t_0-0)\leq 0$. Then by Corollary 3.1, we have $[\psi_y](t_0)<0$. Consequently,

$$\psi_y(t_0+0)=\psi_y(t_0-0)+[\psi_y](t_0).$$

This implies that $\psi_y(t)<0$ in a right half-neighborhood of the point t_0. This contradicts condition (23). Therefore, $[\mu](t_0)=0$.

Similarly, let us show that $[\mu](t_1)=0$. Assume the contrary: let $[\mu](t_1)>0$. Since $y(t_1)=-a$ and $y(t)>-a$ in a right half-neighborhood of the point t_1 , the linear function $y(t)$ increases in a right half-neighborhood of this point, hence $\dot{y}(t)=u(t)=1$ in this right half-neighborhood. Then , in virtue of the condition $u(t)\in \mathrm{Sign}\,\psi_y(t)$, we have: $\psi_y(t)>0$ in the same half-neighborhood. It follows that $\psi_y(t_1+0)\geq 0$. Now assume that the function μ has a jump at the point t_1: $[\mu](t_1)>0$. Then by Corollary 3.1 we have $[\psi_y](t_0)<0$. Consequently,

$$\psi_y(t_1-0)=\psi_y(t_1+0)-[\psi_y](t_1)>0 .$$

This implies that $\psi_y(t)>0$ in a left half-neighborhood of the point t_1. This contradicts condition (23). Therefore, $[\mu](t_1)=0$. The lemma is proved.

Thus, in the case, where the set $\mathcal{M}_0$ is an interval $[t_0,t_1]$, the character of the function $\psi_y(t)$ is the following:

$$\psi_y(t)<0 \text{ for } t\in[0,t_0),\ \psi_y(t)=0 \text{ for } t\in[t_0,t_1],\ \psi_y(t)>0 \text{ for } t\in(t_1,T] \tag{24}$$

Moreover, $\psi_y(t)$ is a continuous, piecewise linear, monotone non-decreasing function. Hence control $u(t)$ has the form:

$$u(t)=-1 \text{ for } t\in(0,t_0),\ u(t)=0 \text{ for } t\in(t_0,t_1),\ u(t)=1 \text{ for } t\in(t_1,T) \tag{25}$$

REFERENCES

1. Dubovitski, A. Ya. and Milyutin, A. A. *Problems for extremum under constraints,* Zh. Vychislit. Mat. i Mat. Fiz., 5, No. 3 (1965), 395-453; English transl. in U.S.S.R. Comput. Math. and Math. Phys.5 (1965).
2. Ioffe, A.D. and Tikhomirov, V.M. *Theory of Extremal Problems,* North-Holland Publishing Company, Amsterdam, 1979, Russian Edition: Nauka, Moscow, 1974.
3. Milyutin, A. A. *Maximum Principle in the General Optimal Control Problem* [in Russian], Fizmatlit, Moscow, 2001.

4. Milyutin, A.A., Dmitruk, A.V., Osmolovskii, N.P., *Maximum principle in optimal control*, Moscow State University, Moscow, 2004 (in Russian)
5. Milyutin, A.A. and Osmolovskii, N.P. *Calculus of Variations and Optimal Control*, Translations of Mathematical Monographs, Vol. 180, American Mathematical Society, Providence, 1998.
6. L. S. Pontryagin, V. G. Boltyanski, R. V. Gramkrelidze, and E. F. Miscenko, *The Mathematical Theory of Optimal Processes*, Fitzmatlit, Moscow; English translation: Pergamon Press, New York, 1964.

ZADANIA MINIMALIZACJI CZASU DLA DWUWYMIAROWYCH LINIOWYCH UKŁADÓW STEROWANIA Z OGRANICZENIAMI FAZOWYMI I-GO RZĘDU

Streszczenie

W artykule dokonano analizy zadania minimalizacji czasu ruchu punktu materialnego, wzdłuż osi poziomej, który odbywa się bez uwzględnienia tarcia. Punkt jest kontrolowany za pomocą siły skierowanej wzdłuż osi poziomej. Wartość siły co do modułu nie przekracza jeden. Prędkość w kierunku przeciwnym jest również ograniczona. Analiza odbywa się na podstawie zasady maksimum Pontryagina.

***Autorzy**:*
prof. dr hab. **Nikolay Osmolovskii –** University of Technology and Humanities Radom; Systems Research Institute, Polish Academy of Sciences Warszawa; University of Natural Sciences and Humanities Siedlce; Moscow State University of Civil Engineering, Moscow, Russia,
dr **Marek Wójtowicz –** University of Technology and Humanities Radom,
dr **Szymon Janiszewski –** University of Technology and Humanities Radom.