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AN EXTENDED FINITE DIFFERENCE METHOD FOR SINGULAR PERTURBATION PROBLEMS ON A NON-UNIFORM MESH

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An extended second order finite difference method on a variable mesh is proposed for the solution of a singularly perturbed boundary value problem. A discrete equation is achieved on the non uniform mesh by extending the first and second order derivatives to the higher order finite differences. This equation is solved efficiently using a tridiagonal solver. The proposed method is analysed for convergence, and second order convergence is derived. Model examples are solved by the proposed scheme and compared with available methods in the literature to uphold the method.

Key words: non uniform grid, finite difference method, singular perturbation, boundary layer.

1. Introduction

The requirement for singularly perturbed boundary value problems (SPBVP) is useful in the various fields of science and engineering such as nuclear engineering, fluid mechanics, control theory, elasticity, optimal control, reaction diffusion process and many other fields (Bigge and Bohl [1]; Ackerberg and O' Malley [2]; Ardema, [3]). In most of the numerical schemes, the presence of sharp boundary layers generates difficulty when the coefficient of highest derivative tends to zero. For example, as the perturbation parameter tends to zero, there will be a discontinuous limit in the solution of singularly perturbed problem and interior layers appear. Hence, there is necessity to frame parameter (value ε) independent accuracy based on numerical schemes which convergent ε - uniformly. The very small perturbation parameter is responsible for arising computational difficulties in the numerical treatment of singularly perturbed differential equations. Mohammadi, [4] discussed a numerical method for SPBVP using adaptive cubic spline on the uniform mesh. The convection-diffusion boundary value problems are presented with two small parameters using a non-polynomial spline technique in [5]. The authors in [6] have given a variable mesh difference scheme of second order for the solution of SPBVP. Surla K. *et al.* [7] applied a quadratic spline collocation method to solve a SPBVP. Doolan *et al.* [8] elucidated several uniform numerical methods for the solution of SPBVPs. The authors in [9] suggested the finite difference methods for second order singularly perturbed delay differential equations and revealed the size effect of the delay argument, the coefficient of the delay term. Hemker and Miller [10], described briefly the numerical analysis of a singular perturbation problem. The authors in [11] made a numerical analysis of singularly perturbed delay differential equations with layer behavior. O'Malley *et al.* [12] explained briefly singular perturbations. Miller *et al.* [13] worked on fitted numerical methods for singular perturbation problems. The researchers in [14] employed a cubic spline compression method, whereas the authors in [15] used finite difference methods based on a ε -uniformly convergent fitted mesh.

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In this paper section 2, gives a description of the problem. In section 3, the method of solution is discussed. The convergence analysis is given in section 4. The numerical examples, graphs, results along with the discussions and conclusions are given in the subsequent sections.

2. Description of the problem

Consider a singular perturbed differential equation as:

$$\varepsilon u''(s)+a(s)u'(s)+b(s)u(s)=f(s), \qquad 0\le s\le 1, \tag{2.1}$$

subject to the boundary conditions;

$$u(0)=\alpha_0 \quad \text{and} \quad u(1)=\alpha_1. \tag{2.2}$$

Here $0<\varepsilon<<1$ is a very small positive parameter and $a(s), b(s)$ and $f(s)$ are smooth functions over the domain. When the perturbation parameter $\varepsilon\to 0$, then the solution of Eq.(2.1) contains layer behaviour based on the sign of convective coefficient $a(s)$. If $a(s)\ge K>0$ where K is positive constant, then problem (2.1) possesses boundary layer at $s=0$. If $a(s)\le Q<0$ where Q is a negative constant, then problem (2.1) possesses boundary layer $s=1$.

3. Numerical scheme

Let the interval [*0, 1*] be split into *N* sub intervals with size $h_i=s_i-s_{i-1}$ for $i=1$ to *N* and $h_{i+1}=r_i h_i$. For the computational implementation, the value h_1 has to be determined. Denote $R=s_N-s_0$. Then $R=(s_N-s_{N-1})+(s_{N-1}-s_{N-2})+....+(s_1-s_0)=h_N+h_{N-1}+....+h_1=(r_1+r_1r_2+....+r_1r_2r_3....r_{N-1})h_1$. Then $h_1=\dfrac{R}{(r_1+r_1r_2+....+r_1r_2r_3....r_{N-1})}$ shows the value of the initial step length which is used to determine the next step lengths h_2, h_3, etc. In case of singular perturbation problems, the presence of a layer at the left end boundary $s=0$ requires generally a large cluster of nodal points near the end point. Similarly, a large cluster of nodal points at this boundary is needed if the layer is at the right end. The nodal points can be distributed with the following process:

$$r_i=r=const.$$

is chosen for $i=1,2,\ldots,N$.
Hence, the step length h_1 reduces to

$$h_1=\frac{R(1-r)}{\left(1-r^N\right)}.$$

We choose $r>1$ for the boundary layer at left end point and thus we have a larger number of nodal points near the left end. Similarly, we choose $r<1$ for the boundary layer at the right end point which guaranties a large number of nodal points at the right end boundary. We can have a symmetric mesh with a large number of nodal points at both ends when the boundary layer is at both the end points of the interval.
A non-uniform higher order finite difference approximation is considered for first and second derivatives as:

$$u_i^{'} = \tilde{u}_i^{'} - \frac{r_i h_i^2}{6} u_i^{'''} - \frac{\left(r_i^2 - r_i\right) h_i^3}{24} u_i^{(iv)} + \tau_1(i), \tag{3.1}$$

$$u_i^{''} = \tilde{u}_i^{''} + \frac{(1-r_i) h_i}{3} u_i^{'''} - \frac{\left(r_i^2 - r_i + 1\right) h_i^2}{12} u_i^{(iv)} + \tau_2(i) \tag{3.2}$$

where:

$$\tilde{u}_i^{'} = \frac{u_{i+1} - r_i^2 u_{i-1} + \left(r_i^2 - 1\right) u_i}{r_i (1+r_i) h_i}, \qquad \tilde{u}_i^{''} = \frac{2\left[u_{i+1} + r_i u_{i-1} - (1+r_i) u_i\right]}{r_i (1+r_i) h_i^2},$$

$$\tau_1(i) = -\frac{\left(r_i^4 + r_i\right)}{120(1+r_i)} h_i^4 u_i^{(v)}, \ldots\ldots \tau_2(i) = -\frac{\left(r_i^4 - r_i\right)}{(1+r_i)} h_i^3 u_i^{(v)}.$$

Enumerating $u_i^{'''}$ and $u_i^{(iv)}$ using Eq.(2.1) and utilising them in Eqs (3.1)-(3.2), we get:

$$\begin{gathered} u_i^{'} = \tilde{u}_i^{'} - C_i \left\{ \frac{f_i^{'} - a_i u_i^{''} - k_3(i) u_i^{'} - b_i^{'} u_i}{\varepsilon} \right\} + \\ -D_i \left\{ \frac{f_i^{''} - a_i\left(f_i^{'} - a_i u_i^{''} - k_3(i) u_i^{'} - b_i^{'} u_i\right) - k_4(i) u_i^{''} - k_5(i) u_i^{'} - b_i^{''} u_i}{\varepsilon} \right\}, \end{gathered} \tag{3.3}$$

$$\begin{gathered} u_i^{''} = \tilde{u}_i^{''} + A_i \left\{ \frac{f_i^{'} - a_i u_i^{''} - k_3(i) u_i^{'} - b_i^{'} u_i}{\varepsilon} \right\} + \\ -B_i \left\{ \frac{f_i^{''} - a_i\left(f^{'} - a_i u_i^{''} - k_3(i) u_i^{'} - b_i^{'} u_i\right) - k_4(i) u_i^{''} - k_5(i) u_i^{'} - b_i^{''} u_i}{\varepsilon} \right\}; \end{gathered} \tag{3.4}$$

where:

$$A_i = \frac{(1-r_i) h_i}{3}, \qquad B_i = \frac{\left(r_i^2 - r_i + 1\right) h_i^2}{12}, \qquad C_i = \frac{r_i h_i^2}{6}, \qquad D_i = \frac{\left(r_i^2 - r_i\right) h_i^3}{24},$$

$$k_1(i) = r_i (1+r) h_i^2, \qquad k_2(i) = r_i (1+r_i) h_i, \qquad k_3(i) = \left(a_i^{'} + b_i\right),$$

$$k_4(i) = \left(2a_i^{'} + b_i\right), \qquad k_5(i) = \left(a_i^{''} + 2b_i^{'}\right).$$

Now inserting Eqs (3.3) and (3.4) in Eq.(2.1), we get;

$$E_i u_{i-1} + F_i u_i + G_i u_{i+1} = R_i \qquad \text{for} \qquad i = 1, 2, \ldots, N-1, \tag{3.5}$$

where:

$$E_i = \left(2L_i r_i - r_i^2 M_i h_i\right), \qquad F_i = \left(M_i\left(r_i^2 - 1\right)h_i - 2L_i\left(1 + r_i\right) + N_i r_i\left(1 + r_i\right)h_i^2\right), \qquad G_i = \left(2L_i + M_i h_i\right),$$

$$L_i = \varepsilon - a_i A_i - \frac{a_i^2 B_i}{\varepsilon} + B_i k_4(i) + \frac{a_i^2 C_i}{\varepsilon} - \frac{a_i^3 D_i}{\varepsilon^2} + \frac{a_i D_i k_4(i)}{\varepsilon},$$

$$M_i = -A_i k_3(i) - \frac{a_i B_i k_3(i)}{\varepsilon} + B_i k_5(i) + a_i + \frac{a_i C_i k_3(i)}{\varepsilon} - \frac{a_i^2 D_i k_3(i)}{\varepsilon^2} + \frac{a_i D_i k_5(i)}{\varepsilon},$$

$$N_i = -\frac{a_i B_i b_i'}{\varepsilon} + B_i b_i'' + \frac{a_i C_i b_i'}{\varepsilon} - \frac{a_i^2 D_i b_i'}{\varepsilon^2} + \frac{a_i D_i b_i''}{\varepsilon} + b_i - A_i b_i',$$

$$R_i = \left(f_i - \left(A_i + \frac{a_i B_i}{\varepsilon} - \frac{a_i C_i}{\varepsilon} + \frac{a_i^2 D_i}{\varepsilon^2}\right)f_i' - \left(-B_i - \frac{a_i D_i}{\varepsilon}\right)f_i''\right)k_1(i).$$

The system of equations Eq.(3.5) is solved by using the tridiagonal solver.

4. Convergence analysis

The proposed scheme has a truncation error which is

$$T_i(h_i) = \frac{(r_i + 1)(r_i - 1)^2 a_i}{9} u_i''' h_i^4 + O\left(h_i^5\right). \tag{4.1}$$

Let the tridiagonal system of Eq.(3.5) in matrix form be

$$YU = V, \tag{4.2}$$

where; $Y = \left(y_{i,j}\right)$ for $1 \le i, j \le N - 1$ is a tridiagonal matrix with $y_{i,i+1} = G_i$, $y_{i,i} = F_i$, $y_{i,i-1} = E_i$ and $V = (v_i)$ is a column vector with $v_i = R_i$.
We have:

$$Y\overline{U} - T_i(h_i) = V \tag{4.3}$$

where $\overline{U} = \left(\overline{u_0}, \overline{u_1}, \ldots, \overline{u_N}\right)^t$ represents the actual solution and $T_i(h_i) = \left(T_0(h_0), T_1(h_1), \ldots, T_N(h_N)\right)^t$ is the local truncation error.
From Eq.(4.2) and Eq.(4.3), we get

$$Y\left(\overline{U}-U\right)=T_i\left(h_i\right). \tag{4.4}$$

Thus the error equation is

$$YE=T_i\left(h_i\right), \tag{4.5}$$

where

$$E=\overline{U}-U=\left(e_0,e_1,...,e_N\right)^t.$$

Let S_i be the sum of elements of the i^{th} row of matrix Y, then we have

$$S_i=\sum_{j=1}^{N-1}m_{i\,j}=-2r_i\varepsilon+\left(a_ir_i^2+\frac{2a_ir_i\left(1-r_i\right)}{3}\right)h_i+O\left(h_i^2\right)\quad \text{for}\quad i=1,$$

$$S_i=\sum_{j=1}^{N-1}m_{i\,j}=r_i\left(r_i+1\right)b_ih_i^2+\left(\frac{r_i\left(r_i+1\right)\left(r_i-1\right)b_i'}{3}\right)h_i^3+$$
$$+O\left(h_i^4\right)=\beta_ih_i^2+O\left(h_i^3\right)\quad \text{for}\quad i=2,3,...,N-2,$$

$$S_i=\sum_{j=1}^{N-1}m_{i\,j}=-2\varepsilon+\left(\frac{\left(1-r_i\right)a_i}{3}-a_i\right)h_i+O\left(h_i^2\right)\quad \text{for}\quad i=N-1.$$

Since $0<\varepsilon<<1$, the matrix Y^{-1} exists and it will have non-negative elements. Hence, from Eq.(4.5), we get

$$E=Y^{-1}T_i\left(h_i\right) \tag{4.6}$$

and

$$E\leq Y^{-1}T_i\left(h_i\right). \tag{4.7}$$

Let $\overline{m}_{ki}$ be the $\left(ki\right)^{th}$ element of Y^{-1} and since all its elements are non-negative, we have

$$\sum_{i=1}^{N-1}\overline{m}_{k,i}\,S_i=1\;,\; k=1,2,...,N-1\;. \tag{4.8}$$

Therefore,

$$\sum_{i=1}^{N-1}\overline{m}_{k,i}\leq\frac{1}{\min\limits_{1\leq i\leq N-1}S_i}=\frac{1}{\beta_i}\leq\frac{1}{\left|\beta_i\right|}. \tag{4.9}$$

We define

$$Y^{-1} = \max_{1\le k\le N-1} \sum_{i=1}^{N-1} \left|\bar{m}_{ki}\right| \text{ and } T_i\left(h_i\right) = \max_{1\le i\le N-1} \left|T_i\left(h_i\right)\right| .$$

From Eqs.(4.6), (4.8) and (4.9), we get

$$e_j = \sum_{i=1}^{N-1} \bar{m}_{ki} T_i\left(h_i\right), \quad j = 1,2,3,\ldots, N-1 ,$$

which gives

$$e_j \le \frac{kh_i^4}{\left|\beta_i\right| h_i^2} , \qquad j = 1,2,\ldots,N-1 \tag{4.10}$$

where

$$k = \frac{\left(r_i+1\right)\left(r_i-1\right)^2 a_i}{9} u_i'''$$

is a constant independent of h_i.

Therefore, using Eq.(4.10), we have $E = O\left(h_i^2\right)$ i.e., the method is quadratic convergent on the non-uniform mesh.

5. Numerical experiments

Four boundary value problems are considered for the computational demonstration of the proposed method. In the solution, the maximum absolute errors are calculated by $E_{N,\varepsilon_1} = \max_{0\le i\le N}\left|u\left(s_i\right) - u_i\right|$ where $u\left(s_i\right)$ is an exact solution and u_i is the computed solution.

Example 1. $\varepsilon u''\left(s\right) + u'\left(s\right) = 0$ with $u\left(0\right) = 1, \quad u\left(1\right) = e^{\frac{-1}{\varepsilon}}$.

The exact solution is given by

$$u\left(s\right) = e^{\frac{-s}{\varepsilon}} .$$

Example 2. $-\varepsilon u''\left(s\right) + u'\left(s\right) = e^s, \; u\left(0\right) = 0, u\left(1\right) = 0$.

The exact solution is given by

$$u\left(s\right) = \frac{1}{\varepsilon - 1}\left[1 - \frac{e-1}{e^{\frac{1}{\varepsilon}} - 1} + \frac{\left(e-1\right)e^{\frac{s}{\varepsilon}}}{e^{\frac{1}{\varepsilon}} - 1} - e^s\right].$$

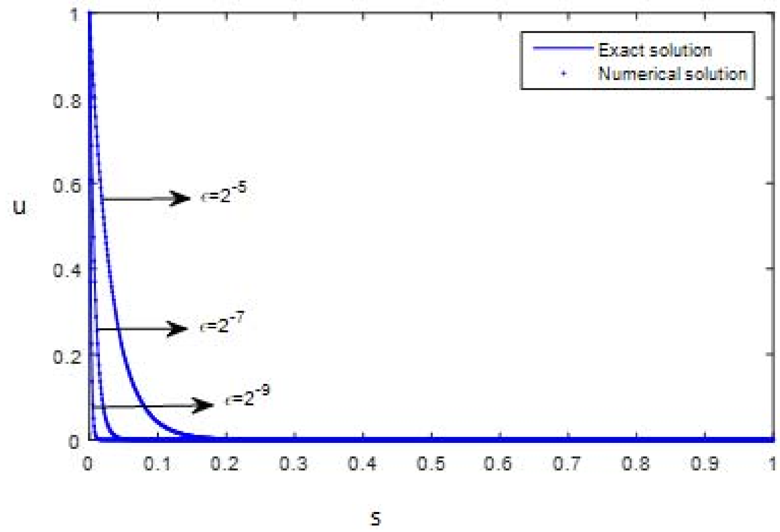

Fig.1. Exact and approximate solution for Example 1 at various values of ε with $N = 1024$.

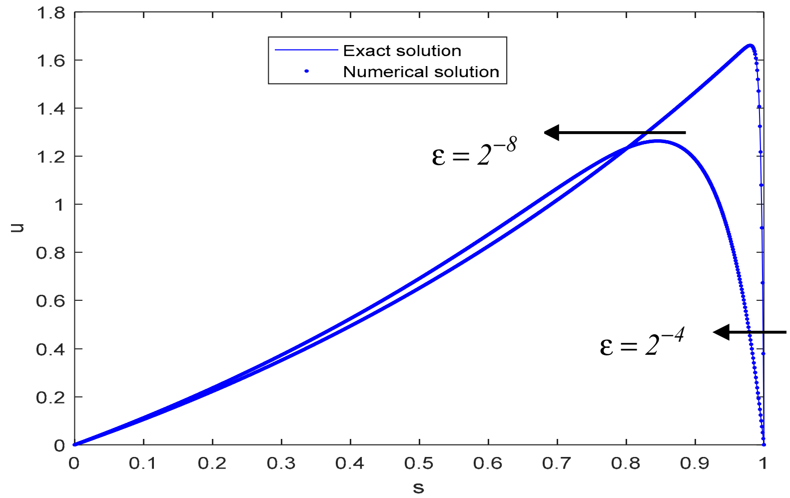

Fig.2. Exact and approximate solution for Example *2* at various values of ε with $N = 1024$.

Example 3. $\varepsilon u'' - \frac{1}{s+1}u' - \frac{1}{s+2}u = f(s)$, $\quad u(0) = 1 + 2^{\frac{-1}{\varepsilon}}, u(1) = e + 2$,

where

$$f(s) = e^{s}\left(\varepsilon - \frac{1}{s+1} - \frac{1}{s+2}\right) - 2^{\frac{-1}{\varepsilon}} \frac{(s+1)^{1+\frac{1}{\varepsilon}}}{s+2}.$$

The exact solution is given by

$$u(s) = e^{s} + 2^{\frac{-1}{\varepsilon}}(s+1)^{1+\frac{1}{\varepsilon}}.$$

Example 4. $\varepsilon u'' - u' - u = -1$, $\quad u(0) = 0, \quad u(1) = 0$.

The exact solution is given by

$$u(s)=\frac{\left(e^{c_2}-1\right)e^{c_1 s}}{e^{c_1}-e^{c_2}}+\frac{\left(1-e^{c_1}\right)e^{c_2 s}}{e^{c_1}-e^{c_2}}+1,$$

where $c_1=\frac{1+\sqrt{1+4\varepsilon}}{2\varepsilon}$ and $c_2=\frac{1-\sqrt{1+4\varepsilon}}{2\varepsilon}$.

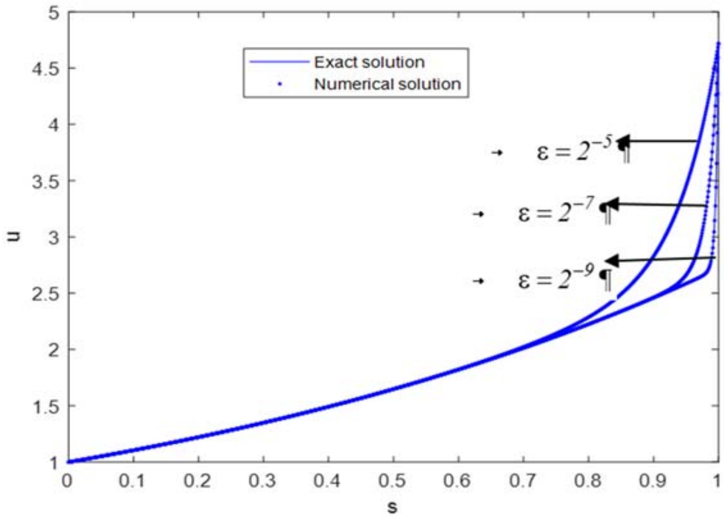

Fig.3. Exact and approximate solution for Example *3* at various values of ε with $N=1024$.

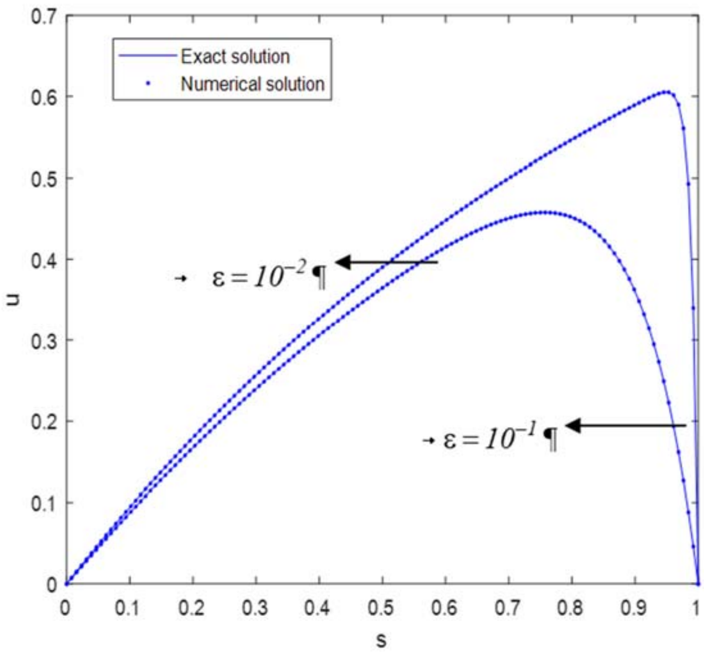

Fig.4. Exact and approximate solution for Example 4 at various values of ε with $N=128$.

Table 1. Maximum absolute errors for various values of ε .

ε	N = 64	N = 128	N = 256	N = 512	N = 1024
Present Method					
2^{-4}	5.6413(-6)	3.7664(-6)	3.6497(-6)	3.6425(-6)	3.6420(-6)
2^{-5}	3.6000(-5)	5.6388(-6)	3.7660(-6)	3.6497(-6)	3.6424(-6)
2^{-6}	5.4445(-4)	3.5958(-5)	5.6337(-6)	3.7654(-6)	3.6496(-6)
2^{-7}	7.5000(-3)	7.5000(-3)	3.5876(-5)	5.6236(-6)	3.7641(-6)
2^{-8}	5.8600(-2)	7.5000(-3)	5.4234(-4)	3.5711(-5)	5.6035(-6)
2^{-9}	2.2540(-1)	5.8500(-2)	7.5000(-3)	5.3953(-4)	3.5385(-5)
Mohammadi [4]					
2^{-4}	1.90(-3)	4.79(-4)	1.16(-4)	2.93(-5)	7.43(-6)
2^{-5}	7.87(-3)	1.90(-3)	4.77(-4)	1.17(-4)	2.96(-5)
2^{-6}	3.44(-2)	7.85(-3)	1.90(-3)	4.75(-4)	1.15(-4)
2^{-7}	1.35(-1)	3.45(-2)	7.84(-3)	1.90(-3)	4.72(-4)
2^{-8}	3.49(-1)	1.33(-1)	3.43(-2)	7.83(-3)	1.90(-3)
2^{-9}	5.98(-1)	3.50(-1)	1.30(-1)	3.41(-2)	7.81(-3)

Table 2. Maximum absolute error for various values of ε .

ε	N = 64	N = 128	N = 256	N = 512	N = 1024
Present Method					
2^{-1}	3.0844(-9)	2.8784(-9)	2.4632(-9)	2.4452(-9)	2.4342(-9)
2^{-4}	3.6761(-6)	2.3835(-7)	2.7397(-8)	2.7169(-8)	2.5270(-8)
2^{-8}	1.01110(-1)	1.2975(-2)	9.3445(-4)	5.6078(-5)	3.6112(-6)
Mohammadi [4]					
2^{-1}	1.71(-5)	4.28(-5)	1.06(-5)	2.67(-6)	2.08(-7)
2^{-4}	1.43(-3)	3.50(-3)	8.72(-4)	2.17(-4)	1.37(-5)
2^{-8}	1.03(-1)	6.06(-2)	2.33(-2)	6.95(-3)	3.32(-4)

Table 3. Maximum absolute error for various values of ε.

Present Method					
ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-2}	9.5742(-9)	8.9062(-9)	7.6031(-9)	8.1440(-9)	7.4875(-9)
2^{-3}	4.3741(-8)	1.3496(-8)	1.1458(-8)	1.2316(-8)	1.1345(-8)
2^{-4}	4.4865(-7)	3.6238(-8)	1.9106(-8)	2.0564(-8)	1.9015(-8)
2^{-5}	5.4604(-6)	3.5358(-7)	3.8485(-8)	3.6770(-8)	3.4178(-8)
2^{-6}	7.6121(-5)	4.7287(-6)	3.1763(-7)	7.0271(-8)	6.3879(-8)
2^{-7}	1.1771(-3)	7.0409(-5)	4.3946(-6)	3.2595(-7)	1.2294(-7)
2^{-8}	1.5697(-2)	1.1298(-3)	6.7656(-5)	4.2758(-6)	3.7101(-7)
2^{-9}	1.1964(-1)	1.5369(-2)	1.1065(-3)	6.6385(-5)	4.2743(-6)
Mohammadi [4]					
2^{-2}	2.47(-5)	6.19(-6)	1.54(-6)	3.87(-7)	9.67(-8)
2^{-3}	1.71(-4)	4.28(-5)	1.07(-5)	2.67(-6)	6.69(-7)
2^{-4}	8.12(-4)	2.03(-4)	5.07(-5)	1.26(-5)	3.17(-5)
2^{-5}	3.53(-3)	8.79(-4)	2.19(-4)	5.48(-5)	1.37(-5)
2^{-6}	1.50(-2)	3.68(-3)	9.17(-4)	2.29(-4)	5.72(-5)
2^{-7}	6.75(-2)	1.54(-2)	3.77(-3)	9.37(-4)	2.34(-4)
2^{-8}	2.66(-1)	6.83(-2)	1.55(-2)	3.81(-3)	9.48(-4)
2^{-9}	6.92(-1)	2.68(-1)	6.87(-2)	1.56(-2)	3.83(-3)

Table 4. Comparison of point wise errors of Example 4 at various values of s for $\varepsilon = 10^{-2}$.

	$N = 32$		$N = 128$	
s	Present Method	Pooja Khandelwal [5]	Present Method	Pooja Khandelwal [5]
1/16	9.4625(-8)	4.55(-6)	5.7740(-8)	2.84(-7)
2/16	1.7789(-7)	8.55(-6)	1.0855(-7)	5.30(-7)
4/16	3.1436(-7)	1.51(-5)	1.9182(-7)	9.45(-7)
6/16	4.1664(-7)	2.00(-5)	2.5423(-7)	1.25(-6)
12/16	5.7426(-7)	3.07(-5)	3.5074(-7)	1.73(-6)
14/16	1.6118(-5)	1.41(-3)	3.4675(-7)	7.31(-7)

6. Discussions and conclusion

A finite difference method on a non-uniform mesh for the solution of a singular perturbation boundary value problem is proposed. The first and second order derivatives are approximated by the higher order finite differences on a variable mesh for acquiring the discretization equation. Four examples are implemented to demonstrate the proposed scheme. The justification of the proposed method is given by comparing the numerical results with the other methods reported in the literature. Better results are observed with the proposed method. It is observed from graphical representation that the width of the boundary layer decreases as the perturbation parameter ε decreases. Convergence of the method is established and it converges uniformly. The accurate results with little computational effort are produced with the proposed method.

Nomenclature

E – error
h_i – mesh size
K – positive constant
N – number of sub intervals
Q – negative constant
r_i – mesh ratio
s – independent variable
s_i – mesh points
u – solution
U – solution matrix
V – right hand side matrix
Y – tridiagonal matrix
ε – perturbation parameter
τ_i – truncation error

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