Impact of roguing and insecticide spraying on mosaic disease in *Jatropha curcas*[*]

by

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Abstract: *Jatropha curcas* plant is greatly impaired by mosaic disease, caused by the viruses (*Begomovirus*), transmitted by whiteflies, which act as the vector. Roguing (i.e. removal of infected plant) and spraying of insecticides are common methods, employed in order to get rid of the disease. In this article, a mathematical model has been developed to study the mosaic disease dynamics while considering preventive measures of roguing and insecticide spraying. Sufficient conditions for the stability of equilibrium points of the system are among the results obtained through qualitative analysis. We obtain the basic reproduction number R_0 and show that the disease free system is stable for $R_0 < 1$ and unstable for $R_0 > 1$. The region of stability of equilibrium points in different parameter spaces have also been analysed. Hopf bifurcation at the endemic steady state has been studied subsequently, as well. Finally, by formulating an optimal control problem, optimal application of roguing and spraying techniques has been determined, keeping in mind the cost effective control of the mosaic disease. Pontryagin minimum principle has been utilized to solve the optimal control problem. Numerical simulations illustrate the validity of the analytical outcomes.

Keywords: *Jatropha curcas*, mosaic disease, roguing, spraying, Hopf bifurcation, optimal control problem (OCP), minimum principle

1. Introduction

A vector borne plant disease, namely, in this case, the mosaic disease, is an important constraint to crop production worldwide, causing serious losses in yield and quality of food (Chakraborty and Newton, 2011). *Jatropha curcas L.* (known as physic or purging nut) is a multipurpose and drought resistant crop and is grown in marginal lands with lesser input. *Jatropha* plants natively occur in tropical and subtropical areas of India, Africa and North America. *Jatropha curcas* is easily affected by mosaic virus *Begomovirus* (Melgarejo et al., 2015; Mulenga et al., 2016). It heavily affects the *Jatropha* plants, causing leaf damage, such as yellowing of leaves and sap drainage and, in particular, the fruits of the plant are also affected. It is responsible, as well, for a reduction in the production of seeds. This plant virus is transmitted by infected vectors - whiteflies *Bemisia tabaci* (Gennadius), (Priya and Selvan, 2006; Basir et al., 2017).

One possible protection method of the *Jatropha curcas* plants from the mosaic virus is to control these vectors - whiteflies. This can be achieved through spraying of insecticides on the plant (Makkouk et al., 2014; Venturino et al., 2016; Roy et al., 2015). Viral disease in a plant population can also be controlled with roguing of infected plants and replanting (replacing the infected palnts with healthy ones), see Basir and Roy (2017), Jeger et al. (2004), Jeger and Chan (1994), Allen (1978), Holt and Chancellor (2006), Thresh et al. (2005).

Roy et al. (2015), as well as Venturino et al. (2016), have studied the dynamics of mosaic disease in *Jatropha curcas* plants using different mathematical models. In both these articles, the authors have considered insecticide as the disease controlling agent. Basir and Roy (2017) have studied the effect of roguing in controlling mosaic disease in *Jatropha* plant. In this present study, consideration of both roguing and spraying as controlling factors as well as investigation of the dynamics of the disease have been carried out. The basic reproduction number, essential for the analysis of the disease dynamics, has been determined as a result of this investigation.

The basic reproduction number, R_0, is defined as the expected number of secondary cases, caused by introduction of one infected individual into the native population. It is a measure of the success of invasion in a population. R_0 is a threshold quantity, where for the value of R_0 higher than 1 ($R_0 > 1$), an outbreak of the infectious agent is possible if the pathogen is introduced; whereas if R_0 is below 1 ($R_0 < 1$), the infection will wither away (Diekmann and Heesterbeek, 2000; Anderson et al., 1990). When an outbreak is possible, then R_0 is also a measure of the risk that an outbreak will actually occur. It determines the fraction of the population that needs to be controlled in order to avoid an outbreak. A method exists in mathematical biology literature to characterize R_0 even in systems with definite complexity: the next-generation matrix method (Diekmann et al., 1990). This approach has the advantage of providing an estimate of R_0 and the elements of the next-generation matrix have a clear biological basis.

It has been previously observed that the *Indian Cassava mosaic virus* can cause the mosaic disease in *Jatropha curcas* plant (Kashina et al., 2013; Ramkat et al., 2011), while the phylogenetic nature of *Cassava* and *Jatropha* are not the same, thus their growth and death rates differ, but the plantation methods are very similar. For this reason, we have estimated some of the parameter values from the available literature (Venturino et al., 2016; Halt et al., 1997) for numerical simulations.

This article is organised as follows: in the next section, a mathematical model is formulated for mosaic disease in *Jatropha curcas* plant with consideration of roguing and spraying. Existence and stability of equilibria and Hopf bifurcation are studied in Section 3. Optimal roguing and spraying is obtained for cost effective control of mosaic disease by formulating and solving an optimal control problem in Section 4. Finally, in Section 5, we provide discussion of our results with a conclusion.

2. The mathematical model

Jatropha curcas mosaic disease is caused by mosaic virus. The virus is carried through the whitefly vector. Therefore, we only consider the vector population in the study of the mosaic disease dynamics. Let $y(t)$ and $v(t)$ denote the infected plant and infected whitefly populations, respectively, and their corresponding healthy counterparts are denoted by $x(t)$ and $u(t)$. Logistic growth is assumed for healthy plant population due to finite area of plantation with r as the growth rate of plant density and k as the maximum plant density. Let λ be the contact rate from an infected vector with respect to a susceptible plant, while β is the rate of disease transmission from an infected plant to a healthy whitefly, and m is the sum of the natural and virus-related mortalities of the infected plants. Further, a is the maximum vector abundance on a plant and b is the healthy vector growth rate, while μ is the mortality of vector population.

Roguing or removal of infected plant and spraying of insecticides are considered as the mosaic disease control measures. It is assumed that cutting is proportional to the density of the infected plant, $y(t)$, at the maximum rate h. On the other hand, insecticide affects both the infected and uninfected vectors by increasing their death rate by a constant, c.

From the above assumption the following model is obtained:

$$
\begin{aligned}
\frac{dx}{dt} &= rx\left(1 - \frac{x+y}{k}\right) - \lambda xv, \\
\frac{dy}{dt} &= \lambda xv - my - hy, \\
\frac{du}{dt} &= b(u+v)\left(1 - \frac{u+v}{a(x+y)}\right) - \beta uy - cu, \\
\frac{dv}{dt} &= \beta uy - \mu v - cv.
\end{aligned}
\tag{1}
$$

with the initial conditions:

$$x(0) > 0, \quad y(0) > 0, \quad u(0) > 0, \quad v(0) > 0. \tag{2}$$

Remark: We do not consider the case when $x + y = 0$, otherwise this study would be meaningless. Also, c can be considered to represent the rate of spraying on the plantation.

The region of attraction is given by the set

$$\mathcal{D} \;\; = \;\; \{(x, y, u, v) \in \mathbb{R}^4_+ : 0 \le x + y \le M, 0 \le u + v \le aM\}$$

where $M = max \{J(0), k\}$, $J(t)$ being the total plant volume at any time t, and the region is positively invariant, attracting all solutions initiating inside the interior of the positive octant and all the solutions are bounded therein.

Table 1. Values of the parameters used in the numerical simulations (Venturino et al., 2016; Holt et al., 1997)

Parameters	Definition	Values
r	growth rate of plant	0.1 day^{-1}
k	density of healthy plants	1 m^{-2}
λ	plant infection rate	0.002 vector^{-1} day^{-1}
m	infected plant death rate	0.03 day^{-1}
b	vector growth rate	0.8 day^{-1}
β	vectors infection rate	0.12 plant^{-1}day^{-1}
a	maximum vector abundance	100 plant^{-1}
μ	vector mortality rate	0.12 day^{-1}

3. Stability of the equilibria

In this section, we have analyzed the dynamics of the system with constant control.

3.1. Equilibria and stability

There are only three possible equilibria for system (2):

    i) the infected plant-vector-free equilibrium $E_1 = (k, 0, 0, 0)$,

    ii) the disease-free equilibrium $E_2 = (k, 0, ak, 0)$,

    iii) and the endemic equilibrium $E^* = (x^*, y^*, u^*, v^*)$,

where

$$x^* = \frac{-A + \sqrt{A^2 - 4B}}{2} \quad y^* = \frac{rx^*(k - x^*)}{ak + rx^*}$$

$$u^* = \frac{mc}{\beta\lambda x^*}, \quad v^* = \frac{mr(k - x^*)}{\lambda(mk + rx^*)},$$

$$\text{with } A = \frac{-mbr(\lambda\beta + c)}{k\lambda\beta a(m + r)(b - c) + mb\beta r} \text{ and } B = \frac{Amck}{rk\beta + rc}. \tag{3}$$

The Jacobian at any point (x, y, u, v) is:

$$M = [m_{ij}] =$$

$$\begin{bmatrix} m_{11} & -\frac{rx}{k} & 0 & -\lambda x \\ \lambda v & -m - h & 0 & \lambda x \\ \frac{b(u+v)^2}{a(x+y)^2} & -\beta u + \frac{b(u+v)^2}{a(x+y)^2} & b - \beta y - \frac{2b(u+v)}{a(x+y)} - c & b - \frac{2b(u+v)}{a(x+y)} \\ 0 & \beta u & \beta y & -\mu - c \end{bmatrix},$$

where $m_{11} = r[1 - \frac{(2x+y)}{k}] - \lambda v$.

LEMMA 1 *The infected plant-vector-free system is unstable.*

PROOF: Eigenvalues of the Jacobian matrix at E_1 are r, $-m - h$, $b - c$, $-\mu - c$. Thus, the equilibrium E_1 is unstable.
We have also the following lemma.

LEMMA 2 *The system (2) is stable at the disease-free equilibrium, E_2, if $R_0 < 1$ and becomes unstable if $R_0 > 1$.*

PROOF: At E_2, M consists of two eigenvalues, which are easily obtained as $-b$ and $-r$ and the remaining characteristic equation is:

$$\xi^2 - (m + h + \mu + c)\xi + (\mu + c)(m + h) - ak^2\beta\lambda = 0, \tag{4}$$

and its roots are negative or have negative real parts if $m + h + \mu + c > 0$, which is obvious, and $(\mu + c)(m + h) - ak^2\beta\lambda > 0$.

Thus, E_2 is stable if

$$(\mu + c)(m + h) - ak^2\beta\lambda > 0$$

and unstable if
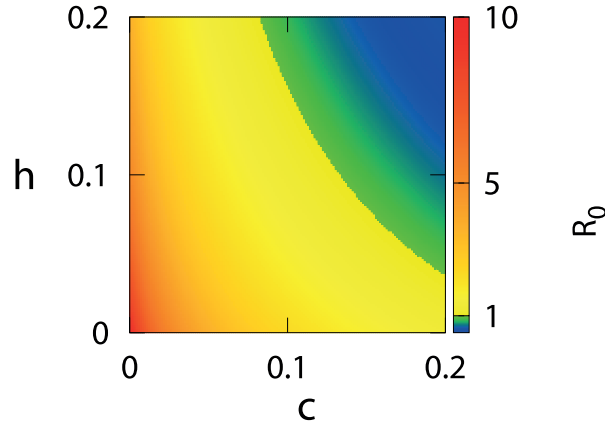
$$(\mu + c)(m + h) - ak^2\beta\lambda < 0.$$

Figure 1. Region of stability of the disease free steady state E_2 in $(c - h)$ space. Colour bar denotes the values of R_0 (for interpretation of the references to colours in the figure, the Reader is asked to consult the web version of the paper). Other parameter values are taken as in Table 1. For $R_0 < 1$, E_2 is stable

From the above discussion we can determine the basic reproduction number, R_0, as follows:

$$R_0 = \frac{ak^2\beta\lambda}{(\mu + c)(m + h)}. \tag{5}$$

3.2. Stability of endemic equilibria E^*

At E^* the characteristic equation is given by:

$$\rho^4 + \sigma_1\rho^3 + \sigma_2\rho^2 + \sigma_3\rho + \sigma_4 = 0, \tag{6}$$

where $\sigma_i, i = 1, 2, 3, 4$ are given by:

$$\begin{aligned}
\sigma_1 &= -(m_{11} + m_{22} + m_{33} + m_{44}), \\
\sigma_2 &= m_{44}(m_{33} + m_{22}) + m_{11}m_{44} - m_{34}m_{43} - m_{24}m_{42} \\
&\quad + m_{22}m_{33} + m_{11}(m_{33} + m_{22}) - m_{12}m_{21}, \\
\sigma_3 &= -m_{33}m_{44}(m_{22} + m_{11}) - m_{44}(m_{11}m_{22} - m_{12}m_{21}) + m_{34}m_{43}(m_{22} + m_{11}) \\
&\quad - m_{24}m_{32}m_{43} - m_{14}m_{31}m_{43} + m_{24}m_{42}(m_{33} + m_{11}) \\
&\quad - m_{14}m_{21}m_{42} - m_{33}(m_{11}m_{22} - m_{12}m_{21}), \\
\sigma_4 &= m_{33}m_{44}(m_{11}m_{22} - m_{12}m_{21}) - m_{34}m_{43}(m_{11}m_{22} - m_{12}m_{21}) \\
&\quad + m_{32}m_{43}(m_{11}m_{24} - m_{14}m_{21}) - m_{31}m_{43}(m_{12}m_{24} \\
&\quad - m_{14}m_{22}) - m_{33}m_{42}(m_{11}m_{24} - m_{14}m_{21}).
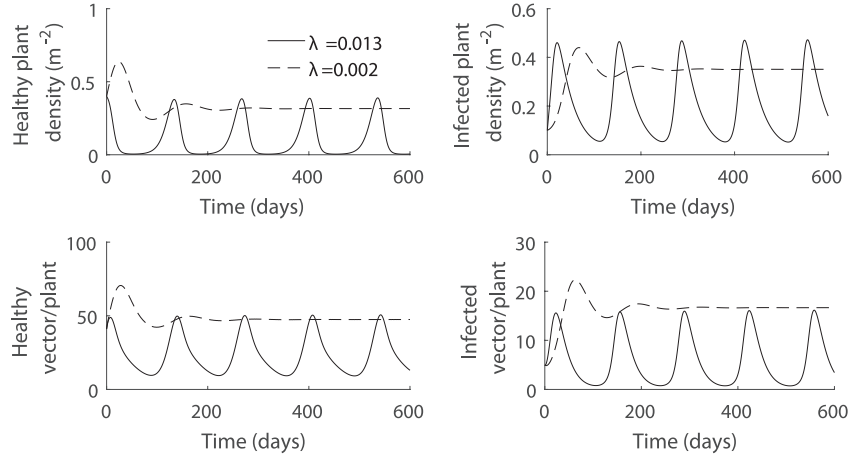\end{aligned} \tag{7}$$

Figure 2. Numerical solution of the system (2) for $\lambda = 0.002$ (dotted line) and $\lambda = 0.013$ (solid line), while the other parameters are same as in Table 1 (without control i.e. $h = 0$, $c = 0$). System is stable at lower infection

and

$$
\begin{aligned}
m_{14} &= -\lambda x^*, \quad m_{21} = \lambda v^*, \quad m_{22} = -(m + h), \\
m_{12} &= -\frac{r x^*}{k}, \quad m_{24} = \lambda x^*, \quad m_{31} = \frac{b(u^* + v^*)^2}{a(x^* + y^*)^2}, \\
m_{32} &= -\beta u^* + \frac{b(u^* + v^*)^2}{a(x^* + y^*)^2}, \\
m_{33} &= b - \beta y^* - \frac{2b(u^* + v^*)}{a(x^* + y^*)}, \quad m_{34} = b - \frac{2b(u^* + v^*)}{a(x^* + y^*)} - c, \quad m_{42} = \beta u^*, \\
m_{11} &= -\frac{x^*}{k}, \quad m_{44} = -(\mu + c), \quad m_{43} = \beta y^*.
\end{aligned}
$$

In this context, according to the Routh-Hurwitz criterion, the E^* is stable if the following conditions are satisfied:

$$
\begin{aligned}
&\sigma_1 > 0, \sigma_4 > 0, \\
&\sigma_1 \sigma_2 - \sigma_3 > 0, \text{ and} \\
&\sigma_1 \sigma_2 \sigma_3 - \sigma_3^2 - \sigma_4 \sigma_1^2 > 0.
\end{aligned} \tag{8}
$$

3.3. Hopf bifurcation analysis

For the possible occurrence of Hopf bifurcations in the system (1) at the endemic equilibrium E^*, we consider $\alpha \in \mathbb{R}$ as the general bifurcation parameter.
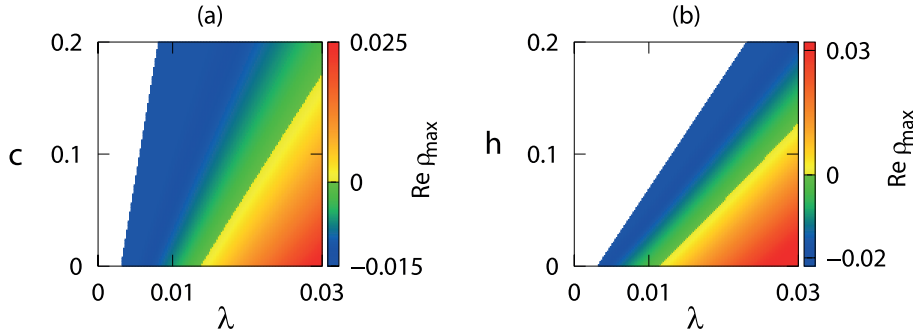
Figure 3. Region of stability of the endemic steady state E^* in (a) $(\lambda - c)$ space with $h = 0$, (b) $(\lambda - h)$ space with $c = 0$. Colour bar denotes $\max[Re(\rho)]$, ρ is the characteristic root of equation (7) (for interpretation of the references to colours in the figure, the Reader is asked to consult the web version of the paper). Other parameter values are the same as in Table 1. In the white region, E^* does not exist

Using the last condition of (8), we define $\phi : (0, \infty) \to \mathbb{R}$ as a continuously differentiable function of α:

$$\phi(\alpha) := \sigma_1(\alpha)\sigma_2(\alpha)\sigma_3(\alpha) - \sigma_3^2(\alpha) - \sigma_4(\alpha)\sigma_1^2(\alpha).$$

Hopf bifurcation will occur if the characteristic equation (6) has a pair of complex eigenvalues $\rho(\alpha^*), \bar{\rho}(\alpha^*)$ such that

$$\mathrm{Re}\rho(\alpha^*) = 0, \quad \mathrm{Im}\rho(\alpha^*) = \omega_0 > 0, \tag{9}$$

and the transversality condition

$$\left. \frac{dRe\rho(\alpha)}{d\alpha} \right|_{\alpha^*} > 0$$

is satisfied. Furthermore, all other eigenvalues must have negative real parts. Thus, we have the following theorem:

Theorem 1 *The endemic equilibrium E^* enters into Hopf bifurcation at $\alpha = \alpha^* \in (0, \infty)$ if and only if $\phi(\alpha^*) = 0$ and*

$$\sigma_1^3 \sigma_2' \sigma_3 (\sigma_1 - 3\sigma_3) > 2(\sigma_2 \sigma_1^2 - 2\sigma_3^2)(\sigma_3' \sigma_1^2 - \sigma_1' \sigma_3^2) \tag{10}$$

and all other eigenvalues have negative real parts, where $\rho(\alpha)$ is purely imaginary at $\alpha = \alpha^*$.

Proof: Using conditions (10), the characteristic equation (6) can be equivalently rewritten in the form

$$\left(\rho^2 + \frac{\sigma_3}{\sigma_1} \right) \left(\rho^2 + \sigma_1 \rho + \frac{\sigma_1 \sigma_4}{\sigma_3} \right) = 0. \tag{11}$$
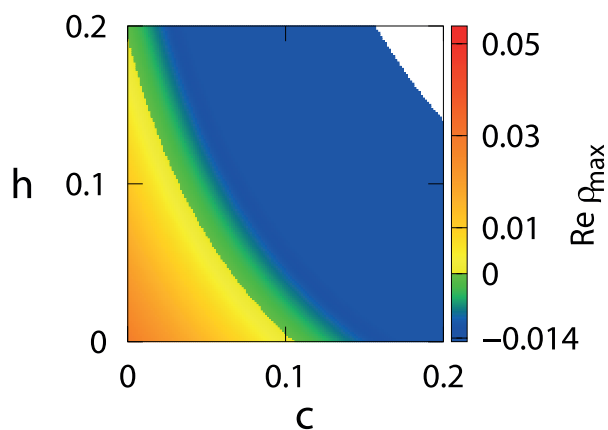
Figure 4. Region of stability of the endemic steady state E^* in $(c - h)$ space. Colour bar denotes $\max[Re(\rho)]$, ρ is the characteristic root of equation (7) (for interpretation of the references to colours in the figure, the Reader is asked to consult the web version of the paper). Other parameter values are taken as in Table 1. In the white region, E^* does not exist
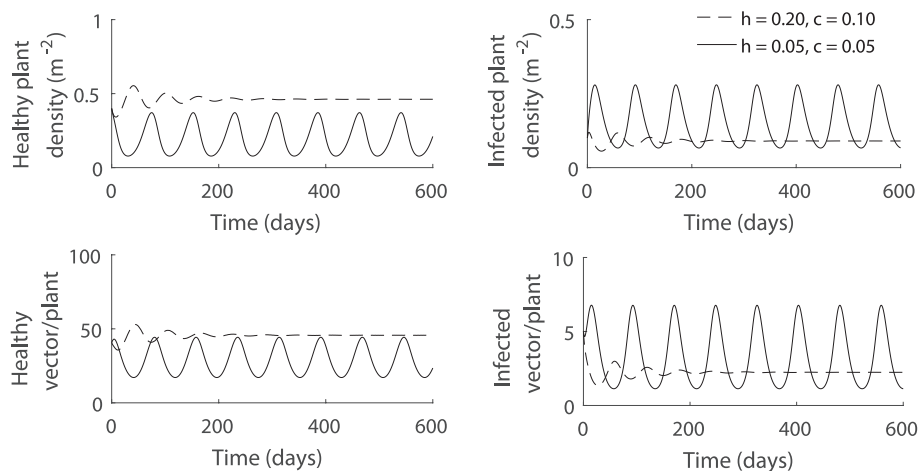

Figure 5. Numerical solution of the system (2) for the values of parameters as in Table 1 (without control, $\lambda = 0.02$)

Two roots of this equation are given by

$$\rho_{1,2} = \pm i\omega_0, \quad \omega_0 = \sqrt{\frac{\sigma_3}{\sigma_1}},$$

while the other two roots, ρ_3 and ρ_4, satisfy the equation

$$\rho^2 + \sigma_1\rho + \frac{\sigma_1\sigma_4}{\sigma_3} = 0,$$

and from the Routh-Hurwitz criterion, they both have a negative real part.

To verify the transversality condition, we first note that $\Phi(\alpha^*)$ is a continuous function of its argument, and hence there exists an open interval

$$\alpha \in (\alpha^* - \epsilon, \alpha^* + \epsilon)$$

where ρ_1 and ρ_2 are complex conjugate roots of the characteristic equation, which can be written in the general form as

$$\rho_{1,2}(\alpha) \quad = \quad \zeta(\alpha) \pm i\nu(\alpha),$$

with $\rho_{1,2}(\alpha^*) = \pm i\omega_0$.

Substituting $\rho_j(\alpha) = \zeta(\alpha) \pm i\nu(\alpha)$ into the characteristic equation (6), differentiating with respect to α and separating real and imaginary parts gives

$$P(\alpha)\zeta'(\alpha) - Q(\alpha)\nu'(\alpha) + R(\alpha) = 0, \qquad\qquad (12)$$
$$Q(\alpha)\zeta'(\alpha) + P(\alpha)\nu'(\alpha) + S(\alpha) = 0,$$

where

$$
\begin{aligned}
P(\alpha) &= 4\zeta^3 - 12\zeta\nu^2 + 3\sigma_1(\zeta^2 - \nu^2) + 2\sigma_2\zeta + \sigma_3, \\
Q(\alpha) &= 12\zeta^2\nu + 6\sigma_1\zeta\nu - 4\zeta^3 + 2\sigma_2\zeta, \\
R(\alpha) &= \sigma_1\zeta^3 - 3\sigma_1'\zeta\nu^2 + \sigma_2'(\zeta^2 - \nu^2) + \sigma_3'\zeta, \\
S(\alpha) &= 3\sigma_1'\zeta^2\nu - \sigma_1'\nu^3 + 2\sigma_2'\zeta\nu + \sigma_3'\zeta.
\end{aligned}
$$

Solving the system (12) for $\zeta'(\alpha^*)$ and using the condition (10) yields

$$\left[\frac{d\mathrm{Re}[\rho_j(\alpha)]}{d\alpha}\right]_{\alpha=\alpha^*} = \zeta'(\alpha^*) = -\frac{Q(\alpha^*)S(\alpha^*) + P(\alpha^*)R(\alpha^*)}{P^2(\alpha^*) + Q^2(\alpha^*)}$$

$$= \frac{\sigma_1^3\sigma_2'\sigma_3(\sigma_1 - 3\sigma_3) - 2(\sigma_2\sigma_1^2 - 2\sigma_3^2)(\sigma_3'\sigma_1^2 - \sigma_1'\sigma_3^2)}{\sigma_1^4(\sigma_1 - 3\sigma_3)^2 + 4(\sigma_2\sigma_1^2 - 2\sigma_3^2)^2} \neq 0.$$

Thus, the transversality condition holds, and, consequently, a Hopf bifurcation occurs at $\alpha = \alpha^*$.

Based on the analytical outcomes above, we perform some numerical simulations. We take $x(0) = 0.4$, $y(0) = 0.1$, $u(0) = 10$, $v(0) = 5$ as initial conditions.

Region of stability of the disease free equilibrium, E_2, is shown in Fig. 1, in $c - h$ space. For $R_0 < 1$, the disease free steady state, E_2, is stable. At $R_0 > 1$, this state loses its stability and the system becomes endemically infected and

the endemically infected steady state can be either stable, or unstable. The time series solution of system (2), without roguing and spraying, is shown in Fig. 2, with parameter values as in Table 1. Here, $R_0 > 1$ and the endemic steady state E^* exists and is stable for lower value of infection rate ($\lambda = 0.002$) and unstable for higher value of infection rate. Periodic solution is observed for ($\lambda = 0.013$).

Figure 3 shows that both roguing (h) and spraying (c) can affect the system, especially with regard to stability of E^*, for $\lambda = 0.02$ and other parameters from Table 1. Colour code represents the maximum value of the real parts of roots of the characteristic equation (for colour code interpretation a Reader is referred to the web version of the article). In yellow region the value is positive i.e. the system is unstable and in green area the value is negative, meaning that the system is stable. Hopf bifurcating periodic solution can be observed on the boundary of the two regions.

Figures 1-3 show that λ, c and h are the most important parameters. Stability switches of endemic equilibrium, E^*, depend on these parameters. When one considers the effects of disease transmission rate, λ, as shown in Fig. 3, for lower values of h and c, the endemic steady state is unstable. On the other hand, for higher values of c or h, E^* has a stabilizing role. As c or h increases, the critical value of λ, at which the Hopf bifurcation occurs, is increasing, but this effect reverses starting with some value of c or h. Periodic solution is observed for $h = 0.05$ and $c = 0.05$; the system is asymptotically stable for $h = 0.2$ and $c = 0.1$ when taking $\lambda = 0.02$ (see Fig. 5).

4. The Optimal Control Problem (OCP)

In this section, an optimal control problem is formulated aimed to minimize the costs involved in insecticide spraying and roguing. It is assumed that all the vectors of a particular region fall possibly under the control of spraying of insecticide. We reformulate the model system (2) introducing the control $0 \leq \gamma_i(t) \leq 1$, $i = 1, 2$ as follows:

$$
\begin{aligned}
\frac{dx}{dt} &= rx\left(1 - \frac{x+y}{k}\right) - \lambda xv, \quad (13) \\
\frac{dy}{dt} &= \lambda xv - my - \gamma_1 hy, \\
\frac{du}{dt} &= b(u+v)\left(1 - \frac{u+v}{a(x+y)}\right) - \beta uy - \gamma_2 cu, \\
\frac{dv}{dt} &= \beta uy - \mu v - \gamma_2 cv.
\end{aligned}
$$

with initial conditions: $x(0) > 0, y(0) > 0, u(0) > 0, v(0) > 0$.

Control parameter γ_1 corresponds to the control of roguing and γ_2 is for the control on spraying. For $\gamma_i \approx 0, i = 1, 2$, there is no control input on infected

plant and vectors, while $\gamma_i \approx 1, i = 1, 2$, implies the maximum use of control.

The cost function is taken in a quadratic form to ensure the existence of both optimal spraying and roguing, as follows:

$$J_c(\gamma(t)) = \int_0^{t_f} [Py + R\gamma_1(t)^2 + S\gamma_2(t)^2]dt. \tag{14}$$

Here, P, R, S are positive constants. The objective functional is taken in such a way that we can account for the costs of spraying and roguing, as expressed by the last two terms inside the integral. Here, the objective is to minimize the cost by finding a suitable pair $\gamma_i(t)$, $i = 1, 2$. In this way, the costs of insecticide spraying and roguing can be accounted for in designing the optimal protection control. Also, the damage of the crop due to infected plants, whose presence needs to be minimized, has been expressed by the first term.

Hence, the objective is to find the optimal control pair $\gamma^*(t) = (\gamma_1^*(t), \gamma_2^*(t))$ such that

$$J_c(\gamma_1^*, \gamma_2^*) = min \ (J(\gamma_1, \gamma_2) : (\gamma_1, \gamma_2) \in U) \ where \ U = U_1 \times U_2,$$

$$U_1 = (\gamma_1(t) : \ \gamma_1 \ is \ measurable \ and \ 0 \le \gamma_1 \le 1, \ t \in [0, t_f]) \ and$$
$$U_2 = (\gamma_2(t) : \ \gamma_2 \ is \ measurable \ and \ 0 \le \gamma_2 \le 1, \ t \in [0, t_f]).$$

For this, Pontryagin Minimum Principle (Fleming and Rachel, 1975) has been used to find the optimal control pair $(\gamma_1^*(t), \ \gamma_1^*(t))$. To solve the optimal control problem, we take the Hamiltonian as:

$$\begin{aligned} H \ &= \ Py^2 + R\gamma_1(t)^2 + S\gamma_2(t)^2 + \xi_1 \left[rx \left(1 - \frac{x+y}{k} \right) - \lambda xv \right] \\ &+ \xi_2 [\lambda xv - my - \gamma_1 hy] + \xi_3 \left[b(u+v)[1 - \frac{u+v}{a(x+y)}] \right. \\ &\left. - \beta uy - \gamma_2 cu \right] + \xi_4 [\beta yu - \mu v - \gamma_2 cv], \end{aligned} \tag{15}$$

where the ξ_i, $i = 1, \ldots, 4$ are adjoint variables.

By applying the Pontryagin Minimum Principle for the existence of the optimal control, we obtain the following theorem:

THEOREM 2 *If the given optimal control pair $\gamma_i^*(t)$, i=1,2, and the solution (x^*, y^*, u^*, v^*) of the corresponding system (8) minimize $J(\gamma)$ over U, then there exist the adjoint variables ξ_1, ξ_2, ξ_3, ξ_4, which satisfy the following equations:*

$$\frac{d\xi_1}{dt} = -\xi_1 r\left(1 - \frac{2x+y}{k}\right) + \xi_1\lambda v - \xi_2\lambda v - \xi_3\frac{b(u+v)^2}{a(x+y)^2}, \tag{16}$$

$$\frac{d\xi_2}{dt} = -2Py + \frac{\xi_1 rx}{k} + \xi_2(m+\gamma_1 h) - \xi_3\frac{b(u+v)^2}{a(x+y)^2} - \xi_4\beta u,$$

$$\frac{d\xi_3}{dt} = \xi_3 b\left(1 - \frac{2(u+v)}{a(x+y)}\right) + \xi_3(\beta y + \gamma_2 c) - \xi_4\beta y,$$

$$\frac{d\xi_4}{dt} = \xi_1\lambda x - \xi_2\lambda x - \xi_3 b\left(1 - \frac{2(u+v)}{a(x+y)}\right) + \xi_4(\mu + \gamma_2 c),$$

with the transversality condition satisfying $\xi_i(t_f) = 0$, $i = 1,\ldots,4$. Moreover, the optimal control policy is given by:

$$\gamma_1^*(t) = \max\left\{0, \min\left\{1, \frac{hy\xi_2}{2R}\right\}\right\}, \tag{17}$$

$$\gamma_2^*(t) = \max\left\{0, \min\left\{1, \frac{c(\xi_3 u + \xi_4 v)}{2S}\right\}\right\}. \tag{18}$$

PROOF: Applying the Pontryagin Minimum Principle, we can posit that the optimal control variable $\gamma^*(t)$ satisfies:

$$\frac{\partial H}{\partial\gamma^*(t)} = 0. \tag{19}$$

From (15) and (19) we can get the following expression for γ_1 and γ_2:

$$\gamma_1^*(t) = \frac{hy\xi_2}{2R},$$

$$\gamma_2^*(t) = \frac{c(\xi_3 u + \xi_4 v)}{2S}.$$

For the boundedness of the optimal control, we have

$$\gamma_1^*(t) = \max\left\{0, \min\left\{1, \frac{hy\xi_2}{2R}\right\}\right\},$$

$$\gamma_2^*(t) = \max\left\{0, \min\left\{1, \frac{c(\xi_3 u + \xi_4 v)}{2S}\right\}\right\}.$$

According to Pontryagin Minimum Principle (Pontryagin et al., 1986), adjoint variables satisfy the relations:

$$\frac{d\xi_i}{dt} = -\frac{\partial H}{\partial s_i}, \quad i = 1, 2, 3, 4, \tag{20}$$

where $s_i \equiv (x, y, u, v)$ and the necessary conditions to be satisfied by the optimal control pair $\gamma^*(t)$ are

$$H(s_i(t), \gamma^*(t), \xi_i(t), t) = \min_{u\in U}(H(x_i(t), \gamma(t), \xi_i(t), t)), \ i = 1, 2, 3, 4. \tag{21}$$

The above equations are the necessary conditions that are satisfied by the optimal control $\gamma(t)$ and also the state variables of the system. From (15) we have the following equalities:

$$\frac{d\xi_1}{dt} = -\frac{\partial H}{\partial x}, \quad \frac{d\xi_2}{dt} = -\frac{\partial H}{\partial y},$$
$$\frac{d\xi_3}{dt} = -\frac{\partial H}{\partial u}, \quad \frac{d\xi_4}{dt} = -\frac{\partial H}{\partial v}. \tag{22}$$

From the set of equations (22), equations (16) can be obtained. Thus, equations (14), (17), (18) and (19) represent the optimal system. This system is a two point boundary value problem. The state system, system (14), is an initial value problem, and adjoint system (17) is a boundary value problem with boundary $\xi_i(t_f) = 0$, $i = 1, 2, 3, 4$.

The optimal control problem, formulated above, is a two point boundary value problem. State equation is an initial value problem and adjoint equation system is a boundary value problem. We solve the OCP using the code given in **Appendix A**. The code is developed from the article by Wang (2009). Here, we fixed the final time and obtain the optimal profile of roguing and spraying.
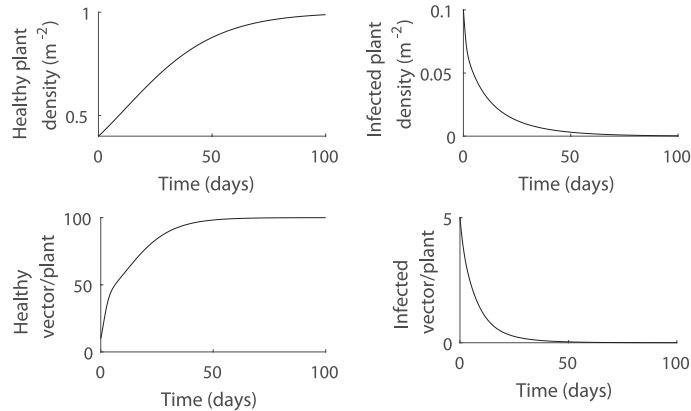
Figure 6. Effect of optimal roguing and optimal spraying on the system (2)

Based on the analytical outcomes above, again some numerical simulations for the OCP have been performed. Like before, $x(0) = 0.4$, $y(0) = 0.1$, $u(0) = 10$, $v(0) = 5$ are taken as initial conditions for the OCP. Also, $P = 0.1$, $R = 0.01$. The values have been varied and it was observed that these are sensitive parameters, but the respective results are not shown in the figures.

The effect of optimal control on the system is shown in Fig. 6. We apply the control through roguing and insecticide spraying for a time period of 100
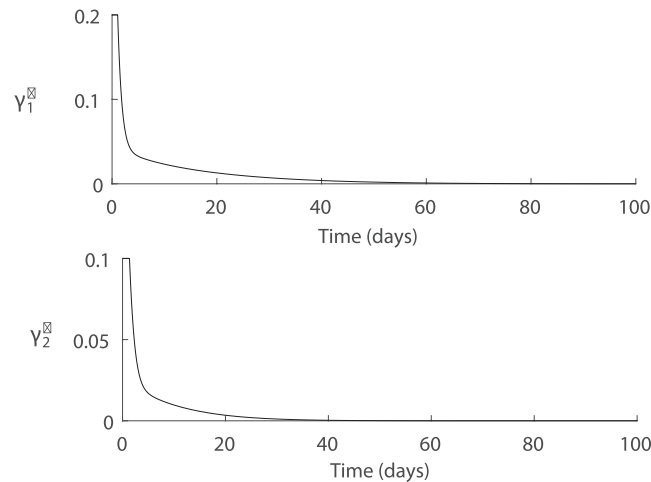
Figure 7. Optimal control pair is plotted as function of time.

days. In this figure, we notice that due to the application of the optimal control pair (γ_1^*, γ_2^*), healthy plant population achieves its maximum value in 100 days and the infected plant volume is also minimized in this time period. Non-infected vector population increases initially, but due to the use of insecticide it ultimately decreases. Infected vector population is reduced significantly within 100 days and finally tends to extinction.

It has also been observed that initially roguing is required for fifty days. Also, control through spraying for the first 40 days of the disease outbreak (see Fig. 7) is required, and the cost is also minimized through the optimal control policy. Optimal control policy, implemented through the use of both roguing and insecticide spraying exerts great influence in terms of making the system disease free and maintaining the stable nature in the remaining time period. Figure 7 shows that optimal roguing and spraying is needed to control the mosaic virus and to minimize the cost of cultivation. Hence, optimal levels of roguing and insecticide can be applied on the host plants systematically to eradicate the disease.

5. Discussion and conclusion

Jatropha curcas is considered to be the most suitable alternative renewable energy resource owing to its capacity to produce biodiesel from its oil (Roy et al., 2014; Al Basir et al., 2015). The plant is infected severely by mosaic virus which is carried by the whitefly vector. Insecticide spraying can remove the vector from the plantation (Roy et al., 2015), but excessive use of insecticides is harmful to the environment. The disease can be controlled with roguing, but it is much more time consuming. Therefore, we adopted both of these techniques,

i.e. roguing and spraying, to eradicate the disease in order to ensure smooth supply of *Jatropha* oil to the industry for biodiesel production. Thus, the main aim of this study is to analyse how roguing and spraying can affect the dynamics of mosaic disease.

A nonlinear mathematical model is formulated to study *Jatropha curcas* plant mosaic disease dynamics, considering roguing and spraying as controlling strategies. The model exhibits three equilibria, explicitly: (i) the disease and vector free, (ii) disease-free, and (iii) endemic equilibrium. The region for stability of disease-free equilibrium has been shown. The system is stable for $R_0 < 1$ and unstable if $R_0 > 1$. For endemic steady state, the region of stability is also shown in the figures. Finally, the analysis of the optimal control problem shows that optimal spraying and roguing can make the system free of disease, in a cost effective way.

Another factor that can be included in the plantation system is the use of nutrients. The use of nutrients along with spraying and rouging gives yet more fruitful results (Eraslan et al., 2007; Perring et al., 1999). It helps the plants to recover from disease quickly by increasing their growth rate (Dordas, 2008). If N is the amount of nutrient used, then the effect of nutrient can be incorporated in the model by modifying the growth rate r as follows:

$$r = r_0[1 + a/(1 + exp\{-b(N - c)\}]$$

where a, b, c are constants. Nutrients facilitate faster crop growth and are beneficial when used in small controlled quantities, but can also lead to plant deficiency and cause plant death due to toxicity, when large amounts of nutrients are applied.

In conclusion, while predicting the future course of an epidemic outbreak, for purposes of controlling the mosaic disease, the potential impact of roguing and spraying must be considered. Also, optimal control theory can be applied for determining cost effective conduct of the control process. In a nutshell, the modeling approach and optimal control problem, presented here, can serve as an integrating measure to identify and design the appropriate disease control strategies.

Acknowledgements

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Appendix A

Matlab code for solving the OCP:

```
function Optimal_control_problem

solinit = bvpinit([0 100], [0.4 0.1 10 5 0 0 0 0]); % initial guess

options = bvpset('Stats','on','RelTol',1e-10);
```

```
global R P Q h c S

h=0.2; c=0.1; R=.01; P=0.1; S=0.01;
sol = bvp4c(@BVP_ode, @BVP_bc, solinit, options);

t = sol.x; y = sol.y;

u1 =max(min(((h.*y(2,:).*y(6,:))/R),1),0);

u2 =max(min(((c.*(y(7,:).*y(3,:)+y(4,:).*y(8,:)))/S),1),0);

n = length(t);

% calculation of cost

J=0.5*(P*y(2,:)+R*u1*u1'+S*u2*u2')/n;

FinalCost = strcat('Final cost is: J=',num2str(J(end,1)))

function dydt = BVP_ode(t,y)

global a b r k lambda c beta m delta P Q R S mu h

r =0.1; lambda=0.02; a =100; b =0.8;

beta =0.12; k =1; m =0.03; mu =0.12 ;

%calculation of optimal control
u1 =max(min(((h.*y(2,:).*y(6,:))/R),1),0);

u2 =max(min(((c.*(y(7,:).*y(3,:)+y(4,:).*y(8,:)))/S),1),0);

dydt=[%state equation

    r*y(1)*(1-(y(1)+y(2))/k)-lambda*y(1)*y(4);

lambda*y(1)*y(4)-(m+u1*h)*y(2);

    b*(y(3)+y(4))*(1-(y(3)+y(4))/(a*(y(1)+y(2))))-beta*y(3)*y(4)-u2*c*y(3);

 beta*y(3)*y(4)-u2*c*y(4)-mu*y(4);

% adjoint equation

-(y(4)*r*(1-(2*y(1)+y(2)/k))-y(5)*lambda*y(4)+y(6)*lambda*y(4)...

+y(7)*(b*(y(3)+y(4))^2/(a * (y(1) + y(2))^2)));
```

-2*P*y(2)-(-y(5)*r*y(1)/k-y(6)*(m+u1*h)...

$+y(7) * (b * (y(3) + y(4))^2/(a * (y(1) + y(2))^2)) + y(8) * beta * y(3));$

y(7)*2*b*(1-(y(3)+y(4))/(a*(y(1)+y(2))))+y(7)*(beta*y(2)+u2*c)-y(8)*beta*y(2);

lambda*y(5)*y(1)-(y(6)*y(1)+y(7)*2*b*(1-(y(3)+y(4))/(a*(y(1)...

+y(2))))+y(8)*(mu+u2*c))];

```
      function res = BVP_bc(ya,yb)
% boundary conditions
res = [ ya(1) - 0.4
        ya(2) - 0.1
        ya(3) - 10
        ya(4) - 5
        ya(5) - 0
        yb(6) - 0
        yb(7) - 0
        yb(8) - 0 ];
```