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EXAMPLES OF GRAPHICAL REPRESENTATIONS REALISED BY SUBSPACE PROJECTIONS WITH BUNDLE DISPERSED CENTRES

Bogusław JANUSZEWSKI

Rzeszow University of Technology
Department of Engineering Geometry and Graphics
2 Poznańska st., 35-084 Rzeszów, Poland, phone +48-17 865 13 07
email: banjanus@prz.rzeszow

Abstract. The following paper analyses examples of R subspace projections with bundle dispersed centers, in which R are graphical representations, including reversible transformations. The result of the analysis is a list of examples of the R mapping apparatuses, when the R projections are graphical representations of three or four-dimensional projective spaces. The final section of the paper presents examples of constructions of images of a straight line, a plane and a hyper plane derived with the help of distinguished types of the R mapping.

Key Words : projection, projection apparatus, subspace projection with bundle dispersed centres, image of subspace

1. Introduction

A so-called *subspace projection with bundle dispersed centres* was defined in [1]. This projection, here marked by **R**, can be realised in an n-dimensional $\boldsymbol{P}_n$ projective or an affine space ($\mathbf{n} \geq 2$), when a so-called *apparatus of the* **R** *projection* is selected in $\boldsymbol{P}_n$. This apparatus consists of (Fig.1):

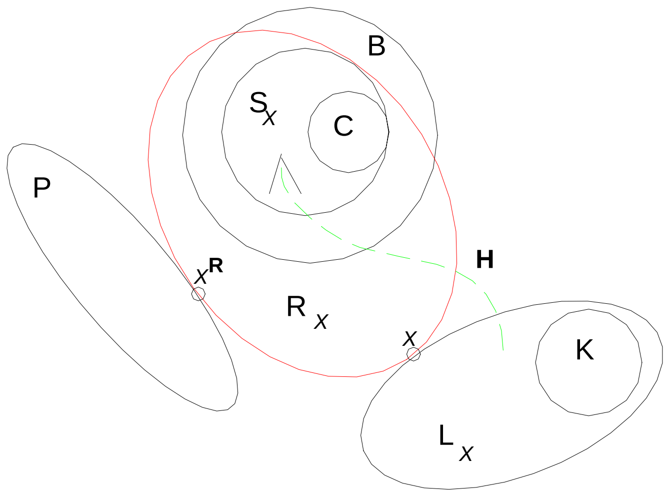

Figure 1: Ideogram of the structure of the **R** projection apparatus and its method of operation

- a ⟨C,B⟩ bundle of subspaces with C core and B field, called a *base of the* **R** *projection centres*,
- a ⟨K,P_n⟩ bundle with a (n - dim B + dimC)-dimensional core K and the P_n field,
- an H projective relation, which transforms the ⟨K,P_n⟩ bundle onto the ⟨C,B⟩ bundle and may be a collineation or a correlation,
- a P subspace of projections.

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The defined **R** subspace $\{\langle \mathbf{C},\mathbf{B}\rangle, \langle \mathbf{K},\boldsymbol{P}_{\mathbf{n}}\rangle, \mathbf{H}, \mathbf{P}\}$ projection apparatus allows us to assign each point $X \in \boldsymbol{P}_{\mathbf{n}}$ respectively to $(X\,\mathrm{O}\,\mathbf{K})$ element of the $\langle \mathbf{K},\boldsymbol{P}_{\mathbf{n}}\rangle$ bundle, when $(X\,\mathrm{O}\,\mathbf{K})$ is a symbol of a junction of the X point and the **K** subspace. The $(X\,\mathrm{O}\,\mathbf{K})$ subspace corresponds in the **H** relation to the $\mathbf{S}_X$ element in the $\langle \mathbf{C},\mathbf{B}\rangle$ centre base of the **R** projection; $\mathbf{S}_X$ is the **R** projection centre for the X point. Finally, the $(X\,\mathrm{O}\mathbf{S}_X)$ junction, called the $\mathbf{R}_X$ *projection formation*, with the **P** subspace of projections gives the $X^{\mathbf{R}} = \mathbf{R}_X \cap \mathbf{P}$ product, which is the *image – projection of the X point in the* **R** *subspace projection of* $\boldsymbol{P}_{\mathbf{n}}$ *onto the* **P** *subspace of projections*.

The analysis of the structure of the apparatus and operating features of the **R** projection leads to a conclusion that the **R** projection can be, in some cases, a reversible transformation of the points of the $\boldsymbol{P}_{\mathbf{n}}$ space, generally positioned against **C**, **B** and **K**. It occurs when $\dim\mathbf{C} \leq 1/2(\dim \mathbf{B} - 3)$, where **H** is a collineation or $\dim\mathbf{C} \leq -1$ and $\mathbf{B} \neq \boldsymbol{P}_{\mathbf{n}}$, where **H** is a correlation.

The properties of the **R** projection with bundle dispersed centres differ depending on dimensions of common parts for the **P** subspace of projection and projection centres assigned to the points of the $[X]$ set, generally positioned against **C**,**B** and **K**. This issue was analysed in [2], where the following three (fundamentally different) types of the **R** projection were distinguished:

- **UG** *projections*, where $\dim(\mathbf{S}_X \cap \mathbf{P}) = \text{const.} \geq 0$ for all points of $[X]$ set,
- **oMg** *projections*, where $\dim(\mathbf{S}_{Xi} \cap \mathbf{P}) = -1$ and $\dim(\mathbf{S}_{Xj} \cap \mathbf{P}) = \text{const.}\geq 0$, where X_{i} and X_{j} are points constituting two disjoint sets $[X_i]$ and $[X_j]$ such, that $[X_i]\cup[X_j] = [X]$,
- **gMg** *projections*, where $\dim(\mathbf{S}_{Xi} \cap \mathbf{P}) = \text{const.}\geq 0$ and $\dim(\mathbf{S}_{Xj} \cap \mathbf{P}) = \text{const.}\geq 0$, but $\dim(\mathbf{S}_{Xi} \cap \mathbf{P}) \neq \dim(\mathbf{S}_{Xi} \cap \mathbf{P})$, where X_{i} and X_{j} are points constituting two disjoint sets $[X_i]$ and $[X_j]$ such, that $[X_i]\cup[X_j] = [X]$.

Each of the distinguished types of the **R** projection can take form of a graphical representation, under the assumption that its subspace of projections is the π plane. Such projections have particularly wide possibilities for application in technically utilised mappings. Moreover, thanks to the simplicity of their analytical description, these **R** projections can be realised using computer software. The following Table presents a numerical attempt to describe conditions which guarantee a possibility of creation of **UG, oMg, gMg** projections apparatuses as mappings of the $\boldsymbol{P}_{\mathbf{n}}$ space.

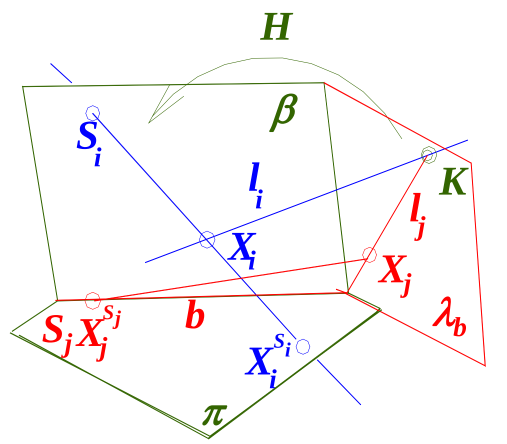
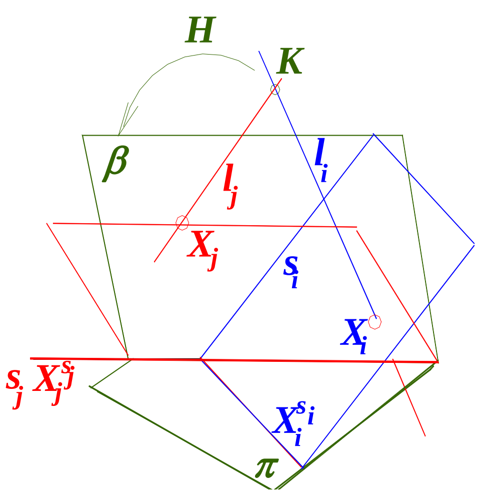
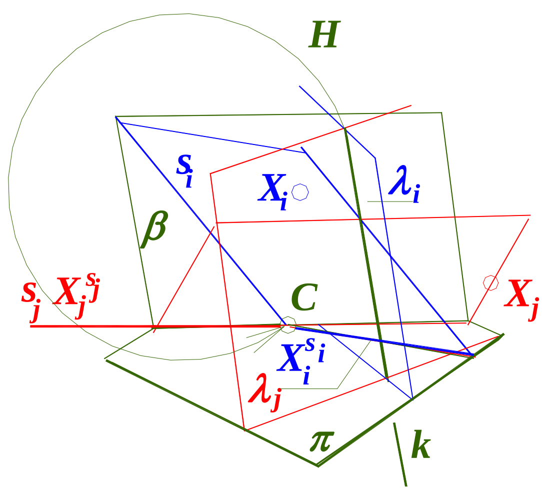
Table 1

Type of **R** projection	Information on elements of the **R** projection apparatus			
	H - collineation		**H** - correlation	
	dim **C***	dim **B***	dim **C***	dim **B***
UG	**n** – 3	$\in \{\mathbf{n}-1, \mathbf{n}\}$	**n** – 3	**n** – 1
oMg	**n** – 4	$\in \langle \mathbf{n}-2, \mathbf{n}\rangle$	$\in \langle -1, \mathbf{n}-4\rangle$	**n** – 2
gMg	**n** – 3**	$\in \{\mathbf{n}-1, \mathbf{n}\}$**	$\in \langle -1, \mathbf{n}-3\rangle$	**n** – 1

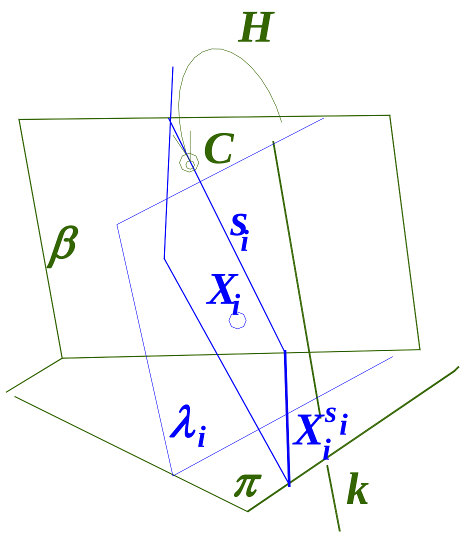
* $\mathbf{n} - \dim \mathbf{B} + \dim\mathbf{C} = \dim \mathbf{K}$, **$\mathbf{C} \cap (\pi \cap \mathbf{B}) \neq \varnothing$ is required

The next main part of the considerations covers in details properties of more interesting examples of projection apparatuses of **UG, oMg, gMg** projections when these projections are mappings of the $\boldsymbol{P}_3$ and the $\boldsymbol{P}_4$ spaces. Additionally, it shows cases where the distinguished **R** mappings are reversible transformations in the $[X]$ set of the representation space, generally positioned against the elements of **R** projection apparatus. The demonstrative schemes of the structures of the distinguished **R** mapping apparatuses and their methods of operation are shown in Table 2 (for the 3-dimensional space $\boldsymbol{P}_3$) and in Table 3 (for the 4-dimensional space $\boldsymbol{P}_4$).

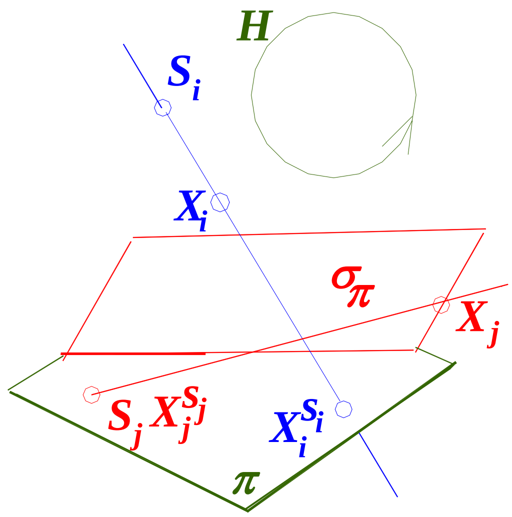
Table 2: Examples of the structure of the **R** mapping apparatuses and their methods of operation in the $\boldsymbol{P}_3$ space (green lines – graphical symbols of elements of the **R** mapping apparatus, blue lines - graphical symbols of the subspaces applied in projecting of $X_i \in [X_i]$, red lines - graphical symbols of the subspaces applied in projecting of $X_j \in [X_j]$)

<table>
<tr><td colspan="10">Mapped space – $\boldsymbol{P}_3$</td></tr>
<tr><td>$\boldsymbol{P}_3/1$</td><td>dim B= 1</td><td>dim C = - 1</td><td>dim K = 1</td><td>H–collin or correl</td><td>$\boldsymbol{P}_3/2$</td><td>dim B =2</td><td>dim C =-1</td><td>dim K = 0</td><td>H -collin</td></tr>
<tr><td colspan="2">Type of R– OMG</td><td colspan="3">R is a reversible mapping for $X \in \boldsymbol{P}_3 - \lambda_i$</td><td colspan="2">Type of R– OMG</td><td colspan="3">R is a reversible mapping for $X \in \boldsymbol{P}_3 - \lambda_b$</td></tr>
<tr><td colspan="5"></td><td colspan="5">
</td></tr>
<tr><td>$\boldsymbol{P}_3/3$</td><td>dim B= 2</td><td>dim C = -1</td><td>dim K= 0</td><td>H - correl</td><td>$\boldsymbol{P}_3/4$</td><td>dim B= 2</td><td>dim C = 0</td><td>dim K = 1</td><td>H-collin or correl</td></tr>
<tr><td colspan="2">Type of R- GMG</td><td colspan="3">R is a reversible mapping for $X \in \boldsymbol{P}_3 - l_i$</td><td colspan="2">Type of R- GMG</td><td colspan="3">R is a non-reversible mapping</td></tr>
<tr><td colspan="5">
</td><td colspan="5">
</td></tr>
</table>

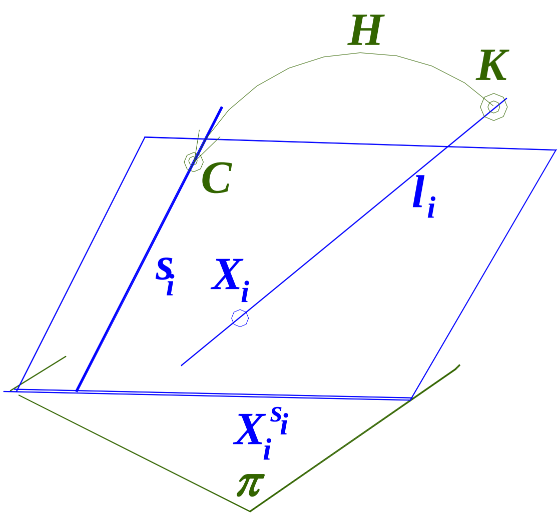
$\boldsymbol{P}_3/5$	dim **B**= 2	dim **C** = 1	dim **K** = 1	**H**–collin or correl
Type of **R**– UG	**R** is a non-reversible mapping			



$\boldsymbol{P}_3/6$	dim **B**= 3	dim **C** = 1	dim **K**= -1	**H** -collin
Type of **R**– oMG	**R** is a reversible mapping for $X \in \boldsymbol{P}_3 - \sigma_\pi$			



$\boldsymbol{P}_3/7$	dim **B**= 3	dim **C** = 0	dim **K** = 0	**H** - collin
Type of **R**- UG	**R** is a reversible mapping for $X \in \boldsymbol{P}_3$			



$\boldsymbol{P}_3/8$	dim **B**= 3	dim **C** = 0	dim **K** = 0	**H** - collin
Type of **R** - MG	**R** is a reversible mapping for $X \in \boldsymbol{P}_3 - \lambda_\pi$, $(C \in \pi)$			

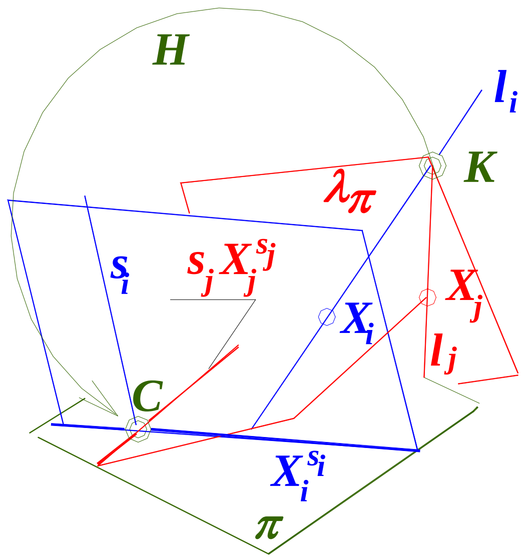
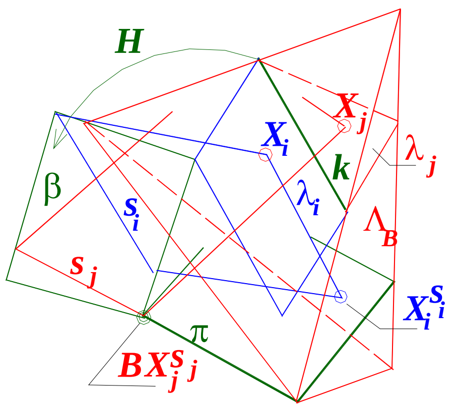
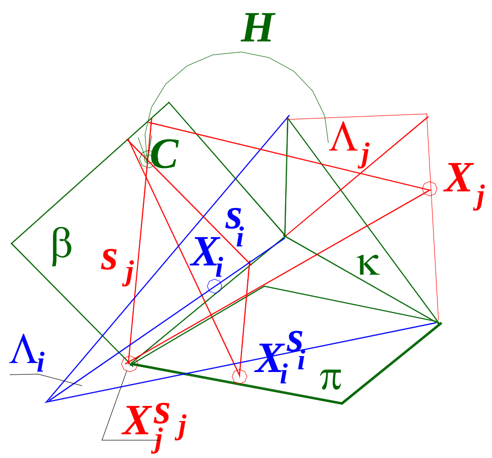
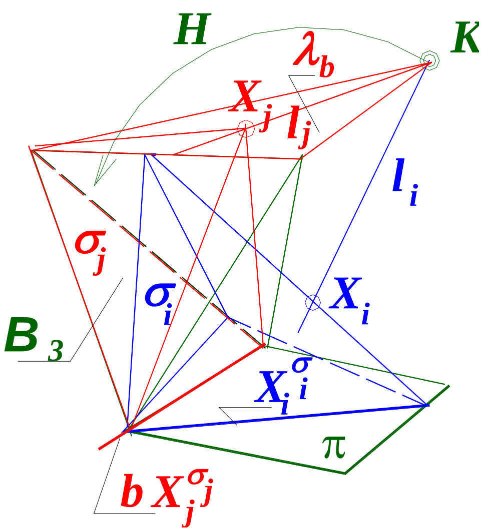
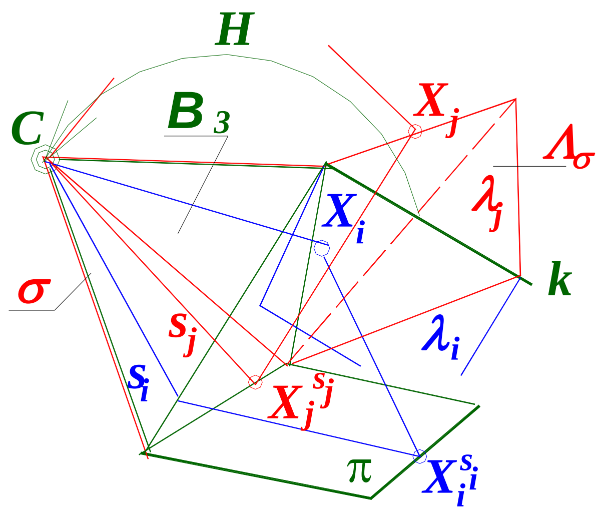
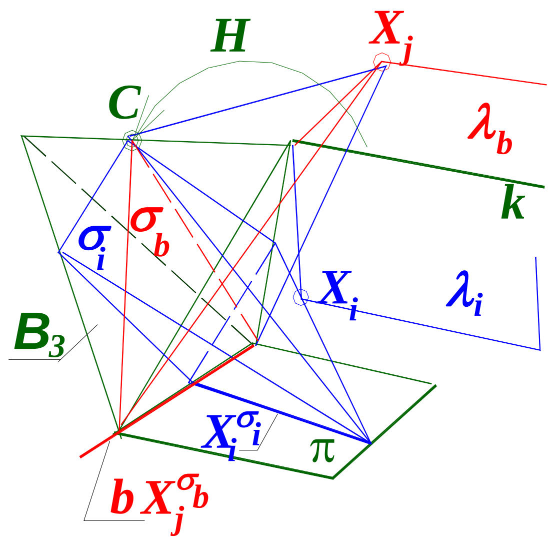


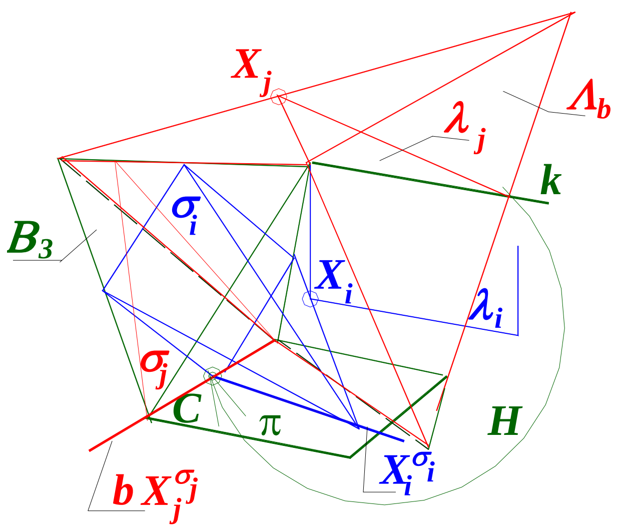
Table 3.:Examples of the structure of the **R** mapping apparatuses and their methods of operation in the $\boldsymbol{P}_4$ space (green lines – graphical symbols of elements of the **R** mapping apparatus, blue lines - graphical symbols of the subspaces applied in projecting of $X_i \in [X_i]$, red lines - graphical symbols of the subspaces applied in projecting of $X_j \in [X_j]$)

<table>
<tr><td colspan="10">Mapped space – $\boldsymbol{P}_4$</td></tr>
<tr><td>$\boldsymbol{P}_4/1$</td><td>dim B = 2</td><td>dim C = - 1</td><td>dim K= 1</td><td>H – correl</td><td>$\boldsymbol{P}_4/2$</td><td>dim B= 2</td><td>dim C = 0</td><td>dim K = 2</td><td>H –collin or correl</td></tr>
<tr><td colspan="2">Type of R– OMG</td><td colspan="3">R is a reversible mapping for $X \in \boldsymbol{P}_4 - \Lambda_B$</td><td colspan="2">Type of R– OMG</td><td colspan="3">R is a non-reversible mapping</td></tr>
<tr><td colspan="5">
</td><td colspan="5">
</td></tr>
<tr><td>$\boldsymbol{P}_4/3$</td><td>dim B= 3</td><td>dim C = -1</td><td>dim K= 0</td><td>H - correl</td><td>$\boldsymbol{P}_4/4$</td><td>dim B= 3</td><td>dim C = 0</td><td>dim K = 1</td><td>H - collin</td></tr>
<tr><td colspan="2">Type of R- MG</td><td colspan="3">R is a reversible mapping for $X \in \boldsymbol{P}_3 - l_b$</td><td colspan="2">Type of R- OMG</td><td colspan="3">R is a reversible mapping for $X \in \boldsymbol{P}_3 - \Lambda_\sigma$</td></tr>
<tr><td colspan="5">
</td><td colspan="5">
</td></tr>
</table>

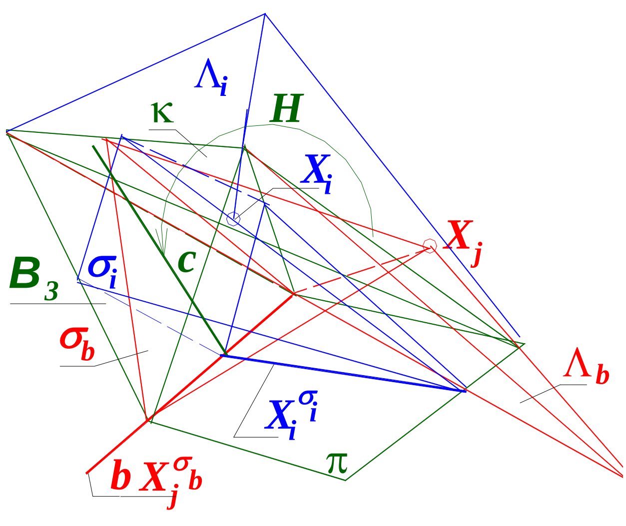
$P_4/5$	dim **B**= 3	dim **C** = 0	dim **K** = 1	**H** – correl
Type of **R– M**G		**R** is a non-reversible mapping		



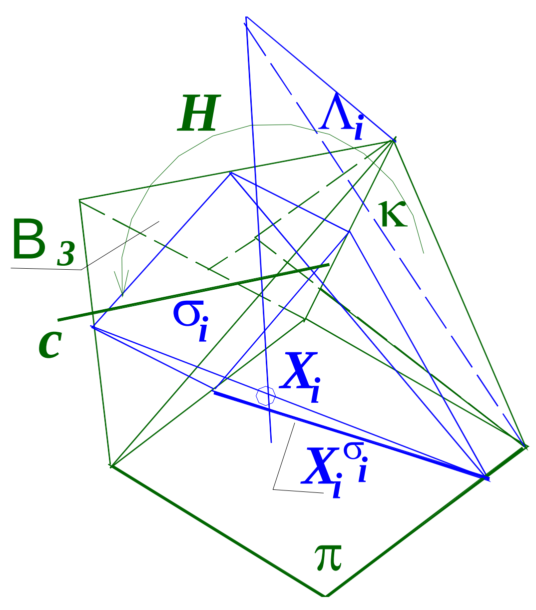
$P_4/6$	dim **B** =3	dim **C** =0	dim **K** =1	**H** –collin or correl
Type of **R– M**G		**R** is a non-reversible mapping ($C \in \pi$)		



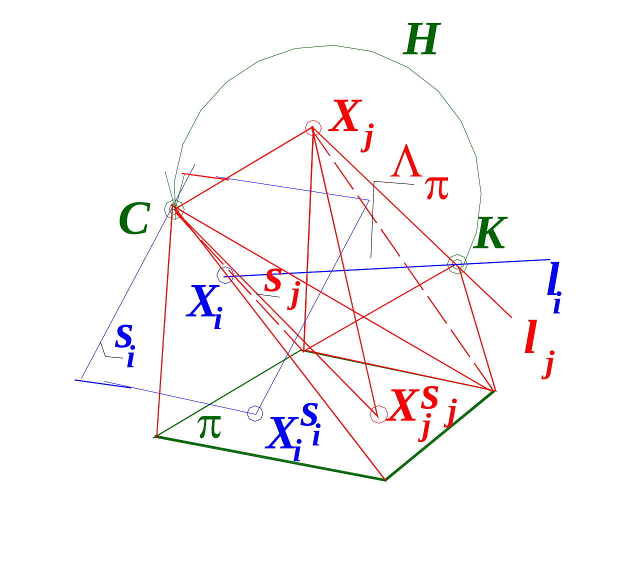
$P_4/7$	dim **B**= 3	dim **C** = 1	dim **K** = 2	**H**-collin or correl
Type of **R- M**G		**R** is a non-reversible mapping ($c \neq c \cap \pi \neq \emptyset$)		



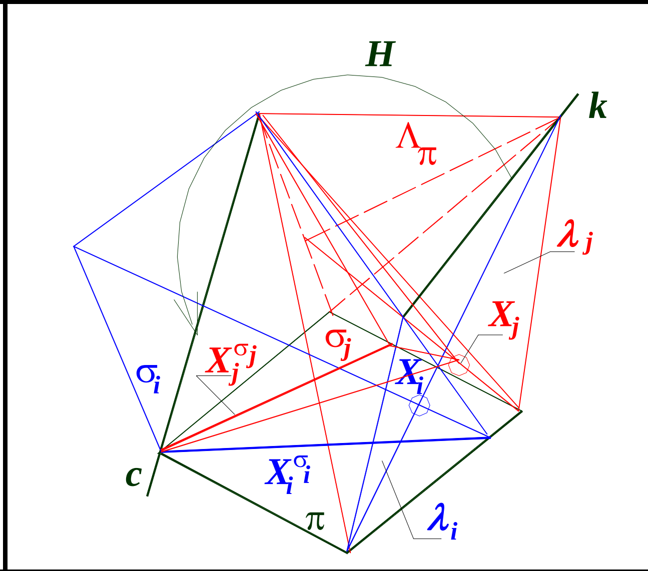
$P_4/8$	dim **B**= 3	dim **C** = 1	dim **K** = 2	**H**-collin or correl
Type of **R- U**G		**R** is a non-reversible mapping ($c \cap \pi = \emptyset$)		



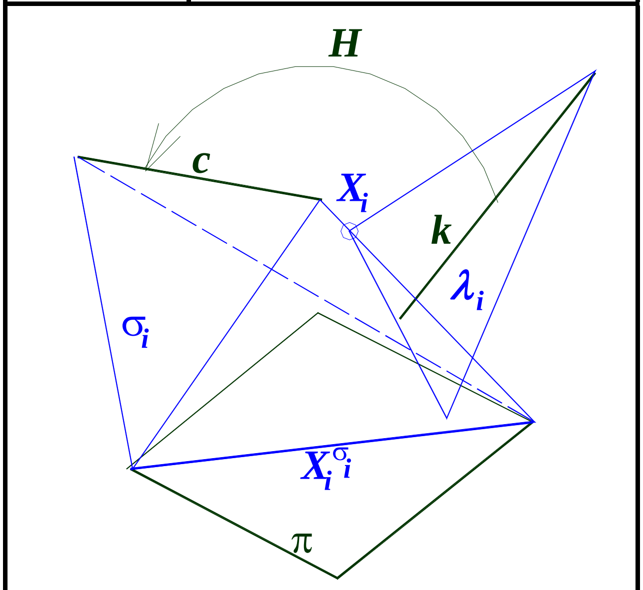
$\boldsymbol{P}_4/9$	dim **B** = 4	dim **C** = 0	dim **K** = 0	**H** – collin
Type of **R– OMG**		**R** is a reversible mapping $X \in \boldsymbol{P}_4 - \Lambda_\pi$		



$\boldsymbol{P}_4/10$	dim **B**=4	dim **C** =1	dim **K** =1	**H** – collin
Type of **R– GMG**		**R** is a non-reversible mapping ($c \neq c \cap \pi \neq \emptyset$)		



$\boldsymbol{P}_4/11$	dim **B**= 4	dim **C** = 1	dim **K**= 1	**H** - collin
Type of **R- UG**		**R** is a non-reversible mapping ($c \cap \pi = \emptyset$)		

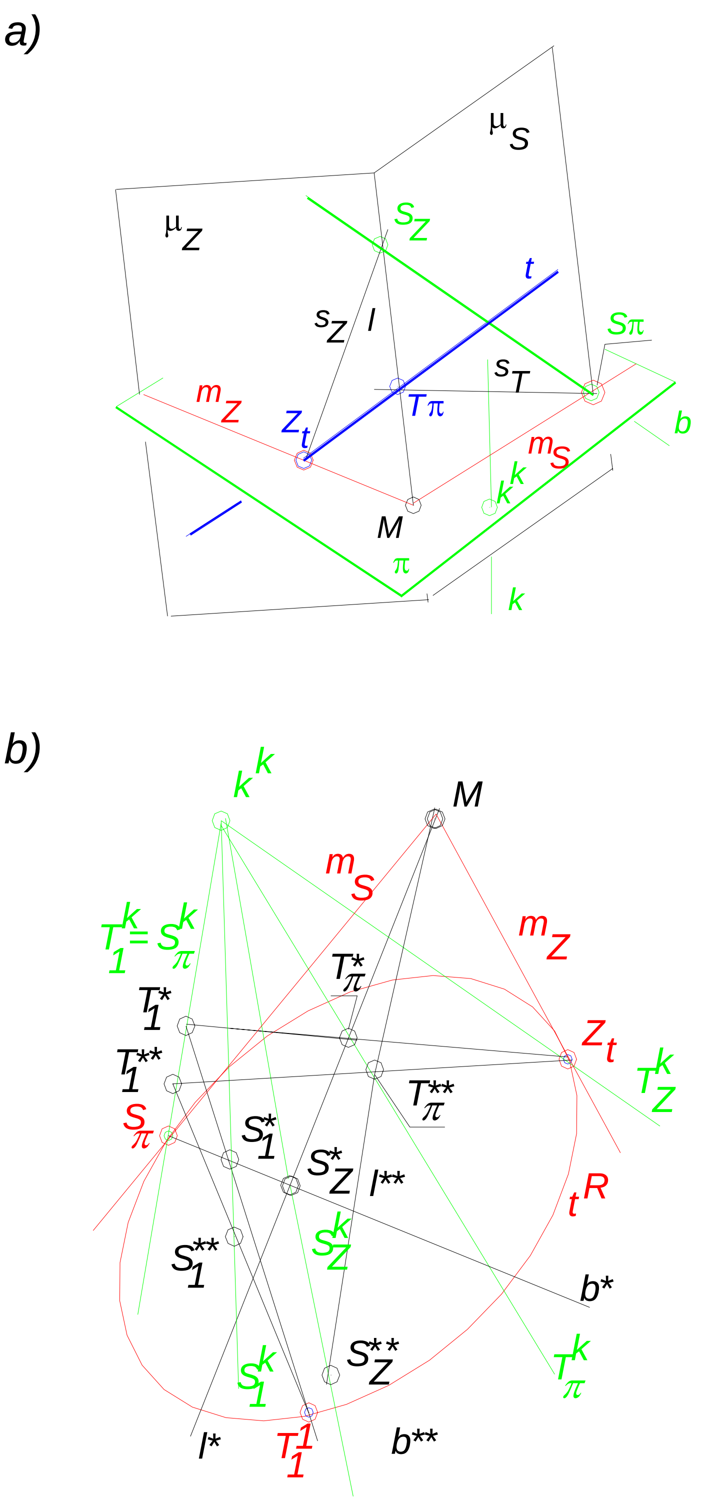


To sum up, the images of a straight line, a plane and a hyper plane, received in the **R** mappings described in Table 2, case $P_3/1$ and case $P_3/3$, also in Table 2, case $P_4/1$, are shown respectively on Figures 2, 3 and 4. The selection of images of the subspaces signals a variety of possible structures of the images. Namely:

- the image of a *t* straight line, which is drawn on the Figure 2b, is a conic defined by the common points of homologous elements of the $\langle S_\pi, \pi\rangle$ and $\langle Z_t, \pi\rangle$ collineation bundles,
- the image of a τ plane, which is drawn on the Figure 3b, is a core conic of the $\langle\emptyset,\pi\rangle_\beta$ and $\langle\emptyset,\pi\rangle_\tau$ correlation bundles,

- the image of a T hyper plane, which is drawn on the Figure 4b, is the conic tangent to the all straight lines $K^{\mathbf{R}}$ and $[t_i^{\mathbf{R}}]$.

The above mentioned construction solutions prove that the reversible **R** mappings are effective methods of graphic representations for multidimensional projective spaces.



COMMENTS TO THE SOLUTION

$\{\langle\emptyset,b\rangle,\langle k,P_3\rangle,\mathbf{H}_{\mathrm{CL}},\pi\}$ – the apparatus of **R** projection,

$t(Z_t,T^{\pi})\,\tau$ - the mapped straight line,

Ω_t – the projection formation of the t straight line – the warped quadric defined by the collineation bundles $\langle\emptyset,b\rangle$ and $\langle\emptyset,t\rangle$, b, s_T, t, $s_Z \subset \Omega_t$,

$\mu_S(b,s_T) = \mu_S(b,T_\pi)$ – the tangent plane to Ω_t in the $S_\pi = b\cap\pi$ point,

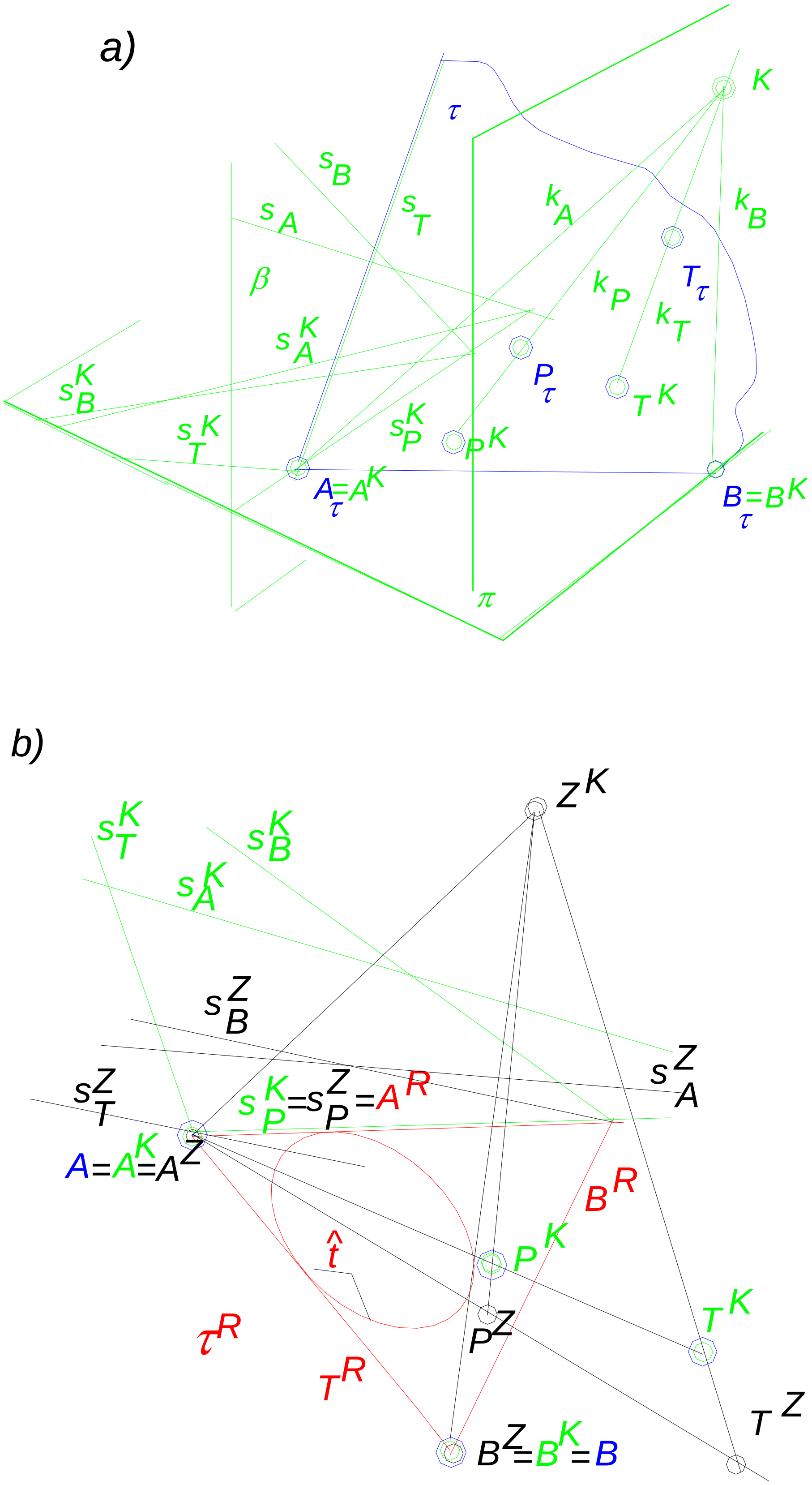
$\mu_Z(t,s_Z) = \mu_Z(t,S_Z)$ – the tangent plane to Ω_t in the $Z_t = t\cap\pi$ point,

$\Omega_t \cap \pi = t^{\mathbf{R}}$ – the image of the t straight line – a conic,

$m_Z = \mu_Z \cap \pi$ - the tangent to $t^{\mathbf{R}}$ in the Z_t point,

$m_S = \mu_S \cap \pi$ - the tangent to $t^{\mathbf{R}}$ in the S_π point.

Figure 2: The structure of the image of a t straight line in the **R** projection defined by $\{\langle\emptyset,b\rangle, \langle k, \boldsymbol{P}_3\rangle, \mathbf{H}_{\mathrm{CL}},\pi\}$ apparatus: a) the spatial situation, b) the construction of the image

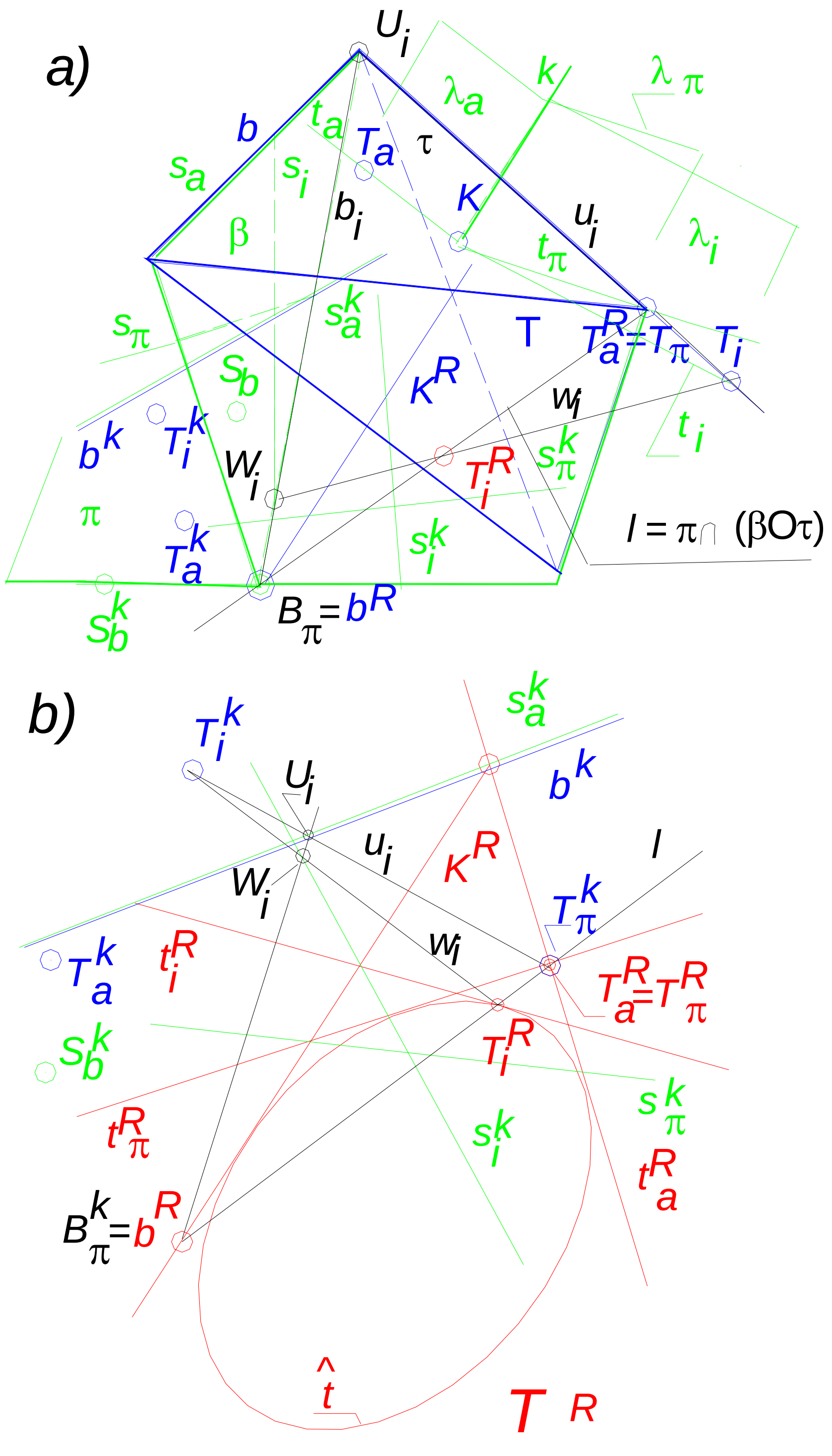


COMMENTS TO THE FIGURE

$\{\langle\varnothing,\beta\rangle,\langle K,P_3\rangle,\mathbf{H}_{CR},\pi\}$ – the apparatus of the **R** projection,
τ - the mapped plane,
$\langle\varnothing,\tau\rangle = \tau\cap\langle K,P_3\rangle$,
$\mathbf{H}_{CR}(\langle\varnothing,\tau\rangle) = \langle\varnothing,\beta\rangle$,
Ω - the enveloped quadric defined by the $\langle\varnothing,\tau\rangle$ and $\langle\varnothing,\beta\rangle$ correlation bundles,
Z – the pole of the π projection plane in relation to the Ω quadric,
$Z\ O\ \langle\varnothing,\tau\rangle = \langle Z,P_3\rangle_\tau$,
$Z\ O\ \langle\varnothing,\beta\rangle = \langle Z,P_3\rangle_\beta$,
$\langle Z,P_3\rangle_\tau \ \overline{\wedge}\ \langle Z,P_3\rangle_\beta$,
$\langle Z,P_3\rangle_\tau\cap\pi = \langle\varnothing,\pi\rangle_\tau$,
$\langle Z,P_3\rangle_\beta\cap\pi = \langle\varnothing,\pi\rangle_\beta$,
$\hat{t}$ - the core conic of the $\langle\varnothing,\pi\rangle_\beta$ and $\langle\varnothing,\pi\rangle_\tau$ correlation bundles – the outline of **R** projection of the τ plane

Figure 3: The structure of an image of a τ plane in the **R** projection defined by the $\{<, \beta>,<K, \mathbf{P}_3>, \mathbf{H}_{CR},\pi\}$ apparatus:

a) the spatial situation, b) the construction of the image



COMMENTS TO THE FIGURE

$\{\langle\varnothing,\beta\rangle,\langle k,P_4\rangle,\mathbf{H}_{CR},\pi\}$ – the apparatus of the **R** projection,

T – the mapped hyper plane,

$b = T \cap \beta$, □ $= b$ O $T_{□}$,

$l = \pi \cap (\beta$ O □$)$,

$\mathbf{H}_{CR}(\langle k,P_4\rangle) = \langle\varnothing,\beta\rangle$,

$K = k \cap T$,

$\langle k,P_4\rangle \cap T = \langle K, T\rangle$,

$\lambda_i \in \langle k,P_4\rangle \Rightarrow \lambda_i \cap T = t_i$,

if $t_i \in \langle K, T\rangle$, then $s_i \in \langle\varnothing,\beta\rangle$ and $s_i = \mathbf{H}_{CR}(t_i)$ is the projection centre for all points of the $\bigcup_i t_i - K$ set,

β is the projection centre for K,

$(s_i$ O $t_i) \cap \pi = t_i^{\mathbf{R}}$,

$t_i^{\mathbf{R}} = T_i^{\mathbf{R}}$ O $(s_i^k \cap K^{\mathrm{R}})$, when

$T_i^{\mathbf{R}} = l \cap w_i(W_i$ O $T_i^{\mathrm{k}})$,

$T^{\mathbf{R}} = K^{\mathbf{R}} \cup [t_i^{\mathbf{R}}]$,

$\hat{\boldsymbol{t}}$ - the conic which is tangent to all straight lines $K^{\mathbf{R}}$ and $t_i^{\mathbf{R}}$ – the outline of **R** projection of T hyper plane

Figure 4: The structure of the image of a T hyper plane in the **R** projection defined by the $\{<ȷ,\beta>,<k, \boldsymbol{P}_4>, \mathbf{H}_{CR}, \pi\}$ apparatus: a) the spatial situation, b) the construction of the image

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[2] Januszewski B.: *Rodzaje podprzestrzeniowych rzutowań o wiązkowo rozproszonych środkach*. Lecture at the International Conference 'Contemporary problems with geometrical modelling', Lvov 2003.

PRZYKŁADY ODWZOROWAŃ GRAFICZNYCH REALIZOWANYCH ZA POMOCĄ RZUTOWAŃ PODPRZESTRZENIOWYCH Z WIĄZKOWO ROZPROSZONYCH ŚRODKÓW

W artykule zawarto analizę przykładów rzutowań podprzestrzeniowych R z wiązkowo rozproszonych środków. Wzięto pod uwagę te spośród rzutowań R, które prowadzą do odwzorowań wykreślnych, w tym wzajemnie jednoznacznych. W rezultacie przeprowadzonych analiz zestawiono przykłady budowy aparatów rzutowań typu R, dla trój- i czterowymiarowych przestrzeni rzutowych.

W ostatniej części artykułu pokazano przykłady konstrukcji obrazów prostej, płaszczyzny i hiperpłaszczyzny uzyskane za pomocą wyróżnionych rodzajów rzutowań typu R. Przykłady te dowodzą, że analizowane rzutowania dają możliwość efektywnego zapisu figur zawartych w trój- lub czterowymiarowej przestrzeni rzutowej.

Reviewer: Prof. Bogusław GROCHOWSKI, DSc

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