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REVERSE LIEB–THIRRING INEQUALITY FOR THE HALF-LINE MATRIX SCHRÖDINGER OPERATOR

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Abstract. We prove a reverse Lieb–Thirring inequality with a sharp constant for the matrix Schrödinger equation on the half-line.

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1. INTRODUCTION

In their celebrated work Lieb and Thirring [21] introduced a family of inequalities that are now known as Lieb–Thirring inequalities. These inequalities, or more precisely the Lieb–Thirring inequalities in spectral form, are concerned with the negative eigenvalues, λ_j, of the one particle Schrödinger operator $-\Delta + V$. They bound the Riesz means $\sum_j |\lambda_j|^\gamma$ in terms of L^p norms of the potential V. Lieb and Thirring introduced their inequalities in their study of the stability of matter. However, these inequalities have found numerous applications in other problems in functional analysis and mathematical physics. The recent monograph [12] contains an extensive study of Lieb–Thirring inequalities. Here we will content ourselves of stating results in one dimension, that are more closely related to our work. It was proved in [21] for $\gamma > 1/2$, and by [26] for $\gamma = 1/2$, that,

$$\sum_j |\lambda_j|^\gamma \leq L_{\gamma,1} \int_{\mathbb{R}} V_-(x)^{1/2+\gamma}\, dx,$$

with $V = V_+ - V_-$, where $V_\pm := 1/2(|V| \pm V)$ are, respectively, the positive and the negative parts of the potential V. Further, $V_- \in L^{1/2+\gamma}(\mathbb{R})$, and $V_+ \in L^1_{\text{loc}}(\mathbb{R})$. Moreover, $L_{1,\gamma}$ is a constant independent of V. In the case $\gamma = 1/2$, $L_{1/2,1} = 1/2$, and this value is sharp [15]. In [11] Lieb–Thirring inequalities were proved for the matrix Schrödinger operator on the half-line. Namely, they consider the selfadjoint Schrödinger operator $-\frac{d^2}{dx^2} + V(x)$ in $L^2(\mathbb{R}^+, \mathbb{C}^n)$, for $n = 1, \ldots$, with the boundary condition $\psi'(0) = B\psi(0)$, where $V(x)$ is a selfadjoint, $n \times n$ matrix that satisfies, $V \geq 0$,

$$\int_0^\infty \mathrm{Tr}[V^2(x)]\, dx < \infty,$$

and B is a selfadjoint $n \times n$ matrix.

One of their main results is the following. They prove that the negative spectrum of the Schrödinger operator consists of eigenvalues λ_j, with multiplicity m_j. Further, the following Lieb–Thirring estimate holds [11],

$$\frac{3}{4}|\lambda_1|\ \mathrm{Tr}[B] + \frac{1}{2}(2m_1 - n)|\lambda_1|^{3/2} + \sum_{j\geq 2} m_j|\lambda_j|^{3/2} \leq \frac{3}{16}\int_0^\infty \mathrm{Tr}[V^2(x)]\, dx + \frac{1}{4}\,\mathrm{Tr}[B^3].$$

For further results on Lieb–Thirring inequalities on the half-line in the scalar case see [10] and [24].

In this paper we are interested in reverse Lieb–Thirring inequalities in one dimension. Namely, in inequalities where one bounds from below a Riesz mean of the absolute value of the negative eigenvalues by the integral of the potential. This type of inequality was first proved independently by Glaser et al. [14], and by Schminke [25]. It was proved by these authors that

$$\sum_j \sqrt{|\lambda_j|} \geq -\frac{1}{4}\int_{\mathbb{R}} V(x)dx,$$

where the potentials V is integrable. Furthermore, the constant $1/4$ is sharp. See also [9] for a further reverse Lieb–Thirring inequality. Moreover, see [3] for two sided Lieb–Thirring inequalities in terms of the landscape function.

The aim of this paper is to prove a reverse Lieb–Thirring inequality for the matrix Schrödinger operator on the half-line. For the results below in the half-line matrix Schrödinger operator the reader can consult [1]. Let us consider the formal matrix Schrödinger operator in $L^2(\mathbb{R}^+, \mathbb{C}^n)$, for $n = 1, \ldots,$

$$-\frac{d^2}{dx^2} + V(x), \tag{1.1}$$

where the potential V is an $n \times n$ selfadjoint matrix-valued function. We assume that the potential V is integrable, i.e. it satisfies

$$\int_0^\infty dx\ |V(x)| < \infty, \tag{1.2}$$

where $|V(x)|$ denotes the operator norm of the matrix $V(x)$. We obtain a selfadjoint Schrödinger operator on the half line by supplementing the formal matrix Schrödinger operator (1.1) with the general selfadjoint boundary condition at $x = 0$, which is written as

$$-B^\dagger\psi(0) + A^\dagger\psi'(0) = 0, \tag{1.3}$$

where the $n \times n$ matrices A and B satisfy

$$B^\dagger A = A^\dagger B, \tag{1.4}$$

$$A^\dagger A + B^\dagger B > 0, \tag{1.5}$$

and we refer to A and B as the boundary matrices. Postmultiplying the boundary matrices on the right by an invertible $n \times n$ matrix T does not change (1.3). Thus, even though the boundary condition (1.3) is uniquely determined by the boundary-matrix pair (A, B), the matrix pair (AT, BT) with any invertible matrix T also yields the same boundary condition (1.3). Actually, as proved in Proposition 2.2.1 of [1], this is the only freedom that we have in choosing the matrices A, B. We denote by $H_{A,B}(V)$ the selfadjoint realization in $L^2(\mathbb{R}^+, \mathbb{C}^n)$ of the formal Schrödinger operator $-\frac{d^2}{dx^2} + V(x)$ with the boundary condition (1.3), where the boundary matrices A, B satisfy (1.4), (1.5). For the details in the construction of $H_{A,B}(V)$ see Sections 3.3, and 3.5 of [1]. As proved in Sections 3.4 and 3.6 of [1], we can unitarily transform the operator $H_{A,B}(V)$ into the operator $H_{\hat{A},\hat{B}}(\hat{V}) := M H_{A,B}(V) M^\dagger$, $\hat{V} := M V M^\dagger$, where M is a unitary matrix, and $A = M\hat{A}T_1 M^\dagger T_2$, $B = M\hat{B}T_1 M^\dagger T_2$, for some invertible matrices T_1, T_2, and where

$$\begin{aligned} &\hat{A} = -\operatorname{diag}\{\sin\theta_1, \ldots, \sin\theta_n\}, \quad \hat{B} = \operatorname{diag}\{\cos\theta_1, \ldots, \cos\theta_n\}, \\ &0 < \theta_j \le \pi, j = 1, \ldots, n. \end{aligned} \tag{1.6}$$

With the boundary matrices (1.6) the boundary condition (1.3) takes the form

$$(\cos\theta_j)\psi_j(0) + (\sin\theta_j)\psi'(0) = 0, \quad j = 1, \ldots, n. \tag{1.7}$$

By (1.7), we see that in the representation where the boundary matrices are diagonal we have Dirichlet boundary condition when $\theta_j = \pi$, Neumann boundary condition when $\theta_j = \pi/2$, and mixed boundary condition if $\theta_j \neq \pi/2, \pi$. Further, we have no Dirichlet boundary condition in (1.7) if and only if the boundary matrix A is invertible.

For our reverse Lieb–Thirring inequality we consider the boundary condition (1.3) with the boundary matrix A invertible. As mentioned above, this amounts to exclude Dirichlet boundary conditions in the diagonal representation of the boundary matrices. We exclude Dirichlet boundary conditions to obtain a meaningful reverse Lieb–Thirring inequality, as we explain in Remark 4.1 below. Note that if A is invertible we can take $T = A^{-1}$ and transforming (A, B) into (AT, BT) we obtain the boundary matrices (I, BA^{-1}). Hence, in the case where A is invertible there is no loss of generality in considering the operator $H_{I,B}(V)$, with the boundary condition

$$\psi'(0) = B\psi(0). \tag{1.8}$$

Observe that for the pair (I, B) conditions (1.4), (1.5) just amount to require that B is selfadjoint. It follow from Theorems 3.11.1 and 4.3.3 of [1] that if the potential satisfies (1.2) the operator $H_{I,B}(V)$ has no singular continuous spectrum, that its absolutely continuous spectrum is $[0, \infty)$, and that it has no positive eigenvalues. Further, zero can be an eigenvalue, and there are N negative eigenvalues λ_j, with multiplicity $m_j \leq n$, for $j = 1, \ldots$ The number of negative eigenvalues N can be zero, finite, or infinite. If there are an infinite number of negative eigenvalues they accumulate at zero. Our reverse Lieb–Thirring inequality is given in the following theorem.

Theorem 1.1. *Let $H_{I,B}(V)$ be the selfadjoint realization in $L^2(\mathbb{R}^+, \mathbb{C}^n)$ of the formal matrix Schrödinger operator* (1.1) *with the boundary condition* (1.8) *where B is a selfadjoint matrix, and the potential V is selfadjoint and fulfills* (1.2). *Assume that $H_{I,B}(V)$ has negative eigenvalues λ_j, for $j = 1, \ldots$ Then, the following reverse Lieb–Thirring inequality holds,*

$$\sum_j m_j \sqrt{|\lambda_j|} > \frac{1}{4} \left[-\int_0^\infty \mathrm{Tr}\,[V(x)]\; dx - \mathrm{Tr}[B] \right], \tag{1.9}$$

where the constant $1/4$ *is sharp.*

In the scalar case, $n = 1$, Theorem 1.1 is given in [6] assuming that V is integrable and that there is only a finite number of negative eigenvalues, λ_j, for $j = 1, \ldots, N < \infty$. In our Theorem 1.1 the number of negative eigenvalues is allowed to be infinite. The proof of (1.9) in the scalar case given in [6] is based in the classical results of the scalar Gel'fand–Levitan method [13, 19, 20, 22], and among other results, in Lemma 2 of [6]. For the proof of Lemma 2 of [6] it is claimed that the difference between a potential and the potential obtained after removing one eigenvalue is monotonic for large x. See however, the comments in page 55 of [24] concerning the validity of the monotonicity claimed in the proof of Lemma 2 of [6]. In our proof of Theorem 1.1 we proceed in a different way. We first prove that the proof of Theorem 1.1 can be reduced to the proof in the particular case of potentials of compact support. Then, we prove Theorem 1.1 for potentials of compact support using our results in transformations to remove eigenvalues of matrix Schrödinger operators on the half-line [2]. The paper is organized as follows. In Section 2 we state results from [1] on the matrix Schrödinger operator on the half-line that we use. In Section 3 we state the results from [2] in transformations to remove eigenvalues that we need. Finally, in Section 4 we prove Theorem 1.1.

The matrix Schrödinger equations have been studied since the early days of quantum mechanics. They are essential to consider properties of particles, such as spin, as well as to consider collections of particles. They have applications, for example, in nuclear, atomic, and molecular physics. An important example is the Pauli equation, that is the Schrödinger equation of a spin one half particle. For these applications see, for example, the monographs [8] and [18]. The theory of quantum graphs gave a new impetus to the interest in matrix Schrödinger equations. Quantum graphs have important applications in several areas, including nanotechnology, quantum wires, and quantum computing. A star graph, that is to say a quantum graph with only one vertex, and a finite number of semi-infinite edges that meet at the vertex, is the particular case of a matrix Schrödinger equation where the potential V is a diagonal matrix. For a general introduction to quantum graphs, as well as for many results and applications, the readers can consult the monographs [5, 17]. In the monograph [1] the readers can find more information about applications of matrix Schrödinger equations, as well as on the literature.

2. THE HALF-LINE MATRIX SCHRÖDINGER EQUATION

In this section we introduce preliminary results that we need later. In [1] the readers can find further information on the half-line matrix Schrödinger equation. Consider the half-line matrix Schrödinger equation,

$$-\psi(x)'' + V(x)\,\psi(x) = k^2\psi(x), \quad k \in \mathbb{C}, x \in \mathbb{R}^+, \tag{2.1}$$

where the prime denotes the x-derivative, the potential $V(x)$ is an $n \times n$ selfadjoint matrix-valued function of x. The wavefunction $\psi(x)$ is either an $n \times n$ matrix or a column vector with n components. We denote $\mathbb{R}^+ := (0, \infty)$. The selfadjointness of the potential means that,

$$V(x)^\dagger = V(x), \quad x \in \mathbb{R}^+. \tag{2.2}$$

By the dagger we denote the matrix adjoint. We always assume that the potential V is integrable, i.e. it satisfies (1.2). In some cases we suppose that the potential V belongs to the Faddeev class $L^1_1(\mathbb{R}^+)$. Namely, that,

$$\int\limits_0^\infty dx\,(1+x)|V(x)| < \infty.$$

We use $\mathbb{C}^+$ to denote the upper half of the complex plane $\mathbb{C}$, and use $\mathbb{R}$ for the real axis. We let $\overline{\mathbb{C}^+} := \mathbb{C}^+ \cup \mathbb{R}$.

As we already mentioned in the introduction, we denote by $H_{A,B}(V)$ the selfadjoint realization in $L^2(\mathbb{R}^+, \mathbb{C}^n)$ of the formal matrix Schrödinger operator $-\frac{d^2}{dx^2} + V(x)$ with the boundary condition (1.3), where the boundary matrices A, B satisfy (1.4), (1.5).

Of particular importance are the following two matrix solutions to (2.1). The first one is the Jost solution $f(k,x)$ that satisfies the asymptotic condition

$$f(k,x) = e^{ikx}\left[I + o(1)\right], \quad f'(k,x) = e^{ikx}\left[ikI + o(1)\right], \quad x \to \infty, \tag{2.3}$$

for $k \in \overline{\mathbb{C}^+} \setminus \{0\}$. We denote by I the $n \times n$ identity matrix. Further, if $V \in L^1_1(\mathbb{R}^+)$ the Jost solution exists also at $k = 0$. The second important matrix solution to (2.1) is the regular solution $\varphi(k,x), k \in \mathbb{C}$ that satisfies the initial conditions

$$\varphi(k,0) = A, \quad \varphi'(k,0) = B. \tag{2.4}$$

Recall that A and B are the boundary matrices appearing in (1.3). Note that $f(k,x)$ does not satisfy, in general, the boundary condition (1.3). However, the regular solution $\varphi(k,x)$ does satisfy (1.3).

We define the Jost matrix associated with (1.3) (2.1) as follows.

$$J(k) := f(-k^*,0)^\dagger\, B - f'(-k^*,0)^\dagger\, A, \quad k \in \mathbb{R} \setminus \{0\}. \tag{2.5}$$

The asterisk denotes complex conjugation. If $V \in L^1_1(\mathbb{R}^+)$ the Jost matrix can be defined also at $k = 0$. The Jost matrix is an $n \times n$ matrix-valued function of k. Moreover, it has an extension to $\mathbb{C}^+$, where the asterisk in (2.5) is used to indicate how that extension occurs.

We discuss now the bound states of the half-line matrix Schrödinger operator. For a given k a bound-state solution corresponds to a square integrable, column-vector solution to (2.1) that satisfies the boundary condition (1.3). We denote $\lambda := k^2$, $k \in \overline{\mathbb{C}^+}$. The real number $\lambda = k^2$ is an eigenvalue of $H_{A,B}(V)$ if an only if for the corresponding k (2.1) has a bound-state solution. By Theorem 3.11.1 of [1], there are no bound states when $\lambda > 0$, but it is possible that there is a bound state at $\lambda = 0$. Further, if $V \in L^1_1(\mathbb{R}^+)$ by Theorem 3.11.1 of [1] for $\lambda = 0$ there is no bound state, and the number of negative bound states is finite. Moreover, the multiplicity of the bound states is smaller or equal to n. The bound states when $\lambda < 0$ appear at the k-values on the positive imaginary axis of the complex k-plane that correspond to the zeros of $\det[J(k)]$. We use $\det[J(k)]$ to designate the determinant of the Jost matrix $J(k)$. We suppose that there are N zeros of $\det[J(k)]$ that appear when $k = i\kappa_j$ for $j = 1, \ldots,$ with κ_j being distinct positive numbers. Remark that N is equal to the number of negative bound states without counting multiplicities. The quantity N can be zero, a positive number or ∞. Hence, $\det[J(i\kappa_j)] = 0$ and we denote by m_j the dimension of $\operatorname{Ker}[J(i\kappa_j)]$. The quantity m_j coincides with the multiplicity of the bound state at $k = i\kappa_j$.

Following [2], we use the Gel'fand–Levitan theory to analyze the bound-state solutions to (2.1) In this theory the normalization matrices for the bound states are obtained by normalizing the regular solution $\varphi(k,x)$ at the bound states. We denote by C_j and $\Phi_j(x)$ the Gel'fand–Levitan normalization matrix and the corresponding normalized matrix solution at the bound state with $k = i\kappa_j$, respectively. We proceed to define the $n \times n$ matrices C_j and $\Phi_j(x)$ following [2]. Let us use Q_j to denote the orthogonal projection onto $\operatorname{Ker}[J(i\kappa_j)]$. The Gel'fand–Levitan bound-state normalized solution to the Schrödinger equation is defined as follows,

$$\Phi_j(x) := \varphi(i\kappa_j, x)\, C_j,$$

where the Gel'fand–Levitan normalization matrix C_j is defined below. Each of C_j is a nonnegative matrix of rank m_j, such that $\Phi_j(x)$ is square-integrable, and moreover,

$$\Phi_j(x) = O(e^{-\kappa_j x}), \quad x \to \infty.$$

By (2.4), $\Phi_j(x)$ satisfies (1.3). We also have that the following normalization conditions hold,

$$\int_0^\infty dx\, \Phi_j(x)^\dagger\, \Phi_l(x) = Q_j \delta_{j,l}, \quad j,l = 1, \ldots, \tag{2.6}$$

where we denote by $\delta_{j,l}$ the Kronecker delta. To construct the $n \times n$ Gel'fand–Levitan normalization matrix C_j we define the $n \times n$ matrix $\mathbf{G}_j$ as

$$\mathbf{G}_j := \int_0^\infty dx\, Q_j\, \varphi(i\kappa_j, x)^\dagger\, \varphi(i\kappa_j, x)\, Q_j. \tag{2.7}$$

By Theorem 3.11.1 (e) of [1] the integral in the right-hand side of (2.7) is finite. Further, we introduce the matrix $\mathbf{H}_j$ as

$$\mathbf{H}_j := I - Q_j + \mathbf{G}_j. \tag{2.8}$$

Both $\mathbf{G}_j$ and $\mathbf{H}_j$ are selfadjoint. Moreover, $\mathbf{H}_j$ is a positive matrix, and we denote by $\mathbf{H}_j^{1/2}$ its unique positive square root. Furthermore, $\mathbf{H}_j$ sends $Q_j\mathbb{C}^n$ into $Q_j\mathbb{C}^n$ and the restriction of $\mathbf{H}_j$ to $Q_j\mathbb{C}^n$ is also positive. The Gel'fand–Levitan normalization matrix C_j is defined as

$$C_j := \mathbf{H}_j^{-1/2}\, Q_j, \quad j = 1, \ldots \tag{2.9}$$

We have that C_j is selfadjoint and nonnegative, and it has rank equal to m_j, the same as the rank of Q_j. Moreover,

$$Q_j C_j = C_j Q_j = C_j, \quad j = 1, \ldots \tag{2.10}$$

3. TRANSFORMATION TO REMOVE A BOUND STATE

In this section we state results from Section 6 of [2] in a transformation to remove a bound state. We state the results for integrable potentials with compact support, that is the case that we use in this paper. For more general results see Section 6 of [2]. We remove any one of the bound states with $\lambda = \lambda_j$, where $\lambda_j := -\kappa_j^2$, and with the Gel'fand–Levitan normalization matrix C_j. Note that as we do not order the distinct positive constants κ_j in any particular way, without loss of generality we can suppose that we remove the bound state with $k = i\kappa_N$ and the Gel'fand–Levitan normalization matrix C_N. After that, we obtain the perturbed Schrödinger operator with the potential $\tilde{V}(x)$, the boundary matrices $\tilde{A}$ and $\tilde{B}$, the regular solution $\tilde{\varphi}(k, x)$, the Jost matrix $\tilde{J}(k)$, and $N-1$ bound states with eigenvalues $-\tilde{\kappa}_j^2$, the Gel'fand–Levitan normalization matrices $\tilde{C}_j$, the orthogonal projections $\tilde{Q}_j$ onto $\operatorname{Ker}[\tilde{J}(i\kappa_j)]$, the Gel'fand–Levitan normalized bound-state solutions $\tilde{\Phi}_j(x)$, and the multiplicities $\tilde{m}_j$ of the bound states, for $j = 1, \ldots, N-1$. In the next theorem, that summarizes results from Section 6 of [2] in the case of potentials with compact support, we express the perturbed quantities distinguished with a tilde in terms of the unperturbed quantities not containing the tilde and the perturbation identified with κ_N and C_N. However, before we state the theorem we introduce the Moore–Penrose inverse.

We designate by M^+ the Moore–Penrose inverse of a matrix M, [4, 7]. We only deal with Moore–Penrose inverses of square matrices. As stated in Definitions 1.12

and 1.13 and Theorem 1.1.1 of [7], the matrix M^+ is the Moore–Penrose inverse of the matrix M if the following four equalities are fulfilled:

$$\begin{cases} MM^+M = M, & M^+MM^+ = M^+, \\ (MM^+)^\dagger = MM^+, & (M^+M)^\dagger = M^+M. \end{cases} \tag{3.1}$$

Theorem 3.1. *Consider the unperturbed Schrödinger operator with the potential V satisfying (1.2), (2.2), and with support in the interval $[0, x_0]$. Further, the selfadjoint boundary condition (1.3) is described by the boundary matrices A and B satisfying (1.4) and (1.5), with the regular solution $\varphi(k,x)$ satisfying the initial conditions (2.4), the Jost solution $f(k,x)$ satisfying (2.3), the Jost matrix $J(k)$ defined in (2.5), containing $N \geq 1$ bound states with eigenvalues $\lambda_j = -\kappa_j^2$, the Gel'fand–Levitan normalization matrices C_j, the orthogonal projections Q_j onto $\mathrm{Ker}[J(i\kappa_j)]$, and the Gel'fand–Levitan normalized bound-state solutions $\Phi_j(x)$ for $1 \leq j \leq N$. Let us denote by $W_N(x)$,*

$$W_N(x) := \int_x^\infty dz\, \Phi_N(z)^\dagger\, \Phi_N(z), \tag{3.2}$$

and define the matrix-valued perturbed potential $\tilde{V}(x)$ as

$$\tilde{V}(x) := V(x) + 2\,\frac{d}{dx}\left[\Phi_N(x)\, W_N(x)^+\, \Phi_N(x)^\dagger\right], \tag{3.3}$$

where we recall that $W_N(x)^+$ denotes the Moore–Penrose inverse of $W_N(x)$. Then, we have:

(a) *The perturbed potential $\tilde{V}(x)$ appearing in (3.3) satisfies (1.2) and(2.2). Moreover, its support is contained in the interval $[0, x_0]$.*
(b) *The quantity*

$$\varphi(k,x) = \tilde{\varphi}(k,x) + \Phi_N(x) W_N(x)^+ \int_0^x dy\, \Phi_N(y)^\dagger \tilde{\varphi}(k,y) dy,$$

is a solution to (2.1) with the potential (3.3).
(c) *For $k \neq \pm i\kappa_N$, the perturbed quantity $\tilde{\varphi}(k,x)$ can be expressed as*

$$\tilde{\varphi}(k,x) = \varphi(k,x) + \frac{1}{k^2 + \kappa_N^2}\, \Phi_N(x)\, W_N(x)^+ \left[\Phi_N'(x)^\dagger\, \varphi(k,x) - \Phi_N(x)^\dagger\, \varphi'(k,x)\right].$$

(d) *Under the perturbation, the projection matrices Q_j for $1 \leq j \leq N-1$ remain unchanged, i.e. we have*

$$\tilde{Q}_j = Q_j, \quad 1 \leq j \leq N-1.$$

(e) *Under the perturbation, the Gel'fand–Levitan normalization matrices for $1 \leq j \leq N-1$ remain unchanged, i.e. we have*

$$\tilde{C}_j = C_j, \quad 1 \leq j \leq N-1.$$

(f) *The perturbed quantity* $\tilde{\varphi}(k,x)$ *satisfies the initial conditions* (2.4) *with* A, B *replaced by* $\tilde{A}$, $\tilde{B}$, *respectively, and where the matrices* $\tilde{A}$ *and* $\tilde{B}$ *are expressed in terms of the unperturbed boundary matrices* A *and* B *and the Gel'fand–Levitan normalization matrix* C_N *for the bound state at* $k = i\kappa_N$ *as*

$$\tilde{A} = A, \quad \tilde{B} = B + A\,C_N^2 A^\dagger A. \tag{3.4}$$

(g) *The matrices* $\tilde{A}$ *and* $\tilde{B}$ *appearing in* (3.4) *satisfy* (1.4) *and* (1.5). *Hence, as a consequence of* (b) *and* (f), *the quantity* $\tilde{\varphi}(k,x)$ *is the regular solution to the matrix Schrödinger equation with the potential* $\tilde{V}(x)$ *in* (3.3) *and with the selfadjoint boundary condition* (1.3) *with* A *and* B *replaced with* $\tilde{A}$ *and* $\tilde{B}$, *respectively.*

(h) *Under the perturbation, the determinant of the Jost matrix is transformed as*

$$\det[\tilde{J}(k)] = \left(\frac{k + i\kappa_N}{k - i\kappa_N}\right)^{m_N} \det[J(k)], \quad k \in \overline{\mathbb{C}^+},$$

where we recall that m_N *is the multiplicity of the bound state of the unperturbed problem at* $k = i\kappa_N$.

(i) *Under the perturbation, the bound state with eigenvalue* $\lambda_N = -\kappa_N^2$ *is removed without adding any new bound states in such a way that the remaining bound states with eigenvalues* $\lambda_j = -\kappa_j^2$ *and their multiplicities* m_j *for* $1 \le j \le N-1$ *are unchanged.*

(j) *Under the perturbation the absolutely continuous spectrum remains unchanged and equal to* $[0,\infty)$. *Moreover, the spectral measures for the absolutely continuous spectrum of the unperturbed and the perturbed problems are the same. For the definition of the spectral measure see* [2].

4. REVERSE LIEB–THIRRING INEQUALITY

In this section we give the proof of Theorem 1.1. We first prove that we can reduce the problem to the case of potentials with compact support. Recall that by Theorem 3.11.1 (g) of [1] for potentials in $L^1_1(\mathbb{R}^+)$, and in particular for potentials of compact support, the number of negative eigenvalues is finite. Let the potential V belong to $L^1(\mathbb{R}^+)$ and let us denote by $V_\pm : 1/2(|V| \pm V)$, its positive, respectively, negative part. Hence,

$$V(x) = V_+(x) - V_-(x), \quad x \in \mathbb{R}^+.$$

We denote by $\chi_{[0,l]}(x), l = 1, \ldots$, the characteristic function of $[0,l]$, and we define,

$$V_l(x) = V_+(x) - \chi_{[0,l]}(x) V_-(x).$$

Recall that $H_{I,B}(V)$ is the selfadjoint realization of the matrix Schrödinger operator $-\frac{d^2}{dx^2} + V(x)$ with the boundary condition (1.3) where the boundary matrices I, B satisfy (1.4), (1.5). Recall that this just amounts to ask that B is selfadjoint. Similarly, we denote by $H_{I,B}(V_l)$ the matrix Schrödinger operator with the potential V replaced by V_l. Let $\lambda_j := -\kappa_j^2$, for $j = 1, \ldots$ be the negative eigenvalues of $H_{I,B}(V)$, in increasing order. Recall that m_j is the multiplicity of λ_j. Similarly, we denote by $\lambda_j^{(l)} := -\kappa_{j,l}^2$, for $j = 1, \ldots, l = 1, \ldots$, the negative eigenvalues of $H_{I,B}(V_l)$, also in increasing order, and by $m_j^{(l)}$ the multiplicity of $\lambda_j^{(l)}$. Let μ_j, for $j = 1, \ldots$, denote the negative eigenvalues of $H_{I,B}(V)$, in nondecreasing order and repeated according to its multiplicity. Further, let $\mu_j^{(l)}$, for $j = 1, \ldots$, designate the negative eigenvalues of $H_{I,B}(V_l)$ in nondecreasing order and repeated according to its multiplicity. Since $H_{I,B}(V) \le H_{I,B}(V_l)$, it follows from the min-max principle [23] that

$$\mu_j \le \mu_j^{(l)}, \quad j = 1, \ldots. \tag{4.1}$$

Then, by (4.1),

$$\sum_j m_j \sqrt{|\lambda_j|} \ge \sum_j m_j^{(l)} \sqrt{\left|\lambda_j^{(l)}\right|}, \quad l = 1, \ldots. \tag{4.2}$$

Below we use (4.2) to reduce the proof of the reverse Lieb–Thirring inequality for $H_{I,B}(V)$ to the proof of the reverse Lieb–Thirring inequality for $H_{I,B}(V_l)$.

We first prove that $H_{I,B}(V_l)$ has a finite number of negative eigenvalues. Let us denote

$$V_{l,p} := \chi_{[0,p]}(x) V_l(x), \quad l, p = 1, \ldots,$$

and by $H_{I,B}(V_{l,p})$ the Schrödinger operator defined as $H_{I,B}(V)$ with the potential $V_{l,p}$ instead of V. As $V_{l,p}$ has compact support the operator $H_{I,B}(V_{l,p})$ has a finite number of negative eigenvalues. We designate by $\mu_j^{(l,p)}$, for $j = 1, \ldots$, the negative eigenvalues of $H_{I,B}(V_{l,p})$ in nondecreasing order and repeated according to its multiplicity. Since for $p \ge l, H_{I,B}(V_{l,p}) \le H_{I,B}(V_l)$, by the min-max principle [23]

$$\mu_j^{(l,p)} \le \mu_j^{(l)}, \quad p \ge l, j = 1, \ldots. \tag{4.3}$$

Assume that for some fixed $l = 1, \ldots, H_{I,B}(V_l)$ has an infinite number of negative eigenvalues. Then, by min-max principle [23] there would be an infinite number of negative $\mu_j^{(l)}$. But by (4.3) this would imply the existence of an infinite number of negative $\mu_j^{(l,p)}$ for all $p \ge l$. However, this is impossible, again by the the min-max principle [23], because as $V_{l,p}$ has compact support, $H_{I,B}(V_{l,p})$ has only a finite number of negative eigenvalues. It follows that $H_{I,B}(V_l)$ has a finite number of negative eigenvalues.

We now prove that for l large enough $H_{I,B}(V_l)$ has at least one negative eigenvalue, using that $H_{I,B}(V)$ has at least one negative eigenvalue. For this purpose, we prove that $H_{I,B}(V_l)$ converges to $H_{I,B}(V)$ in norm resolvent sense. Let us introduce the polar decomposition for $V(x)$,

$$V(x) = U(x)\hat{V}(x),$$

where $U(x)$ is a partially isometric matrix and $\hat{V}(x)$ is the absolute value of $V(x)$, a selfadjoint nonnegative matrix. For a concrete representation of $U(x)$ and of $\hat{V}(x)$ see equation (4.2.24) to (4.2.26) of [1]. Further, denote

$$V^{(1)} := \sqrt{\hat{V}(x)}, \quad V^{(2)} := U(x)\sqrt{\hat{V}(x)}.$$

We designate by

$$R_{0,I,B}(z) := (H_{I,B}(0) - z)^{-1}, \quad z \in \rho(H_{I,B}(0)),$$

the resolvent of $H_{I,B}(0)$, where $\rho(H_{I,B}(0))$ denotes the resolvent set of $H_{I,B}(0)$. Further, we introduce the resolvent of $H_{I,B}(V)$,

$$R_{I,B,V}(z) := (H_{I,B}(V) - z)^{-1}, \quad z \in \rho(H_{I,B}(V)),$$

where $\rho(H_{I,B}(V))$ denotes the resolvent set of $H_{I,B}(V)$. Then, by equation (4.2.39) of [1],

$$R_{I,B,V}(z) = R_{0,I,B}(z) - R_{0,I,B}(z)\left(I + V^{(1)}R_{0,I,B}(z)V^{(2)}\right)^{-1} R_{0,I,B}(z), \tag{4.4}$$

for $z \in \rho(H_{I,B}(0)) \cap \rho(H_{I,B}(V))$. In a similar way, replacing in the formulae above V by V_l, and $H_{I,B}(V)$ by $H_{I,B}(V_l)$ we get,

$$R_{I,B,V_l}(z) = R_{0,I,B}(z) - R_{0,I,B}(z)\left(I + V_l^{(1)}R_{0,I,B}(z)V_l^{(2)}\right)^{-1} R_{0,I,B}(z), \tag{4.5}$$

for $z \in \rho(H_{I,B}(0)) \cap \rho(H_{I,B}(V_l))$. By equation (4.2.11) of [1], and as $V \in L^1(\mathbb{R}^+)$, the operators

$$V^{(1)}R_{0,I,B}(z)V^{(2)}, \quad V_l^{(1)}R_{0,I,B}(z)V_l^{(2)},$$

are Hilbert–Schmidt, and

$$\lim_{l\to\infty} V_l^{(1)}R_{0,I,B}(z)V_l^{(2)} = V^{(1)}R_{0,I,B}(z)V^{(2)}, \tag{4.6}$$

where the limit is in the Hilbert-Schmidt norm. Then, by (4.4),(4.5), and (4.6) $H_{I,B}(V_l)$ converges in norm resolvent sense to $H_{I,B}(V)$. The smallest negative eigenvalue of $H_{I,B}(V)$ is separated from the rest of the spectrum by a small circle Γ. Then, it follows from Theorems 2.25 and 3.16, and the comments in Section 5, of Chapter IV, of [16], that for l large enough $H_{I,B}(V_l)$ has at least one negative eigenvalue inside the circle Γ.

We now prove that we can reduce the proof of the reverse Lieb–Thirring inequality for $H_{I,B}(V_l)$ to the proof of the reverse Lieb–Thirring inequality for $H_{I,B}(V_{l,p})$. For this purpose we observe that for each fixed l the operator $H_{I,B}(V_{l,p})$ converges to $H_{I,B}(V_l)$ as $p \to \infty$ in norm resolvent sense. The proof is the same as the proof that $H_{I,B}(V_l)$ converges to $H_{I,B}(V)$ as $l \to \infty$ in norm resolvent sense that we gave above. Take any $\varepsilon > 0$ small enough so that each negative eigenvalue $\lambda_j^{(l)}$ is separated by an open disk of center $\lambda_j^{(l)}$ and radius ε from the rest of the spectrum. Note that this is possible because $H_{I,B}(V_l)$ has a finite number of negative eigenvalues. Hence, using again Theorems 2.25 and 3.16, and the comments in Section 5, of Chapter IV, of [16], we get that for p large enough, inside the disk of center $\lambda_j^{(l)}$ and radius ε the operator $H_{I,B}(V_{l,p})$ has a finite number of negative eigenvalues of total multiplicity $m_j^{(l)}$ and outside the union of these open disks $H_{I,B}(V_{l,p})$ has no negative eigenvalues. Since ε can be chosen arbitrarily small we have,

$$\lim_{p\to\infty} \sum_j m_j^{(l,p)} \lambda_j^{(l,p)} = \sum_j m_j^{(l)} \lambda_j^{(l)}, \tag{4.7}$$

where we denote by$\lambda_j^{(l,p)}$, for $j = 1, \dots$ the negative eigenvalues of $H_{I,B}(V_{l,p})$, in increasing order, and by $m_j^{(l,p)}$, for $j = 1, \dots$ the multiplicity of the negative eigenvalue $\lambda_j^{(l,p)}$. Below we prove the reverse Lieb–Thirring inequality for $H_{A,B}(V_{l,p})$, and we use (4.7) to obtain the reverse Lieb–Thirring inequality for $H_{I,B}(V_l)$.

Removing the negative eigenvalues of $H_{I,B}(V_{l,p})$, one by one, as in Theorem 3.1, we obtain the operator $H_{I,\tilde{B}}(\tilde{V}_{l,p})$ with no negative eigenvalues, where

$$\tilde{V}_{l,p}(x) = V_{l,p}(x) + 2\sum_j \frac{d}{dx}\left[\Phi_j^{(l,p)}(x) \left(W_j^{(l,p)}(x)\right)^+ \left(\Phi_j^{(l,p)}(x)\right)^\dagger\right], \tag{4.8}$$

where $\Phi_j^{(l,p)}(x)$ is the Gel-fand–Levitan normalized matrix solution for the eigenvalue $\lambda_j^{(l,p)} = -\left(\kappa_j^{(l,p)}\right)^2$ of $H_{I,B}(V_{l,p})$, and $W_j^{(l,p)}(x)$ is defined as in (3.2), but with $\Phi_j^{(l,p)}(x)$ instead of $\Phi_N(x)$. Further,

$$\tilde{B} = B + \sum_j \left[C_j^{(l,p)}\right]^2, \tag{4.9}$$

where $C_j^{(l,p)}$ is the Gel-fand–Levitan normalization matrix for the negative eigenvalue $\lambda_j^{(l,p)}$ of $H_{I,B}(V_{l,p})$. Note that by (2.6)

$$W_j^{(l,p)}(0) = Q_j^{(l,p)}, \quad j = 1, \dots, \tag{4.10}$$

with $Q_j^{(l,p)}$ the orthogonal projection onto the kernel of $J^{(l,p)}\left(i\kappa_j^{(l,p)}\right)$. The quantity $J^{(l,p)}(k)$ is the Jost matrix of $H_{I,B}(V_{l,p})$. Further, we used that $\left(Q_j^{(l,p)}\right)^+ = Q_j^{(l,p)}$ as it can be easily verified using the definition (3.1) and that $Q_j^{(l,p)}$ is an orthogonal projection. Moreover, by equations (6.98), (6.99), (6.108), and (6.110) of [2],

$$\lim_{x\to\infty}\left[\Phi_j^{(l,p)}(x) \left(W_j^{(l,p)}(x)\right)^+ \Phi_j^{(l,p)}(x)^\dagger\right] = 2\kappa_j^{(l,p)} P_j^{(l,p)}, \quad j = 1, \dots, \tag{4.11}$$

where $P_j^{(l,p)}$ is the orthogonal projection onto the kernel of $\left(J^{(l,p)}\left(i\kappa_j^{(l,p)}\right)\right)^\dagger$. The dimension of Ker $\left[\left(J^{(l,p)}\left(i\kappa_j^{(l,p)}\right)\right)^\dagger\right]$ is $m_j^{(l,p)}$, that is also the dimension of Ker $\left[J^{(l,p)}\left(i\kappa_j^{(l,p)}\right)\right]$.

By (2.10), (4.8), (4.10), and (4.11),

$$\int_0^\infty \tilde{V}_{l,p}(x)\, dx = \int_0^\infty V_{l,p}(x)\, dx - 2\sum_j \left[C_j^{(l,p)}\right]^2 + 4\sum_j \sqrt{\left|\lambda_j^{(l,p)}\right|}\, P_j^{(l,p)}. \tag{4.12}$$

For later use we prove that

$$\int_0^\infty \mathrm{Tr}[\tilde{V}_{l,p}(x)]\, dx + \mathrm{Tr}[\tilde{B}] \geq 0. \tag{4.13}$$

This statement was proved in the scalar case in [6] using results in the Gel'fand-Levitan method that are known in the scalar case. Here we prove that (4.13) is an immediate consequence of the fact that as $H_{I,\tilde{B}}(\tilde{V}_{l,p})$ has no negative eigenvalues it is a nonnegative operator. Let us denote by $H^1(\mathbb{R}^+, \mathbb{C}^n)$ the Sobolev space of all functions in $L^2(\mathbb{R}^+, \mathbb{C}^n)$, with the first derivative in $L^2(\mathbb{R}^+, \mathbb{C}^n)$. Then, the quadratic form of $H_{I,\tilde{B}}(\tilde{V}_{l,p})$ is given by

$$q_{I,\tilde{B},\tilde{V}_{l,p}}(\phi,\psi) := \sum_{i=1}^n \left(\phi_j', \psi_j'\right) + (\tilde{V}_{l,p}\phi, \psi) + \phi^\dagger(0)\tilde{B}\psi(0), \quad \phi, \psi \in H^1(\mathbb{R}^+, \mathbb{C}^n), \tag{4.14}$$

with domain $H^1(\mathbb{R}^+, \mathbb{C}^n)$. Let $f \in C^1([0,\infty))$ be real valued and satisfy $f(x) = 1$, $0 \leq x \leq 1$, $f(x) = 0$, $x \geq 2$. For $r = 1, \ldots, n$, $s = 1, \ldots$, denote

$$\phi^{(r,s)} := (0, \ldots, f(x/s), 0, \ldots, 0)^T,$$

with $f(x/s)$ in the r position. Moreover, since $H_{I,\tilde{B}}(\tilde{V}_{l,p})$ has no negative eigenvalues the quadratic form (4.14) is nonnegative, and hence,

$$q_{I,\tilde{B},\tilde{V}_{l,p}}(\phi^{(r,s)}, \phi^{(r,s)}) := \frac{1}{s}\int_0^\infty (f'(y))^2\, dy + \int_0^\infty \left(\tilde{V}_{l,p}\right)_{r,r}(x)\, f^2(x/s)\, dx + \tilde{B}_{r,r} \geq 0, \tag{4.15}$$

where we denote by $\left(\tilde{V}_{l,p}\right)_{r,r}(x)$ the r,r entry of $\tilde{V}_{l,p}(x)$, and by $\tilde{B}_{r,r}$ the r,r entry of B. Taking the limit as $s \to \infty$ in (4.15) we get,

$$\int_0^\infty dx\, \left(\tilde{V}_{l,p}\right)_{r,r}(x) + \tilde{B}_{r,r} \geq 0, \quad r = 1, \ldots, n. \tag{4.16}$$

Equation (4.13) follows from (4.16). Furthermore, by (4.9), (4.12) and (4.13),

$$\sum_j m_j^{(l,p)} \sqrt{\left|\lambda_j^{(l,p)}\right|} \geq \frac{1}{4}\left[-\int_0^\infty \mathrm{Tr}\left[V_{l,p}(x)\right]\, dx - \mathrm{Tr}[B] + \sum_j \mathrm{Tr}\left[\left(C_j^{(l,p)}\right)^2\right]\right]. \tag{4.17}$$

We now prove that the last term in the right-hand side of (4.17) is bounded below by a positive constant uniformly in l and p. For this purpose, it is enough to prove that this is so for $\mathrm{Tr}\left[\left(C_1^{(l,p)}\right)^2\right]$. Let $v_j^{(l,p)}$, for $j = 1,\dots,m_1^{(l,p)}$ be an orthonormal basis of the kernel of $J^{l,p}\left(i\sqrt{|\lambda_1^{(l,p)}|}\right)$. Then,

$$Q_1^{(l,p)} = \sum_{j=1}^{m_1^{(l,p)}} v_j^{(l,p)} \left(v_j^{(l,p)}\right)^{\dagger}.$$

Let us denote by $\varphi^{(l,p)}(k,x)$, respectively, $f^{(l,p)}(k,x)$, the regular solution and the Jost solution for the potential $V_{l,p}$ with the boundary condition (1.8). By Theorem 3.11.1 of [1],

$$\varphi^{(l.p)}\left(i\sqrt{|\lambda_1^{(l,p)}|},x\right) Q_1^{(l,p)} = \sum_{j=1}^{m_1^{(l,p)}} f^{(l.p)}\left(i\sqrt{|\lambda_1^{(l,p)}|},x\right) \omega_j^{(l,p)} \left(v_j^{(l,p)}\right)^{\dagger}, \tag{4.18}$$

where $w_j^{(l,p)}$ belongs to the kernel of $J^{l,p}\left(i\sqrt{|\lambda_1^{(l,p)}|}\right)^{\dagger}$, and

$$\varphi^{(l.p)}\left(i\sqrt{|\lambda_1^{(l,p)}|},x\right) v_j^{(l,p)} = f^{(l.p)}\left(i\sqrt{|\lambda_1^{(l,p)}|},x\right) w_j^{(l,p)}, \quad j = 1,\dots,m_1^{(l,p)}.$$

It follows from the proof of Proposition 3.2.1 of [1] that the Jost solution $f^{(l,p)}\left(i\sqrt{|\lambda_1^{(l,p)}|},x\right)$ satisfies (2.3) with the $o(1)$ uniform in $l,p = 1,\dots$ Further, it follows from the proof of Proposition 3.2.9 of [1] that the regular solution $\varphi^{(l.p)}\left(i\sqrt{|\lambda_1^{(l,p)}|},x\right)$ is bounded in any interval $[0,x_0], x_0 > 0$, uniformly for all $l,p = 1,\dots$ Take an x_0 so large that $f^{(l,p)}\left(i\sqrt{|\lambda_1^{(l,p)}|},x_0\right)$ is invertible and $|o(1)| < 1/2$. Then,

$$w_j^{(l,p)} = f^{(l,p)}\left(i\sqrt{|\lambda_1^{(l,p)}|},x_0\right)^{-1} \varphi^{(l.p)}\left(i\sqrt{|\lambda_1^{(l,p)}|},x\right) v_j^{(l,p)}. \tag{4.19}$$

It follows from (4.19) that

$$|w_j^{(l,p)}| \leq C, \quad j = 1,\dots m_1^{(l,p)}, \tag{4.20}$$

where the constant C is uniform in $l,p = 1\dots$ Then, by (2.3), (2.7), (4.18), and (4.20),

$$|\mathbf{G}_1^{(l,p)}| \leq C, \tag{4.21}$$

where the constant C is uniform in $l,p = 1\dots$ Further, by (2.8), (2.9), and (4.21),

$$\mathrm{Tr}\left[C_1^{(l,p)}\right]^2 \geq \delta > 0, \tag{4.22}$$

where the positive constant δ is uniform on $l, p = 1, \dots$ By (4.17) and (4.22),

$$\sum_j m_j^{(l,p)} \sqrt{\left|\lambda_j^{(l,p)}\right|} \geq \frac{1}{4} \left[-\int_0^\infty \operatorname{Tr}\left[V_{l,p}(x)\right] \, dx - \operatorname{Tr}[B] + \delta \right]. \tag{4.23}$$

Moreover, by (4.7) and (4.23),

$$\sum_j m_j^{(l)} \sqrt{\left|\lambda_j^{(l)}\right|} \geq \frac{1}{4} \left[-\int_0^\infty \operatorname{Tr}\left[V_l(x)\right] \, dx - \operatorname{Tr}[B] + \delta \right]. \tag{4.24}$$

Finally, by (4.2) and (4.24),

$$\sum_{j=1}^N m_j \sqrt{|\lambda_j|} > \frac{1}{4} \left[-\int_0^\infty \operatorname{Tr}\left[V_l(x)\right] \, dx - \operatorname{Tr}[B] \right].$$

Let us prove that the constant 1/4 in (1.9) is sharp. We consider the matrix Schrödinger operator $H_{I,0}(0)$ with the Neumann boundary condition $\psi'(0) = 0$, and potential identically zero. The operator $H_{I,0}(0)$ has no eigenvalues. Then, using the results in Section 8 of [2] we add to $H_{I,0}(0)$ a negative eigenvalue, $\lambda_1 = -\kappa_1^2, \kappa_1 > 0$, with multiplicity m_1 and with Gel'fand–Levitan norming constant C_1, to obtain the matrix Schrödinger operator $H_{I,B}(V)$ with $B = -C_1^2$, and where V is integrable, and it satisfies $V(x) = O(xe^{-2\kappa_1 x})$, $x \to \infty$. By equations (8.17), (8.22), (8.37), (8.51), and (8.54) of [2], since $B = -C_1^2$, and as we added a bound state to the identically zero potential,

$$\sqrt{|\lambda_1|}\, P_1 = \frac{1}{4} \left[-\int_0^\infty V(x) \, dx - B + C_1^2 \right], \tag{4.25}$$

where we used that the orthogonal projection, Q_1, onto the kernel of the Jost matrix, $J(k)$, of $H_{I,B}(V)$ at $k = i\kappa_1$, satisfies $Q_1 = Q_1^+$. Recall that P_1 is the orthogonal projection onto the kernel of $J(i\kappa_1)^\dagger$. Taking traces in both sides of (4.25) we obtain

$$m_1 \sqrt{|\lambda_1|} = \frac{1}{4} \left[-\int_0^\infty \operatorname{Tr}[V](x) \, dx - \operatorname{Tr}[B] + \operatorname{Tr}[C_1^2] \right]. \tag{4.26}$$

Assume that (1.9) holds with 1/4 replaced by $1/\alpha$, with $0 < \alpha < 4$, that is to say,

$$m_1 \sqrt{|\lambda_1|} > \frac{1}{\alpha} \left[-\int_0^\infty \operatorname{Tr}\left[V(x)\right] \, dx - \operatorname{Tr}[B] \right]. \tag{4.27}$$

Then, by (4.26) and (4.27),

$$\left(\frac{1}{4} - \frac{1}{\alpha} \right) \left[-\int_0^\infty \operatorname{Tr}[V](x) \, dx - \operatorname{Tr}[B] \right] + \frac{1}{4} \operatorname{Tr}[C_1^2] > 0. \tag{4.28}$$

Introducing (4.26) in the left-hand side of (4.28) we get

$$\left(\frac{1}{4}-\frac{1}{\alpha}\right)\left[4m_1\sqrt{|\lambda_1|}-\operatorname{Tr}\left[C_1^2\right]\right]+\frac{1}{4}\operatorname{Tr}\left[C_1^2\right]\right]>0. \tag{4.29}$$

Keeping λ_1 and m_1 fixed, and taking $\operatorname{Tr}[C_1^2]$ small enough we reach a contradiction in (4.29). This proves that the constant $1/4$ in (1.9) is sharp. In the scalar case a similar argument was used in [6]. This completes the proof of Theorem 1.1.

Remark 4.1. As mentioned in the introduction taking the boundary matrix A invertible in the boundary condition (1.3) amounts to exclude Dirichlet boundary conditions in the diagonal representation where the boundary matrices are given by $\hat{A}$, $\hat{B}$. Formally the purely Dirichlet boundary condition $\psi(0)=0$ corresponds to taking $B\to\infty$ in which case the reverse Lieb–Thirring inequality amounts to $\sum_j\sqrt{|\lambda_j|}>-\infty$, which is, of course, trivially always satisfied. Moreover, as is well known, in the case of the purely Dirichlet boundary condition in the small coupling constant limit there are no bound states. For the reader's convenience we give the simple proof of this fact assuming that the potential belongs to the Faddeev class $L_1^1(\mathbb{R}^+)$.

Consider the matrix Schrödinger operator with purely Dirichlet boundary condition $H_{0,I}(\beta\, Q)$, where the coupling constant β is a real number and the selfadjoint matrix potential $Q\in L_1^1(\mathbb{R}^+)$. Let us denote by $H_0^1(\mathbb{R}^+,\mathbb{C}^n)$ the completion of $C_0^\infty(\mathbb{R}^+,\mathbb{C}^n)$ in the norm of $H^1(\mathbb{R}^+,\mathbb{C}^n)$. The quadratic form of $H_{0,I}(\beta\, Q)$ is given by

$$q_{0,I,\beta\, Q}(\phi,\psi):=\sum_{i=1}^{n}\left(\phi_j',\psi_j'\right)+\beta(Q\phi,\psi),\quad \phi,\psi\in H_0^1(\mathbb{R}^+,\mathbb{C}^n). \tag{4.30}$$

As for $\phi\in H_0^1(\mathbb{R}^+,\mathbb{C}^n)$ we have $\phi(0)=0$,

$$\phi_j(x)=\int_0^x\phi_j(y)'\,dy,\quad j=1,\dots,n.$$

Then, by Schwarz's inequality,

$$|\phi_j(x)|\le\|\phi_j'\|_{L^2(\mathbb{R}^+)}\sqrt{x},\quad j=1,\dots,n. \tag{4.31}$$

Moreover, by (4.30) and (4.31)

$$q_{0,I,\beta Q}(\phi,\phi)\ge\left[\sum_{i=1}^{n}\left(\phi_j',\phi_j'\right)\right]\left(1-|\beta|\int_0^\infty x\,|V(x)|\,dx\right),\quad \phi\in H_0^1(\mathbb{R}^+,\mathbb{C}^n).$$

It follows that

$$q_{0,I,\beta\, Q}(\phi,\phi)\ge 0,$$

if

$$1-|\beta|\int_0^\infty x\,|V(x)|\,dx\ge 0.$$

Finally, $H_{0,I}(\beta Q)$ has no negative eigenvalues if (4.1) holds.

In conclusion, we have excluded Dirichlet boundary conditions to obtain a meaningful reverse Lieb–Thirring inequality.

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