ARCHIVES OF MECHANICS
Arch. Mech. **75** (5), 559–590, 2023, DOI: 10.24423/aom.4391

Potential method in the coupled linear quasi-static theory of thermoelasticity for double porosity materials

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This paper concerns the coupled linear quasi-static theory of thermoelasticity for materials with double porosity under local thermal equilibrium. The system of equations of this theory is based on the constitutive equations, Darcy's law of the flow of a fluid through a porous medium, Fourier's law of heat conduction, the equations of equilibrium, fluid mass conservation and heat transfer. By virtue of Green's identity the uniqueness theorems for classical solutions of the internal and external quasi-static boundary value problems (BVPs) are proved. The fundamental solution of the system of steady vibration equations in the considered theory is constructed and its basic properties are established. Then, the surface and volume potentials are presented and their basic properties are given. Finally, on the basis of these results the existence theorems for classical solutions of the above mentioned BVPs are proved by means of the potential method (boundary integral equation method) and the theory of singular integral equations.

Key words: quasi-static, materials with double porosity, uniqueness and existence theorems, potential method.

1. Introduction

For over a century, predicting the mechanical properties of materials with single and multiple porosity has been a prominent area of interest in continuum mechanics. Understanding the mechanical effects of coupling processes is crucial for modern theories of porous media, as many engineering problems involve multiple coupled mechanical concepts. Such processes play a fundamental role in various applications of porous materials in engineering, biology, and geology.

The first quasi-static theory of poroelasticity based on Darcy's law was proposed by Biot in the paper [1] in which a coupling effect between fluid pressure and mechanical stress is introduced. The basic results and historical information on the poroelasticity and thermoporoelasticity for single-porosity materials can be found in the books by Cheng [2], Coussy [3], Selvadurai and Suvorov [4], WANG [5] and the references therein. Later, Biot's classical model was generalized and the mathematical model of double porosity materials based on Darcy's law were developed by WILSON and AIFANTIS [6], and studied by several researchers (see [7–13]).

On the other hand, by using the concept of volume fraction the theory of thermoelasticity for materials with double-porosity structure is presented by IEŞAN and QUINTANILLA [14] as extension to the single-porosity model of NUNZIATO and COWIN [15, 16]. The basic problems of the theories of elasticity and thermoelasticity for materials with double voids are studied by IEŞAN [17], CHIRITA and ARUSOAIE [18], DE CICCO and IESAN [19], DE CICCO [20], KUMAR *et al.* [21], SVANADZE [22, 23]. More general models of the theories of elasticity and thermoelasticity for materials with voids based on the volume fraction concept are introduced and intensively investigated. Basic results in this subject may be found in the books of CIARLETTA and IEŞAN [24], IEŞAN [25], STRAUGHAN [26] and the references therein.

Recently, the coupled linear theories of elasticity and thermoelasticity for materials with single and double porosity have been presented by SVANADZE [27–30] in which the coupled effect of Darcy's law and the concept of the volume fraction is developed. Moreover, the basic BVPs of the coupled quasi-static theories of elasticity and thermoelasticity for solids with single porosity are studied by MIKELASHVILI [31, 32]. More recently, in the paper [33], the same author has investigated the BVPs of steady vibrations of the coupled quasi-static theory of elasticity for double porosity materials by means of the potential method. A wide information on the potential method is given in the books by KUPRADZE *et al.* [34] and SVANADZE [35].

In the present paper, a quasi-static mathematical model for thermoelastic double-porosity materials is introduced in which the coupled phenomenon of Darcy's law and the concept of the volume fractions of two levels of pores (macro- and micropores) is proposed. The goal of this work is to prove existence and uniqueness theorems for classical solutions of the basic internal and external BVPs of steady vibrations in the coupled linear quasi-static theory of double-porosity materials.

This paper is articulated as follows. In Section 2, the system of equations of the quasi-static mathematical model for thermoelastic double-porosity materials is expressed in terms of the displacement vector field, the changes of the volume fractions of pores and fissures, the fluid pressures in pore and fissure networks and the temperature. In Section 3, the basic internal and external BVPs of steady vibrations of the considered theory are formulated, and in Section 4, the uniqueness theorems for classical solutions of these BVPs are proved. Afterwards, in Section 5, the fundamental solution of the system of steady vibration equations is constructed and its basic properties are established. In Section 6, the surface and volume potentials are introduced and their basic properties are given. Moreover, the basic properties of some useful singular integral operators are established. Finally, in Section 7, the existence theorems for classical solutions of the above mentioned BVPs are proved by means of the potential method and the theory of singular integral equations.

2. Governing equations

Let $\mathbf{x} = (x_1, x_2, x_3)$ be a point of the Euclidean three-dimensional space $\mathbb{R}^3$ and let t denotes the time variable, $t \geq 0$. We assume that an isotropic and homogeneous elastic solid with double porosity structure occupies a region of $\mathbb{R}^3$. This structure of materials means that the skeleton of solid consists of pores on the macro scale and pores on a much smaller micro scale (also called fissures). Afterwards, in this section, functions and vectors that depend on the space variable $\mathbf{x}$ and the time t are denoted with hat.

Let $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$ be the displacement vector in solid, $\hat{\varphi}_1$ and $\hat{\varphi}_2$ are the changes of the volume fractions of pores and fissures, respectively; $\hat{p}_1$ and $\hat{p}_2$ are the changes of the fluid pressures in pores and fissures networks, respectively, and $\hat{\theta}$ is the temperature measured from some constant absolute temperature T_0 (> 0). Moreover, throughout this paper, we shall employ the usual summation and differentiation conventions: (i) repeated Latin and Greek indices are summed over the ranges $(1, 2, 3)$ and $(1, 2)$, respectively; (ii) the subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate; (iii) a superposed dot denotes differentiation with respect to t.

Following [32, 33], the governing system of field equations in the coupled linear quasi-static theory of thermoelasticity for materials with double porosity consists of the following six sets of equations:

• *The equilibrium equations*

$$
\begin{aligned}
&\hat{t}_{lj,j} = -\rho \hat{F}'_l, \qquad \hat{\sigma}^{(1)}_{j,j} + \hat{\xi}^{(1)} = -\rho \hat{s}_1, \\
&\hat{\sigma}^{(2)}_{j,j} + \hat{\xi}^{(2)} = -\rho \hat{s}_2, \qquad l = 1, 2, 3,
\end{aligned}
\tag{2.1}
$$

where $\hat{t}_{lj}$ is the component of total stress tensor, $\hat{\mathbf{F}}' = (\hat{F}'_1, \hat{F}'_2, \hat{F}'_3)$ is the body force per unit mass, ρ (> 0) is the reference mass density; $\hat{\sigma}^{(1)}_j, \hat{\xi}^{(1)}, \hat{s}_1$ and $\hat{\sigma}^{(2)}_j, \hat{\xi}^{(2)}, \hat{s}_2$ are the components of the equilibrated stress, the intrinsic equilibrated body force, the extrinsic equilibrated body force associated macro and micro pore networks, respectively:

$$
\begin{aligned}
\hat{\xi}^{(1)} &= -b_1 \hat{e}_{rr} - \alpha_1 \hat{\varphi}_1 - \alpha_3 \hat{\varphi}_2 + m_1 \hat{p}_1 + m_3 \hat{p}_2 + \varepsilon_1 \hat{\theta}, \\
\hat{\xi}^{(2)} &= -b_2 \hat{e}_{rr} - \alpha_3 \hat{\varphi}_1 - \alpha_2 \hat{\varphi}_2 + m_3 \hat{p}_1 + m_2 \hat{p}_2 + \varepsilon_2 \hat{\theta},
\end{aligned}
\tag{2.2}
$$

$\hat{e}_{lj}$ is the component of strain tensor and given by:

$$\hat{e}_{lj} = \tfrac{1}{2}(\hat{u}_{l,j} + \hat{u}_{j,l}), \qquad l,j = 1,2,3. \tag{2.3}$$

- *The constitutive equations*

$$\begin{aligned} \hat{t}_{lj} &= 2\mu\hat{e}_{lj} + \lambda\hat{e}_{rr}\delta_{lj} + (b_\alpha\hat{\varphi}_\alpha - \beta_\alpha\hat{p}_\alpha - \varepsilon_0\hat{\theta})\delta_{lj}, \\ \hat{\sigma}_j^{(1)} &= a_1\hat{\varphi}_{1,l} + a_3\hat{\varphi}_{2,l}, \qquad \hat{\sigma}_j^{(2)} = a_3\hat{\varphi}_{1,l} + a_2\hat{\varphi}_{2,l}, \\ \rho\hat{\eta} &= \varepsilon_0\hat{e}_{rr} + \varepsilon_\alpha\hat{\varphi}_\alpha + \varepsilon_{\alpha+2}\hat{p}_\alpha + a\hat{\theta}, \qquad l,j = 1,2,3, \end{aligned} \tag{2.4}$$

where δ_{lj} is the Kronecker delta, $\hat{\eta}$ is the entropy per unit mass.

- *The equations of fluid mass conservation*

$$\begin{aligned} \hat{v}_{j,j}^{(1)} + \dot{\hat{\zeta}}_1 + \beta_1\dot{\hat{e}}_{rr} + \gamma_0(\hat{p}_1 - \hat{p}_2) &= 0, \\ \hat{v}_{j,j}^{(2)} + \dot{\hat{\zeta}}_2 + \beta_2\dot{\hat{e}}_{rr} - \gamma_0(\hat{p}_1 - \hat{p}_2) &= 0, \end{aligned} \tag{2.5}$$

where $\hat{\mathbf{v}}^{(1)} = (\hat{v}_1^{(1)}, \hat{v}_2^{(1)}, \hat{v}_3^{(1)})$ and $\hat{\mathbf{v}}^{(2)} = (\hat{v}_1^{(2)}, \hat{v}_2^{(2)}, \hat{v}_3^{(2)})$ are the fluid flux vectors associated to the macro and micro pore networks, respectively; γ_0 (≥ 0) is the internal transport coefficient and corresponds to a fluid transfer rate respecting the intensity of the flow between macro and micro pores,

$$\begin{aligned} \hat{\zeta}_1 &= \gamma_1\hat{p}_1 + \gamma_3\hat{p}_2 + m_1\hat{\varphi}_1 + m_3\hat{\varphi}_2 + \varepsilon_3\hat{\theta}, \\ \hat{\zeta}_2 &= \gamma_3\hat{p}_1 + \gamma_2\hat{p}_2 + m_3\hat{\varphi}_1 + m_2\hat{\varphi}_2 + \varepsilon_4\hat{\theta}. \end{aligned} \tag{2.6}$$

- *Darcy's extended law*

$$\begin{aligned} \hat{\mathbf{v}}^{(1)} &= -\frac{\kappa_1'}{\mu'}\nabla\hat{p}_1 - \frac{\kappa_3'}{\mu'}\nabla\hat{p}_2 - \rho_1\hat{\mathbf{s}}_3, \\ \hat{\mathbf{v}}^{(2)} &= -\frac{\kappa_3'}{\mu'}\nabla\hat{p}_1 - \frac{\kappa_2'}{\mu'}\nabla\hat{p}_2 - \rho_2\hat{\mathbf{s}}_4, \end{aligned} \tag{2.7}$$

where κ_j' $(j = 1,2,3)$ is the macro-permeability inside the double porosity material, ρ_1, $\hat{\mathbf{s}}_3$ and ρ_2, $\hat{\mathbf{s}}_4$ are the density of fluid, the external force (such as gravity) for the macro and micro pore networks, respectively; ∇ is the gradient operator.

- *Fourier's law of heat conduction*

$$\hat{\mathbf{q}} = -\kappa\nabla\hat{\theta}, \tag{2.8}$$

where $\hat{\mathbf{q}}$ is the heat flux vector and κ (> 0) is the thermal conductivity of the porous material.

- *The heat transfer equation*

$$\operatorname{div}\hat{\mathbf{q}} = -T_0\dot{\hat{\eta}} + \rho\hat{s}_5, \tag{2.9}$$

where s_5 is the heat supply per unit mass.

Substituting Eqs. (2.2)–(2.4) and (2.6)–(2.8) into (2.1), (2.5) and (2.9) we obtain the following system of equations in the coupled linear quasi-static theory of thermoelastic double-porosity materials expressed in terms of the displacement vector $\hat{\mathbf{u}}$, the changes of the volume fractions $\hat{\varphi}_1, \hat{\varphi}_2$, the changes of the fluid pressures $\hat{p}_1, \hat{p}_2$ and the changes of the temperature $\hat{\theta}$:

$$
\begin{aligned}
&\mu\Delta\hat{\mathbf{u}}+(\lambda+\mu)\nabla \operatorname{div}\hat{\mathbf{u}}+b_\alpha\nabla\hat{\varphi}_\alpha-\beta_\alpha\nabla\hat{p}_\alpha-\varepsilon_0\nabla\hat{\theta}=-\rho\hat{\mathbf{F}}',\\
&(a_1\Delta-\alpha_1)\hat{\varphi}_1+(a_3\Delta-\alpha_3)\hat{\varphi}_2-b_1 \operatorname{div}\hat{\mathbf{u}}+m_1\hat{p}_1+m_3\hat{p}_2+\varepsilon_1\hat{\theta}=-\rho\hat{s}_1,\\
&(a_3\Delta-\alpha_3)\hat{\varphi}_1+(a_2\Delta-\alpha_2)\hat{\varphi}_2-b_2 \operatorname{div}\hat{\mathbf{u}}+m_3\hat{p}_1+m_2\hat{p}_2+\varepsilon_2\hat{\theta}=-\rho\hat{s}_2,\\
&k_1\Delta\hat{p}_1+k_3\Delta\hat{p}_2-\gamma_1\dot{\hat{p}}_1-\gamma_3\dot{\hat{p}}_2-\beta_1 \operatorname{div}\dot{\hat{\mathbf{u}}}-m_1\dot{\hat{\varphi}}_1-m_3\dot{\hat{\varphi}}_2-\varepsilon_3\dot{\hat{\theta}}\\
&\qquad\qquad\qquad\qquad-\gamma_0(\hat{p}_1-\hat{p}_2)=-\rho_1 \operatorname{div}\hat{\mathbf{s}}_3,\\
&k_3\Delta\hat{p}_1+k_2\Delta\hat{p}_2-\gamma_3\dot{\hat{p}}_1-\gamma_2\dot{\hat{p}}_2-\beta_2 \operatorname{div}\dot{\hat{\mathbf{u}}}-m_3\dot{\hat{\varphi}}_1-m_2\dot{\hat{\varphi}}_2-\varepsilon_4\dot{\hat{\theta}}\\
&\qquad\qquad\qquad\qquad+\gamma_0(\hat{p}_1-\hat{p}_2)=-\rho_2 \operatorname{div}\hat{\mathbf{s}}_4,\\
&\kappa\Delta\hat{\theta}-T_0(a\dot{\hat{\theta}}+\varepsilon_0 \operatorname{div}\dot{\hat{\mathbf{u}}}+\varepsilon_\alpha\dot{\hat{\varphi}}_\alpha+\varepsilon_{\alpha+2}\dot{\hat{p}}_\alpha)=-\rho\hat{s}_5,
\end{aligned}
\tag{2.10}
$$

where Δ is the Laplacian operator and $k_l=\frac{k'}{\mu'}$ $(l=1,2,3)$.

If we assume that $\hat{u}_j, \hat{F}'_j, \hat{\varphi}_l, \hat{p}_l,\ \hat{s}_l, \hat{\mathbf{s}}_{l+2}, \hat{s}_5$ and $\hat{\theta}$ $(l=1,2,\ j=1,2,3)$ are postulated to have a harmonic time variation

$$\{\hat{u}_j, \hat{F}'_j, \hat{\varphi}_l, \hat{p}_l,\ \hat{s}_l, \hat{\mathbf{s}}_{l+2}, \hat{s}_5, \hat{\theta}\}(\mathbf{x},t)=\operatorname{Re}[\{u_j, F'_j, \varphi_l, p_l, s_l, \mathbf{s}_{l+2}, s_5, \theta\}(\mathbf{x})\, e^{-i\omega t}],$$

then from (2.10) we obtain the following system of equations of steady vibrations in the theory under consideration:

$$
\begin{aligned}
&\mu\Delta\mathbf{u}+(\lambda+\mu)\nabla \operatorname{div}\mathbf{u}+b_\alpha\nabla\varphi_\alpha-\beta_\alpha\nabla p_\alpha-\varepsilon_0\nabla\theta=-\rho\mathbf{F}',\\
&(a_1\Delta-\alpha_1)\varphi_1+(a_3\Delta-\alpha_3)\varphi_2-b_1 \operatorname{div}\mathbf{u}+m_1p_1+m_3p_2+\varepsilon_1\theta=-\rho s_1,\\
&(a_3\Delta-\alpha_3)\varphi_1+(a_2\Delta-\alpha_2)\varphi_2-b_2 \operatorname{div}\mathbf{u}+m_3p_1+m_2p_2+\varepsilon_2\theta=-\rho s_2,\\
&(k_1\Delta+\gamma'_1)p_1+(k_3\Delta+\gamma'_3)p_2+\beta'_1 \operatorname{div}\mathbf{u}+m'_1\varphi_1+m'_3\varphi_2+\varepsilon'_3\theta=-\rho_1 \operatorname{div}\mathbf{s}_3,\\
&(k_3\Delta+\gamma'_3)p_1+(k_2\Delta+\gamma'_2)p_2+\beta'_2 \operatorname{div}\mathbf{u}+m'_3\varphi_1+m'_2\varphi_2+\varepsilon'_4\theta=-\rho_2 \operatorname{div}\mathbf{s}_4,\\
&(\kappa\Delta+a')\theta+\varepsilon'_0 \operatorname{div}\mathbf{u}+\varepsilon'_\alpha\varphi_\alpha+\varepsilon'_{\alpha+2}T_0p_\alpha=-\rho s_5,
\end{aligned}
\tag{2.11}
$$

where $\mathbf{u}=(u_1,u_2,u_3)$, $\mathbf{F}'=(F'_1,F'_2,F'_3)$, ω (>0) is the oscillation frequency, $\beta'_l=i\omega\beta_l$, $m'_j=i\omega m_j$, $\gamma'_l=i\omega\gamma_l-\gamma_0$, $\gamma'_3=i\omega\gamma_3+\gamma_0$ $(l=1,2,\ j=1,2,3)$.

For further considerations we need the following second order matrix differential operator with constant coefficients:

$$\mathbf{M}(\mathbf{D_x}) = (M_{lj}(\mathbf{D_x}))_{8\times 8}, \quad M_{lj} = \mu\Delta\delta_{lj} + (\lambda+\mu)\frac{\partial^2}{\partial x_l \partial x_j},$$

$$M_{l;r+3} = -M_{r+3;l} = b_r\frac{\partial}{\partial x_l}, \quad M_{l;r+5} = -\beta_r\frac{\partial}{\partial x_l}, \quad M_{l8} = -\varepsilon_0\frac{\partial}{\partial x_l},$$

$$M_{44} = a_1\Delta - \alpha_1, \quad M_{45} = M_{54} = a_3\Delta - \alpha_3, \quad M_{55} = a_2\Delta - \alpha_2, \quad M_{46} = m_1,$$

$$M_{47} = M_{56} = m_3, \quad M_{57} = m_2, \quad M_{r+3;8} = \varepsilon_r, \quad M_{r+5;l} = \beta'_r\frac{\partial}{\partial x_l},$$

$$M_{64} = m'_1, \quad M_{65} = M_{74} = m'_3, \quad M_{75} = m'_2, \quad M_{66} = k_1\Delta + \gamma'_1,$$

$$M_{67} = M_{76} = k_3\Delta + \gamma'_3, \quad M_{77} = k_2\Delta + \gamma'_2, \quad M_{r+5;8} = \varepsilon'_{r+2},$$

$$M_{8l} = \varepsilon'_0\frac{\partial}{\partial x_l}, \quad M_{8;r+3} = \varepsilon'_r, \quad M_{8;r+5} = \varepsilon'_{r+2}T_0, \quad M_{88} = \kappa\Delta + a',$$

$$\mathbf{D_x} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right), \quad l, j = 1, 2, 3, \ r = 1, 2.$$

It is easily seen that the system (2.11) can be rewritten in the following form

$$\mathbf{M}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{F}(\mathbf{x}), \tag{2.12}$$

where

$$\mathbf{U} = (\mathbf{u}, \varphi_1, \varphi_2, p_1, p_2, \theta)$$

and

$$\mathbf{F} = (-\rho\mathbf{F}', -\rho s_1, -\rho s_2, -\rho_1 \operatorname{div}\mathbf{s}_3, -\rho_2 \operatorname{div}\mathbf{s}_4, -\rho s_5)$$

are eight-component vector functions, $\mathbf{x} \in \mathbb{R}^3$.

In what follows, we assume that the following inequalities are fulfilled:

$$\begin{aligned}
&\mu > 0, \quad 3\lambda + 2\mu > 0, \quad a_1 > 0, \quad a_1a_2 - a_3^2 > 0, \quad (3\lambda+2\mu)\alpha_1 > 3b_1^2,\\
&\alpha_1\alpha_2 - \alpha_3^2 > 0, \quad \gamma_1 > 0, \quad \gamma_1\gamma_2 - \gamma_3^2 > 0, \quad k_1 > 0, \quad k_1k_2 - k_3^2 > 0,\\
&\frac{1}{3}(3\lambda+2\mu)(\alpha_1\alpha_2 - \alpha_3^2) > \alpha_1 b_2^2 - 2\alpha_3 b_1 b_2 + \alpha_2 b_1^2, \quad \kappa > 0,\\
&a(\gamma_1\gamma_2 - \gamma_3^2) > \gamma_1\varepsilon_4^2 - 2\gamma_3\varepsilon_3\varepsilon_4 + \gamma_2\varepsilon_3^2.
\end{aligned} \tag{2.13}$$

3. Boundary value problems

Let S be the closed surface surrounding the finite domain Ω^+ in $\mathbb{R}^3$, $S \in C^{1,\nu}$, $0 < \nu \le 1$, $\overline{\Omega^+} = \Omega^+ \cup S$, $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$, $\overline{\Omega^-} = \Omega^- \cup S$; $\mathbf{n}(\mathbf{z})$ is the external (with respect to Ω^+) unit normal vector to S at $\mathbf{z}$.

DEFINITION 1. Vector function $\mathbf{U} = (U_1, U_2, \ldots, U_8)$ is called *regular* in Ω^- (or Ω^+) if

(i) $U_l \in C^2(\Omega^-) \cap C^1(\overline{\Omega^-})$ (or $U_l \in C^2(\Omega^+) \cap C^1(\overline{\Omega^+})$),
(ii)

$$U_l(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \qquad U_{l,j}(\mathbf{x}) = o(|\mathbf{x}|^{-1}) \tag{3.1}$$

for $|\mathbf{x}| \gg 1$, where $l = 1, 2, \ldots, 8$ and $j = 1, 2, 3$.

In the sequel, we use the matrix differential operator

$$\mathbf{R}(\mathbf{D_x}, \mathbf{n}) = (R_{lj}(\mathbf{D_x}, \mathbf{n}))_{8\times 8},$$

where

$$\begin{aligned}
&R_{lj}(\mathbf{D_x}, \mathbf{n}) = \mu\delta_{lj}\frac{\partial}{\partial \mathbf{n}} + \mu n_j \frac{\partial}{\partial x_l} + \lambda n_l \frac{\partial}{\partial x_j}, \qquad R_{lr}(\mathbf{D_x}, \mathbf{n}) = b_{r-3}\, n_l,\\
&R_{l;r+2}(\mathbf{D_x}, \mathbf{n}) = -\beta_{r-3}\, n_l, \quad R_{l8}(\mathbf{D_x}, \mathbf{n}) = -\varepsilon_0\, n_l,\\
&R_{44}(\mathbf{D_x}, \mathbf{n}) = a_1 \frac{\partial}{\partial \mathbf{n}}, \quad R_{45}(\mathbf{D_x}, \mathbf{n}) = R_{54}(\mathbf{D_x}, \mathbf{n}) = a_3 \frac{\partial}{\partial \mathbf{n}},\\
&R_{55}(\mathbf{D_x}, \mathbf{n}) = a_2 \frac{\partial}{\partial \mathbf{n}}, \quad R_{66}(\mathbf{D_x}, \mathbf{n}) = k_1 \frac{\partial}{\partial \mathbf{n}},\\
&R_{67}(\mathbf{D_x}, \mathbf{n}) = R_{76}(\mathbf{D_x}, \mathbf{n}) = k_3 \frac{\partial}{\partial \mathbf{n}},\\
&R_{77}(\mathbf{D_x}, \mathbf{n}) = k_2 \frac{\partial}{\partial \mathbf{n}}, \qquad R_{88}(\mathbf{D_x}, \mathbf{n}) = \kappa \frac{\partial}{\partial \mathbf{n}},\\
&R_{sj}(\mathbf{D_x}, \mathbf{n}) = R_{r;m+2}(\mathbf{D_x}, \mathbf{n}) = R_{r+2;m}(\mathbf{D_x}, \mathbf{n}) = R_{r;8}(\mathbf{D_x}, \mathbf{n})\\
&\qquad = R_{r+2;8}(\mathbf{D_x}, \mathbf{n}) = R_{8;r}(\mathbf{D_x}, \mathbf{n}) = R_{8;r+2}(\mathbf{D_x}, \mathbf{n}) = 0,\\
&\qquad\qquad l, j = 1, 2, 3, \ r, m = 4, 5, \ s = 4, 5, 6, 7, 8
\end{aligned} \tag{3.2}$$

and $\frac{\partial}{\partial \mathbf{n}}$ is the derivative along the vector $\mathbf{n}$.

The basic internal and external BVPs of steady vibrations in the coupled linear quasi-static theory of thermoelasticity for materials with double porosity are formulated as follows.

Find a regular (classical) solution to system (2.12) for $\mathbf{x} \in \Omega^+$ satisfying the boundary condition

$$\lim_{\Omega^+ \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{U}(\mathbf{x}) \equiv \{\mathbf{U}(\mathbf{z})\}^+ = \mathbf{f}(\mathbf{z}) \tag{3.3}$$

in the internal *Problem* $(I)^+_{\mathbf{F},\mathbf{f}}$, and

$$\lim_{\Omega^+ \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{R}(\mathbf{D_x}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{x}) \equiv \{\mathbf{R}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^+ = \mathbf{f}(\mathbf{z}) \tag{3.4}$$

in the internal *Problem* $(II)^+_{\mathbf{F},\mathbf{f}}$, where $\mathbf{F}$ and $\mathbf{f}$ are prescribed eight-component vector functions.

Find a regular (classical) solution to system (2.12) for $\mathbf{x} \in \Omega^-$ satisfying the boundary condition

$$\lim_{\Omega^- \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{U}(\mathbf{x}) \equiv \{\mathbf{U}(\mathbf{z})\}^- = \mathbf{f}(\mathbf{z}) \tag{3.5}$$

in the external *Problem* $(I)^-_{\mathbf{F},\mathbf{f}}$, and

$$\lim_{\Omega^- \ni \mathbf{x} \to \mathbf{z} \in S} \mathbf{R}(\mathbf{D_x}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{x}) \equiv \{\mathbf{R}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^- = \mathbf{f}(\mathbf{z}) \tag{3.6}$$

in the external *Problem* $(II)^-_{\mathbf{F},\mathbf{f}}$, where $\mathbf{F}$ and $\mathbf{f}$ are prescribed eight-component vector functions and $\operatorname{supp}\mathbf{F}$ is a finite domain in Ω^-.

Our objective is to establish the existence and uniqueness of classical solutions for the basic BVPs associated with steady vibrations, denoted as $(I)^\pm_{\mathbf{F},\mathbf{f}}$ and $(II)^\pm_{\mathbf{F},\mathbf{f}}$, through the utilization of the potential method. To establish the uniqueness theorems for classical solutions, we rely on Green's first identity. Furthermore, the verification of the existence theorems necessitates the basic properties of surface and volume potentials.

With these outcomes in mind, we are able to reduce the BVPs $(I)^\pm_{\mathbf{F},\mathbf{f}}$ and $(II)^\pm_{\mathbf{F},\mathbf{f}}$ to equivalent singular integral equations, which will be amenable to Noether's theorems.

4. Uniqueness theorems

In this section, Green's first identity of the coupled linear quasi-static theory of thermoelasticity for materials with double porosity is obtained. Then, the uniqueness theorems for the regular (classical) solutions of the BVPs $(I)^\pm_{\mathbf{F},\mathbf{f}}$ and $(II)^\pm_{\mathbf{F},\mathbf{f}}$ are proved.

In what follows, the scalar product of two vectors $\mathbf{U} = (U_1, U_2, \ldots, U_8)$ and $\mathbf{U}' = (U_1', U_2', \ldots, U_8')$ is denoted by $\mathbf{U}\cdot\mathbf{U}' = \sum_{j=1}^{8} U_j \overline{U_j'}$, where $\overline{U_j'}$ is the complex conjugate of U_j'.

In the sequel we use the matrix differential operators:

1)

$$\mathbf{M}^{(0)}(\mathbf{D_x}) = (M_{lj}^{(0)}(\mathbf{D_x}))_{3\times 3}, \qquad M_{lj}^{(0)}(\mathbf{D_x}) = \mu\Delta\delta_{lj} + (\lambda+\mu)\frac{\partial^2}{\partial x_l \partial x_j},$$

$$\mathbf{M}^{(1)}(\mathbf{D_x}) = (M_{lr}^{(1)}(\mathbf{D_x}))_{3\times 8}, \qquad M_{lr}^{(1)}(\mathbf{D_x}) = M_{lr}(\mathbf{D_x}),$$

$$\mathbf{M}^{(m)}(\mathbf{D_x}) = (M_{1r}^{(m)}(\mathbf{D_x}))_{1\times 8}, \qquad M_{1r}^{(m)}(\mathbf{D_x}) = M_{m+2;r}(\mathbf{D_x}),$$

$$\mathbf{M}^{(m+2)}(\mathbf{D_x}) = (M_{1r}^{(m+2)}(\mathbf{D_x}))_{1\times 8}, \qquad M_{1r}^{(m+2)}(\mathbf{D_x}) = M_{m+4;r}(\mathbf{D_x}),$$
$$\mathbf{M}^{(6)}(\mathbf{D_x}) = (M_{1r}^{(6)}(\mathbf{D_x}))_{1\times 8}, \qquad M_{1r}^{(6)}(\mathbf{D_x}) = M_{8r}(\mathbf{D_x});$$

2)
$$\mathbf{R}^{(0)}(\mathbf{D_x},\mathbf{n}) = (R_{lj}^{(0)}(\mathbf{D_x},\mathbf{n}))_{3\times 3}, \qquad R_{lj}^{(0)}(\mathbf{D_x},\mathbf{n}) = R_{lj}(\mathbf{D_x},\mathbf{n}),$$
$$\mathbf{R}^{(1)}(\mathbf{D_x},\mathbf{n}) = (R_{lr}^{(1)}(\mathbf{D_x},\mathbf{n}))_{3\times 8}, \qquad R_{lr}^{(1)}(\mathbf{D_x},\mathbf{n}) = R_{lr}(\mathbf{D_x},\mathbf{n}),$$

where $l,j = 1,2,3$, $m = 2,3$ and $r = 1,2,\dots,8$.

We introduce the notation:

$$\begin{aligned}
W^{(0)}(\mathbf{u},\mathbf{u}') &= \frac{1}{3}(3\lambda+2\mu)\,\operatorname{div}\mathbf{u}\,\operatorname{div}\overline{\mathbf{u}'} \\
&\quad + \frac{\mu}{2}\sum_{l,j=1}^{3}\left(\frac{\partial u_j}{\partial x_l} - \frac{\partial u_l}{\partial x_j}\right)\left(\frac{\partial \overline{u_j'}}{\partial x_l} - \frac{\partial \overline{u_l'}}{\partial x_j}\right) \\
&\quad + \frac{\mu}{3}\sum_{l,j=1}^{3}\left(\frac{\partial u_l}{\partial x_l} - \frac{\partial u_j}{\partial x_j}\right)\left(\frac{\partial \overline{u_l'}}{\partial x_l} - \frac{\partial \overline{u_j'}}{\partial x_j}\right),\\
W^{(1)}(\mathbf{U},\mathbf{u}') &= W^{(0)}(\mathbf{u},\mathbf{u}') + (b_\alpha\,\varphi_\alpha - \beta_\alpha p_\alpha - \varepsilon_0\theta)\,\operatorname{div}\overline{\mathbf{u}'},\\
W^{(2)}(\mathbf{U},\varphi_1') &= (a_1\nabla\varphi_1 + a_3\nabla\varphi_2)\cdot\nabla\varphi_1' \\
&\quad + (b_1\,\operatorname{div}\mathbf{u} + \alpha_1\varphi_1 + \alpha_3\varphi_2 - m_1p_1 - m_3p_2 - \varepsilon_1\theta)\overline{\varphi_1'},\\
W^{(3)}(\mathbf{U},\varphi_2') &= (a_3\nabla\varphi_1 + a_2\nabla\varphi_2)\cdot\nabla\varphi_2' \\
&\quad + (b_2\,\operatorname{div}\mathbf{u} + \alpha_3\varphi_1 + \alpha_2\varphi_2 - m_3p_1 - m_2p_2 - \varepsilon_2\theta)\overline{\varphi_2'},\\
W^{(4)}(\mathbf{U},p_1') &= (k_1\nabla p_1 + k_3\nabla p_2)\cdot\nabla p_1' \\
&\quad - (\beta_1'\,\operatorname{div}\mathbf{u} + m_1'\varphi_1 + m_3'\varphi_2 + \gamma_1'p_1 + \gamma_3'p_2 + \varepsilon_3'\theta)\overline{p_1'},\\
W^{(5)}(\mathbf{U},p_2') &= (k_3\nabla p_1 + k_2\nabla p_2)\cdot\nabla p_2' \\
&\quad - (\beta_2'\,\operatorname{div}\mathbf{u} + m_3'\varphi_1 + m_2'\varphi_2 + \gamma_3'p_1 + \gamma_2'p_2 + \varepsilon_4'\theta)\overline{p_2'},\\
W^{(6)}(\mathbf{U},\theta') &= \kappa\nabla\theta\nabla\theta' - (a'\theta + \varepsilon_0'\,\operatorname{div}\mathbf{u} + \varepsilon_\alpha'\varphi_\alpha + \varepsilon_{\alpha+2}'T_0p_\alpha)\overline{\theta'}.
\end{aligned} \tag{4.1}$$

The subsequent Lemmas prove valuable in investigating the uniqueness of classical solutions for the BVPs $(I)^{\pm}_{\mathbf{F},\mathbf{f}}$ and $(II)^{\pm}_{\mathbf{F},\mathbf{f}}$.

LEMMA 1. *If* $\mathbf{U} = (\mathbf{u},\varphi_1,\varphi_2,p_1,p_2,\theta)$ *is a regular vector in* Ω^+, $u_j',\varphi_1',\varphi_2',p_1',p_2',\theta' \in C^1(\Omega^+)\cap C(\overline{\Omega^+})$, $j=1,2,3$, *then:*

$$\begin{aligned}
&\int_{\Omega^+}\big[\mathbf{M}^{(1)}(\mathbf{D_x})\,\mathbf{U}\cdot\mathbf{u}' + W^{(1)}(\mathbf{U},\mathbf{u}')\big]\,d\mathbf{x} = \int_S \mathbf{R}^{(1)}(\mathbf{D_z},\mathbf{n})\mathbf{U}\cdot\mathbf{u}'\,d_{\mathbf{z}}S,\\
&\int_{\Omega^+}\big[\mathbf{M}^{(2)}(\mathbf{D_x})\,\mathbf{U}\,\overline{\varphi_1'} + W^{(2)}(\mathbf{U},\varphi_1')\big]\,d\mathbf{x} = \int_S\left(a_1\frac{\partial\varphi_1}{\partial\mathbf{n}} + a_3\frac{\partial\varphi_2}{\partial\mathbf{n}}\right)\overline{\varphi_1'}\,d_{\mathbf{z}}S,
\end{aligned} \tag{4.2}$$

$$\int_{\Omega^+} \big[\mathbf{M}^{(3)}(\mathbf{D_x})\,\mathbf{U}\,\overline{\varphi_2'} + W^{(3)}(\mathbf{U},\varphi_2')\big]\,d\mathbf{x} = \int_S \Big(a_3\frac{\partial \varphi_1}{\partial \mathbf{n}} + a_2\frac{\partial \varphi_2}{\partial \mathbf{n}}\Big)\overline{\varphi_2'}\,d_{\mathbf{z}}S,$$

$$\int_{\Omega^+} \big[\mathbf{M}^{(4)}(\mathbf{D_x})\,\mathbf{U}\,\overline{p_1'} + W^{(4)}(\mathbf{U},p_1')\big]\,d\mathbf{x} = \int_S \Big(k_1\frac{\partial p_1}{\partial \mathbf{n}} + k_3\frac{\partial p_1}{\partial \mathbf{n}}\Big)\overline{p_1'}\,d_{\mathbf{z}}S, \tag{4.2$_{[\text{cont.}]}$}$$

$$\int_{\Omega^+} \big[\mathbf{M}^{(5)}(\mathbf{D_x})\,\mathbf{U}\,\overline{p_2'} + W^{(5)}(\mathbf{U},p_2')\big]\,d\mathbf{x} = \int_S \Big(k_3\frac{\partial p_2}{\partial \mathbf{n}} + k_2\frac{\partial p_2}{\partial \mathbf{n}}\Big)\overline{p_2'}\,d_{\mathbf{z}}S,$$

$$\int_{\Omega^+} \big[\mathbf{M}^{(6)}(\mathbf{D_x})\,\mathbf{U}\,\overline{\theta'} + W^{(6)}(\mathbf{U},\theta')\big]\,d\mathbf{x} = \kappa\int_S \frac{\partial \theta}{\partial \mathbf{n}}\,\overline{\theta'}\,d_{\mathbf{z}}S,$$

where $\mathbf{u}' = (u_1', u_2', u_3')$ *and* $\mathbf{U}' = (\mathbf{u}', \varphi_1', \varphi_2', p_1', p_2', \theta')$.

Proof. On the basis of Green's first identity of the classical theory of elasticity (see, e.g., KUPRADZE *et al.* [34])

$$\int_{\Omega^+} \big[\mathbf{M}^{(0)}(\mathbf{D_x})\,\mathbf{u}(\mathbf{x})\cdot\mathbf{u}'(\mathbf{x}) + W^{(0)}(\mathbf{u},\mathbf{u}')\big]\,d\mathbf{x} = \int_S \mathbf{R}^{(0)}(\mathbf{D_z},\mathbf{n})\mathbf{u}(\mathbf{z})\cdot\mathbf{u}'(\mathbf{z})\,d_{\mathbf{z}}S,$$

we obtain the first relation of (4.2).

On the other hand, the divergence theorem leads to the following identity

$$\int_{\Omega^+} \big[\Delta\varphi_l(\mathbf{x})\,\overline{\varphi_j'(\mathbf{x})} + \nabla\varphi_l(\mathbf{x})\cdot\nabla\varphi_j'(\mathbf{x})\big]\,d\mathbf{x} = \int_S \frac{\partial \varphi_l(\mathbf{z})}{\partial \mathbf{n}(\mathbf{z})}\,\overline{\varphi_j'(\mathbf{z})}\,d_{\mathbf{z}}S. \tag{4.3}$$

Now, in view of the relations (4.1), from (4.3) we may derive the last five relations of (4.2). □

Lemma 1 and the condition at infinity (3.1) lead to the following result.

LEMMA 2. *If* $\mathbf{U} = (\mathbf{u}, \varphi_1, \varphi_2, p_1, p_2, \theta)$ *and* $\mathbf{U}' = (\mathbf{u}', \varphi_1', \varphi_2', p_1', p_2', \theta')$ *are regular vectors in* Ω^-*, then:*

$$\int_{\Omega^-} \big[\mathbf{M}^{(1)}(\mathbf{D_x})\,\mathbf{U}\cdot\mathbf{u}' + W^{(1)}(\mathbf{U},\mathbf{u}')\big]\,d\mathbf{x} = -\int_S \mathbf{R}^{(1)}(\mathbf{D_z},\mathbf{n})\mathbf{U}\cdot\mathbf{u}'\,d_{\mathbf{z}}S, \tag{4.4}$$

$$\int\limits_{\Omega^-}\big[\mathbf{M}^{(2)}(\mathbf{D_x})\,\mathbf{U}\,\overline{\varphi_1'}+W^{(2)}(\mathbf{U},\varphi_1')\big]\,d\mathbf{x}
=-\int\limits_S\Big(a_1\frac{\partial\varphi_1}{\partial\mathbf{n}}+a_3\frac{\partial\varphi_2}{\partial\mathbf{n}}\Big)\overline{\varphi_1'}\,d_{\mathbf{z}}S,$$

$$\int\limits_{\Omega^-}\big[\mathbf{M}^{(3)}(\mathbf{D_x})\,\mathbf{U}\,\overline{\varphi_2'}+W^{(3)}(\mathbf{U},\varphi_2')\big]\,d\mathbf{x}
=-\int\limits_S\Big(a_3\frac{\partial\varphi_1}{\partial\mathbf{n}}+a_2\frac{\partial\varphi_2}{\partial\mathbf{n}}\Big)\overline{\varphi_2'}\,d_{\mathbf{z}}S,$$

(4.4)[cont.]
$$\int\limits_{\Omega^-}\big[\mathbf{M}^{(4)}(\mathbf{D_x})\,\mathbf{U}\,\overline{p_1'}+W^{(4)}(\mathbf{U},p_1')\big]\,d\mathbf{x}
=-\int\limits_S\Big(k_1\frac{\partial p_1}{\partial\mathbf{n}}+k_3\frac{\partial p_1}{\partial\mathbf{n}}\Big)\overline{p_1'}\,d_{\mathbf{z}}S,$$

$$\int\limits_{\Omega^-}\big[\mathbf{M}^{(5)}(\mathbf{D_x})\,\mathbf{U}\,\overline{p_2'}+W^{(5)}(\mathbf{U},p_2')\big]\,d\mathbf{x}
=-\int\limits_S\Big(k_3\frac{\partial p_2}{\partial\mathbf{n}}+k_2\frac{\partial p_2}{\partial\mathbf{n}}\Big)\overline{p_2'}\,d_{\mathbf{z}}S,$$

$$\int\limits_{\Omega^-}\big[\mathbf{M}^{(6)}(\mathbf{D_x})\,\mathbf{U}\,\overline{\theta'}+W^{(6)}(\mathbf{U},\theta')\big]\,d\mathbf{x}=-\kappa\int\limits_S\frac{\partial\theta}{\partial\mathbf{n}}\,\overline{\theta'}d_{\mathbf{z}}S.$$

Obviously, on the basis of Lemmas 1 and 2 the following consequences arise.

THEOREM 1. *If* $\mathbf{U}=(\mathbf{u},\varphi_1,\varphi_2,p_1,p_2,\theta)$ *is a regular vector in* Ω^+, $\mathbf{U}'=(\mathbf{u}',\varphi_1',\varphi_2',p_1',p_2',\theta')\in C^1(\Omega^+)\cap C(\overline{\Omega^+})$, *then*

$$\int\limits_{\Omega^+}\big[\mathbf{M}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x})\cdot\mathbf{U}'(\mathbf{x})+W(\mathbf{U},\mathbf{U}')\big]\,d\mathbf{x}=\int\limits_S\mathbf{R}(\mathbf{D_z},\mathbf{n})\mathbf{U}(\mathbf{z})\cdot\mathbf{U}'(\mathbf{z})\,d_{\mathbf{z}}S,\tag{4.5}$$

where

$$W(\mathbf{U},\mathbf{U}')=W^{(1)}(\mathbf{U},\mathbf{u}')+W^{(2)}(\mathbf{U},\varphi_1')+W^{(3)}(\mathbf{U},\varphi_2')
+W^{(4)}(\mathbf{U},p_1')+W^{(5)}(\mathbf{U},p_2')+W^{(6)}(\mathbf{U},\theta').$$

THEOREM 2. *If* $\mathbf{U}=(\mathbf{u},\varphi_1,\varphi_2,p_1,p_2,\theta)$ *and* $\mathbf{U}'=(\mathbf{u}',\varphi_1',\varphi_2',p_1',p_2',\theta')$ *are regular vectors in* Ω^-, *then*

$$\int\limits_{\Omega^-}\big[\mathbf{M}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x})\cdot\mathbf{U}'(\mathbf{x})+W(\mathbf{U},\mathbf{U}')\big]\,d\mathbf{x}
=-\int\limits_S\mathbf{R}(\mathbf{D_z},\mathbf{n})\mathbf{U}(\mathbf{z})\cdot\mathbf{U}'(\mathbf{z})\,d_{\mathbf{z}}S.\tag{4.6}$$

The formulas (4.5) and (4.6) are Green's first identities in the coupled linear quasi-static theory of thermoelastic double-porosity materials for domains Ω^+ and Ω^-, respectively.

It is easy to verify that from (4.1) we get:

$$\begin{aligned}
W^{(1)}(\mathbf{U},\mathbf{u}) &= \frac{1}{3}(3\lambda+2\mu)\,|\operatorname{div}\mathbf{u}|^2 + W_0(\mathbf{u},\mathbf{u})\\
&\quad + (b_\alpha\,\varphi_\alpha - \beta_\alpha p_\alpha - \varepsilon_1\theta)\operatorname{div}\overline{\mathbf{u}},\\
W^{(2)}(\mathbf{U},\varphi_1) &= (a_1\nabla\varphi_1 + a_3\nabla\varphi_2)\cdot\nabla\varphi_1\\
&\quad + (b_1\operatorname{div}\mathbf{u} + \alpha_1\varphi_1 + \alpha_3\varphi_2 - m_1p_1 - m_3p_2 - \varepsilon_2\theta)\overline{\varphi_1},\\
W^{(3)}(\mathbf{U},\varphi_2) &= (a_3\nabla\varphi_1 + a_2\nabla\varphi_2 + \varepsilon_3'\theta)\cdot\nabla\varphi_2\\
&\quad + (b_2\operatorname{div}\mathbf{u} + \alpha_3\varphi_1 + \alpha_2\varphi_2 - m_3p_1 - m_2p_2 + \varepsilon_4'\theta)\overline{\varphi_2},\\
W^{(4)}(\mathbf{U},p_1) &= (k_1\nabla p_1 + k_3\nabla p_2)\cdot\nabla p_1\\
&\quad - (\beta_1'\operatorname{div}\mathbf{u} + m_1'\varphi_1 + m_3'\varphi_2 + \gamma_1'p_1 + \gamma_3'p_2)\overline{p_1},\\
W^{(5)}(\mathbf{U},p_2) &= (k_3\nabla p_1 + k_2\nabla p_2)\cdot\nabla p_2\\
&\quad - (\beta_2'\operatorname{div}\mathbf{u} + m_3'\varphi_1 + m_2'\varphi_2 + \gamma_3'p_1 + \gamma_2'p_2)\overline{p_2},\\
W^{(6)}(\mathbf{U},\theta) &= \kappa|\nabla\theta|^2 - (a'\theta + \varepsilon_0'\operatorname{div}\mathbf{u} + \varepsilon_\alpha'\varphi_\alpha + \varepsilon_{\alpha+2}'T_0p_\alpha)\overline{\theta},
\end{aligned} \tag{4.7}$$

where

$$W_0(\mathbf{u},\mathbf{u}) = \frac{\mu}{2}\sum_{l,j=1;\,l\neq j}^{3}\left|\frac{\partial u_j}{\partial x_l} + \frac{\partial u_l}{\partial x_j}\right|^2 + \frac{\mu}{3}\sum_{l,j=1}^{3}\left|\frac{\partial u_l}{\partial x_l} - \frac{\partial u_j}{\partial x_j}\right|^2. \tag{4.8}$$

We are now prepared to delve into the examination of uniqueness concerning the regular solutions for the BVPs $(I)^{\pm}_{\mathbf{F},\mathbf{f}}$ and $(II)^{\pm}_{\mathbf{F},\mathbf{f}}$. The resulting outcomes are as follows.

THEOREM 3. *The internal BVP $(I)^{+}_{\mathbf{F},\mathbf{f}}$ admits at most one regular solution.*

Proof. Suppose that there are two regular solutions to the problem $(I)^{+}_{\mathbf{F},\mathbf{f}}$. Then their difference $\mathbf{U}$ is a regular solution of the internal homogeneous BVP $(I)^{+}_{\mathbf{0},\mathbf{0}}$. Hence, $\mathbf{U}$ is a regular solution of the homogeneous equation

$$\mathbf{M}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{0} \tag{4.9}$$

for $\mathbf{x}\in\Omega^+$, satisfying the homogeneous boundary condition

$$\{\mathbf{U}(\mathbf{z})\}^+ = \mathbf{0} \quad \text{for} \quad \mathbf{z}\in S. \tag{4.10}$$

Clearly, by virtue of (4.9) and (4.10), from (4.2) it follows that

$$\tag{4.11} \begin{gathered} \int\limits_{\Omega^+} W^{(1)}(\mathbf{U},\mathbf{u})d\mathbf{x}=0, \qquad \int\limits_{\Omega^+} W^{(l+1)}(\mathbf{U},\varphi_l)d\mathbf{x}=0,\\ \int\limits_{\Omega^+} W^{(l+3)}(\mathbf{U},p_l)d\mathbf{x}=0, \qquad \int\limits_{\Omega^+} W^{(6)}(\mathbf{U},\theta)d\mathbf{x}=0, \qquad l=1,2. \end{gathered}$$

In view of the relations (4.7) we can easily verify that:

$$\begin{aligned} &\operatorname{Im} W^{(1)}(\mathbf{U},\mathbf{u}) = b_\alpha \operatorname{Im}(\varphi_\alpha \operatorname{div}\overline{\mathbf{u}}) - \beta_\alpha \operatorname{Re}(p_\alpha \operatorname{div}\overline{\mathbf{u}}) - \varepsilon_0 \operatorname{Im}(\theta \operatorname{div}\overline{\mathbf{u}}),\\ &\operatorname{Im}[W^{(2)}(\mathbf{U},\varphi_1)+W^{(3)}(\mathbf{U},\varphi_2)] = \operatorname{Im}[b_\alpha(\operatorname{div}\mathbf{u}\overline{\varphi_\alpha}) - (m_1\overline{\varphi_1}+m_3\overline{\varphi_2})p_1\\ &\qquad\qquad - (m_3\overline{\varphi_1}+m_2\overline{\varphi_2})p_2 - \varepsilon_\alpha\overline{\varphi_\alpha}\theta],\\ &\operatorname{Re}[W^{(4)}(\mathbf{U},p_1)+W^{(5)}(\mathbf{U},p_2)] = k_1|\nabla p_1|^2 + 2k_3\operatorname{Re}(\nabla p_1\cdot\nabla p_2) + k_2|\nabla p_2|^2\\ &\qquad + \omega\operatorname{Im}[\operatorname{div}\mathbf{u}\beta_\alpha\overline{p_\alpha} + (m_1\overline{p_1}+m_3\overline{p_2})\varphi_1 - (m_3\overline{p_1}+m_2\overline{p_2})\varphi_2 + \varepsilon_{\alpha+2}\overline{p_\alpha}\theta],\\ &\operatorname{Re} W^{(6)}(\mathbf{U},\theta) = \kappa|\nabla\theta|^2 + \omega T_0\operatorname{Im}[\varepsilon_0\operatorname{div}\mathbf{u} + \varepsilon_\alpha\varphi_\alpha + \varepsilon_{\alpha+2}p_\alpha)\overline{\theta}] \end{aligned}$$

and consequently, we can write:

$$\begin{aligned} &\operatorname{Im}[W^{(1)}(\mathbf{U},\mathbf{u}) + W^{(2)}(\mathbf{U},\varphi_1) + W^{(3)}(\mathbf{U},\varphi_2)]\\ &\qquad - \frac{1}{\omega}\operatorname{Re}[W^{(4)}(\mathbf{U},p_1)+W^{(5)}(\mathbf{U},p_2)] - \frac{1}{\omega T_0}\operatorname{Re} W^{(6)}(\mathbf{U},\theta)\\ &\qquad = -\frac{1}{\omega}[k_1|\nabla p_1|^2 + 2k_3\operatorname{Re}(\nabla p_1\cdot\nabla p_2) + k_2|\nabla p_2|^2 + \frac{\kappa}{T_0}|\nabla\theta|^2] \le 0. \end{aligned}$$

On the basis of the assumption (2.13) and the relation (4.8) from (4.11) we have

$$\tag{4.12} k_1|\nabla p_1|^2 + 2k_3\operatorname{Re}(\nabla p_1\cdot\nabla p_2) + k_2|\nabla p_2|^2 = 0, \qquad |\nabla\theta|^2 = 0.$$

Obviously, by virtue of (2.13) from (4.12) we have:

$$\tag{4.13} p_l(\mathbf{x}) = c_l = \text{const}, \qquad \theta(\mathbf{x}) = c_3 = \text{const}, \qquad l=1,2 \qquad \text{for } \mathbf{x}\in\Omega^+.$$

On the other hand, by virtue of (4.10) from (4.13) it follows that:

$$\tag{4.14} p_l(\mathbf{x}) = \theta(\mathbf{x}) = 0, \qquad l=1,2, \qquad \text{for } \mathbf{x}\in\Omega^+.$$

Now, using the assumption (2.13) and the relation (4.14) in (4.7) we obtain

$$\tag{4.15} \begin{aligned} &W^{(1)}(\mathbf{U},\mathbf{u}) + W^{(2)}(\mathbf{U},\varphi_1) + W^{(3)}(\mathbf{U},\varphi_2)\\ &= \frac{1}{3}(3\lambda+2\mu)|\operatorname{div}\mathbf{u}|^2 + 2b_\alpha\operatorname{Re}(\varphi_\alpha\operatorname{div}\overline{\mathbf{u}}) + \alpha_1|\varphi_1|^2 + 2\alpha_3\operatorname{Re}(\varphi_1\overline{\varphi_2})\\ &\quad + \alpha_2|\varphi_2|^2 + W_0(\mathbf{u}) + a_1|\nabla\varphi_1|^2 + 2\operatorname{Re}(\nabla\varphi_1\cdot\nabla\varphi_2) + a_2|\nabla\varphi_2|^2 \ge 0. \end{aligned}$$

It is clear that from (4.11) and (4.15) we get

$$\begin{aligned}&\frac{1}{3}(3\lambda+2\mu)|\operatorname{div}\mathbf{u}|^2+2b_\alpha\operatorname{Re}(\varphi_\alpha\operatorname{div}\overline{\mathbf{u}})+\alpha_1|\varphi_1|^2\\&+2\alpha_3\operatorname{Re}(\varphi_1\overline{\varphi_2})+\alpha_2|\varphi_2|^2=0,\\&W_0(\mathbf{u})=0,\quad a_1|\nabla\varphi_1|^2+2\operatorname{Re}(\nabla\varphi_1\cdot\nabla\varphi_2)+a_2|\nabla\varphi_2|^2=0.\end{aligned}\tag{4.16}$$

Now, using the assumption (2.11) from (4.16) we obtain

$$\operatorname{div}\mathbf{u}(\mathbf{x})=\varphi_1(\mathbf{x})=\varphi_2(\mathbf{x})=0,\quad \mathbf{x}\in\Omega^+.\tag{4.17}$$

At the same time, using Eqs. (4.14) and (4.17) in (4.9) and (4.10), we get Dirichlet's BVP for the vector $\mathbf{u}$:

$$\Delta\mathbf{u}(\mathbf{x})=\mathbf{0},\quad \mathbf{u}(\mathbf{z})^+=\mathbf{0},\quad \mathbf{x}\in\Omega^+,\quad \mathbf{z}\in S.\tag{4.18}$$

Finally, from (4.18) it follows that $\mathbf{u}(\mathbf{x})\equiv\mathbf{0}$ for $\mathbf{x}\in\Omega^+$. Thus, $\mathbf{U}(\mathbf{x})\equiv\mathbf{0}$ for $\mathbf{x}\in\Omega^+$ and we have the desired result. □

THEOREM 4. *Two regular solutions of the internal BVP* $(II)^+_{\mathbf{F},\mathbf{f}}$ *may differ only for an additive vector* $\mathbf{U}=(\mathbf{u},\varphi_1,\varphi_2,p_1,p_2,\theta)$, *where* φ_l, p_l $(l=1,2)$ *and* θ *satisfy the condition*

$$\varphi_l(\mathbf{x})=p_l(\mathbf{x})=\theta(\mathbf{x})=0,\quad \mathbf{x}\in\Omega^+,\quad l=1,2,$$

the vector $\mathbf{u}$ *is a rigid displacement vector and has the form*

$$\mathbf{u}(\mathbf{x})=\tilde{\mathbf{a}}+\tilde{\mathbf{b}}\times\mathbf{x},\quad \mathbf{x}\in\Omega^+,\tag{4.19}$$

for $\mathbf{x}\in\Omega^+$, *where* $\tilde{\mathbf{a}}$ *and* $\tilde{\mathbf{b}}$ *are arbitrary three-component constant vectors.*

Proof. Suppose that there are two regular solutions of the problem $(II)^+_{\mathbf{F},\mathbf{f}}$. Then their difference $\mathbf{U}$ is a regular solution of the internal homogeneous BVP $(II)^+_{\mathbf{0},\mathbf{0}}$. Hence, $\mathbf{U}$ is a regular solution of the homogeneous equation (4.9) for $\mathbf{x}\in\Omega^+$, satisfying the homogeneous boundary condition

$$\{\mathbf{R}(\mathbf{D}_\mathbf{z},\mathbf{n})\mathbf{U}(\mathbf{z})\}^+=\mathbf{0}\quad \text{for } \mathbf{z}\in S.$$

As in Theorem 3, in a similar way we can obtain the relation (4.13). Taking into account this relation from (4.7) we get:

$$\begin{aligned}W^{(1)}(\mathbf{U},\mathbf{u})&=\frac{1}{3}(3\lambda+2\mu)\,|\operatorname{div}\mathbf{u}|^2+W_0(\mathbf{u},\mathbf{u})\\&\quad+(b_\alpha\,\varphi_\alpha-\beta_\alpha c_\alpha-\varepsilon_0c_3)\operatorname{div}\overline{\mathbf{u}},\\W^{(2)}(\mathbf{U},\varphi_1)&=(a_1\nabla\varphi_1+a_3\nabla\varphi_2)\cdot\nabla\varphi_1\\&\quad+(b_1\operatorname{div}\mathbf{u}+\alpha_1\varphi_1+\alpha_3\varphi_2-m_1c_1-m_3c_2-\varepsilon_2c_3)\overline{\varphi_1},\end{aligned}\tag{4.20}$$

$$
\begin{aligned}
W^{(3)}(\mathbf{U},\varphi_2) &= (a_3\nabla\varphi_1 + a_2\nabla\varphi_2 + \varepsilon_3'\theta)\cdot\nabla\varphi_2\\
&\quad + (b_2 \operatorname{div}\mathbf{u} + \alpha_3\varphi_1 + \alpha_2\varphi_2 - m_3c_1 - m_2c_2 + \varepsilon_4'c_3)\overline{\varphi_2},\\
W^{(4)}(\mathbf{U},p_1) &= -(\beta_1'\operatorname{div}\mathbf{u} + m_1'\varphi_1 + m_3'\varphi_2 + \gamma_1'c_1 + \gamma_3'c_2 + \varepsilon_3'c_3)\overline{c_1},\\
W^{(5)}(\mathbf{U},p_2) &= -(\beta_2'\operatorname{div}\mathbf{u} + m_3'\varphi_1 + m_2'\varphi_2 + \gamma_3'c_1 + \gamma_2'c_2 + \varepsilon_4'c_3)\overline{c_2},\\
W^{(6)}(\mathbf{U},\theta) &= -(a'c_3 + \varepsilon_0'\operatorname{div}\mathbf{u} + \varepsilon_\alpha'\varphi_\alpha + \varepsilon_{\alpha+2}'T_0c_\alpha)\overline{c_3}.
\end{aligned}
\tag{4.20$_{\text{[cont.]}}$}
$$

In view of (4.20) we can easily verify that:

$$
\begin{aligned}
\operatorname{Re}W^{(1)}(\mathbf{U},\mathbf{u}) &= \frac{1}{3}(3\lambda+2\mu)|\operatorname{div}\mathbf{u}|^2 + W_0(\mathbf{u},\mathbf{u})\\
&\quad + \operatorname{Re}\,[(b_\alpha\varphi_\alpha - \beta_\alpha c_\alpha - \varepsilon_0c_3)\operatorname{div}\overline{\mathbf{u}}],\\
\operatorname{Re}\,[W^{(2)}(\mathbf{U},\varphi_1) + W^{(3)}(\mathbf{U},\varphi_2)] &= a_1|\nabla\varphi_1|^2 + 2a_3\operatorname{Re}(\nabla\varphi_1\cdot\nabla\varphi_2) + a_2|\nabla\varphi_2|^2\\
&\quad + \alpha_1|\varphi_1|^2 + 2\alpha_3\operatorname{Re}\,(\varphi_1\overline{\varphi_2}) + \alpha_2|\varphi_2|^2 + \operatorname{Re}\,[b_\alpha(\operatorname{div}\mathbf{u}\overline{\varphi_\alpha})\\
&\quad - (m_1\overline{\varphi_1} + m_3\overline{\varphi_2})c_1 - (m_3\overline{\varphi_1} + m_2\overline{\varphi_2})c_2 - \varepsilon_\alpha\overline{\varphi_\alpha}c_3],\\
-\frac{1}{\omega}\operatorname{Im}\,[W^{(4)}(\mathbf{U},p_1) + W^{(5)}(\mathbf{U},p_2)] &= \operatorname{Re}\,[\beta_\alpha\overline{c_\alpha}\operatorname{div}\mathbf{u} + (m_1\overline{c_1}\\
&\quad + m_3\overline{c_2})\varphi_1 + (m_3\overline{c_1} + m_2\overline{c_2})\varphi_2\\
&\quad + \varepsilon_{\alpha+2}\overline{c_\alpha}c_3] + \gamma_1|p_1|^2 + 2\gamma_3\operatorname{Re}(p_1\overline{p_2}) + \gamma_2|p_2|^2,\\
-\frac{1}{\omega T_0}\operatorname{Im}[W^{(6)}(\mathbf{U},\theta)] &= \operatorname{Re}\,[(\varepsilon_0\operatorname{div}\mathbf{u} + \varepsilon_\alpha\varphi_\alpha + \varepsilon_{\alpha+2}c_\alpha)\overline{c_3}] + a|c_3|^2
\end{aligned}
$$

and consequently, we can write:

$$
\begin{aligned}
&\operatorname{Re}[W^{(1)}(\mathbf{U},\mathbf{u}) + W^{(2)}(\mathbf{U},\varphi_1) + W^{(3)}(\mathbf{U},\varphi_2)]\\
&\qquad - \frac{1}{\omega}\operatorname{Im}[W^{(4)}(\mathbf{U},p_1) + W^{(5)}(\mathbf{U},p_2)] - \frac{1}{\omega T_0}\operatorname{Im}W^{(6)}(\mathbf{U},\theta)\\
&\quad = W_0(\mathbf{u}) + \frac{1}{3}(3\lambda+2\mu)|\operatorname{div}\mathbf{u}|^2 + 2b_\alpha\operatorname{Re}\,(\varphi_\alpha\operatorname{div}\overline{\mathbf{u}})\\
&\qquad + \alpha_1|\varphi_1|^2 + 2\alpha_3\operatorname{Re}(\varphi_1\bar{\varphi}_2) + \alpha_2|\varphi_2|^2 + a_1|\nabla\varphi_1|^2 + 2\operatorname{Re}(\nabla\varphi_1\nabla\varphi_2)\\
&\qquad + a_2|\nabla\varphi_2|^2 + \gamma_1|c_1|^2 + 2\gamma_3\operatorname{Re}(c_1\bar{c_2}) + \gamma_2|c_2|^2 + 2\varepsilon_{\alpha+2}\operatorname{Re}(\bar{c_\alpha}c_3) + a|c_3|^2\\
&\quad \geq 0.
\end{aligned}
$$

On the basis of this relation from (4.11) we have:

$$
\begin{aligned}
&\frac{1}{3}(3\lambda+2\mu)|\operatorname{div}\mathbf{u}|^2 + 2b_\alpha\operatorname{Re}(\varphi_\alpha\operatorname{div}\overline{\mathbf{u}}) + \alpha_1|\varphi_1|^2\\
&\qquad\qquad + 2\alpha_3\operatorname{Re}(\varphi_1\bar{\varphi}_2) + \alpha_2|\varphi_2|^2 = 0,\\
&W_0(\mathbf{u}) = 0, \qquad a_1|\nabla\varphi_1|^2 + 2\operatorname{Re}(\nabla\varphi_1\nabla\varphi_2) + a_2|\nabla\varphi_2|^2 = 0,\\
&\gamma_1|c_1|^2 + 2\gamma_3\operatorname{Re}(c_1\bar{c_2}) + \gamma_2|c_2|^2 + 2\varepsilon_{\alpha+2}\operatorname{Re}(\bar{c_\alpha}c_3) + a|c_3|^2 = 0.
\end{aligned}
$$

Obviously, by virtue of these equalities and the condition (2.13) we obtain the relations (4.14) and (4.17). Finally, just as in the classical theory of elasticity (see [34]), from the equations $\operatorname{div}\mathbf{u}(\mathbf{x}) = W_0(\mathbf{u}) = 0$ we get the formula (4.19). □

Theorem 5. *The external BVP* $(K)^-_{\mathbf{F},\mathbf{f}}$ *has one regular solution, where* $K = I, II$.

Proof. Suppose that there are two regular solutions of problem $(K)^-_{\mathbf{F},\mathbf{f}}$, where $K = I, II$. Then their difference $\mathbf{U}$ is a regular solution of the external homogeneous BVP $(K)^-_{\mathbf{0},\mathbf{0}}$. Hence, $\mathbf{U}$ is a regular solution of the homogeneous equation (4.9) for $\mathbf{x} \in \Omega^-$ satisfying the homogeneous boundary condition

$$\{\mathbf{U}(\mathbf{z})\}^- = \mathbf{0} \tag{4.21}$$

for $K = I$ and

$$\{\mathbf{R}(\mathbf{D}_\mathbf{z}, \mathbf{n})\mathbf{U}(\mathbf{z})\}^- = \mathbf{0} \tag{4.22}$$

for $K = II$.

Clearly, by virtue of (4.9), (4.21) and (4.22) from (4.4) we obtain:

$$\begin{gathered}
\int\limits_{\Omega^-} W^{(1)}(\mathbf{U}, \mathbf{u})\, d\mathbf{x} = 0,\\
\int\limits_{\Omega^-} W^{(l+1)}(\mathbf{U}, \varphi_l)\, d\mathbf{x} = 0,\\
\int\limits_{\Omega^-} W^{(l+2)}(\mathbf{U}, p_l)\, d\mathbf{x} = 0,\\
\int\limits_{\Omega^-} W^{(6)}(\mathbf{U}, \theta)\, d\mathbf{x} = 0, \qquad l = 1, 2.
\end{gathered} \tag{4.23}$$

In a similar manner as in Theorem 4, from (4.23) we obtain the relations:

$$\begin{aligned}
&\mathbf{u}(\mathbf{x}) = \tilde{\mathbf{a}} + \tilde{\mathbf{b}} \times \mathbf{x}, \quad p_l(\mathbf{x}) = c_l = \text{const}, \quad \theta(\mathbf{x}) = c_3 = \text{const},\\
&\operatorname{div}\mathbf{u}(\mathbf{x}) = \varphi_l(\mathbf{x}) = 0, \quad l = 1, 2
\end{aligned} \tag{4.24}$$

for $\mathbf{x} \in \Omega^-$, where $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ are arbitrary three-component constant vectors. In view of the condition at infinity (3.1) from (4.24) we get $\mathbf{u}(\mathbf{x}) \equiv \mathbf{0}$ for $\mathbf{x} \in \Omega^-$. Thus, we have the desired result. □

5. Fundamental solution

In this section, the fundamental solution of the system of equations (2.11) is constructed explicitly and its basic properties are established.

DEFINITION 2. The fundamental solution of system (2.11) is the matrix $\mathbf{G}(\mathbf{x}) = (G_{lj}(\mathbf{x}))_{8\times 8}$ satisfying the following equation in the class of generalized functions

$$\mathbf{M}(\mathbf{D_x})\mathbf{G}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{J},$$

where $\delta(\mathbf{x})$ is the Dirac delta, $\mathbf{J} = (\delta_{lj})_{8\times 8}$ is the unit matrix, $\mathbf{x} \in \mathbb{R}^3$.

We now construct the matrix $\mathbf{G}(\mathbf{x})$. We consider the system of nonhomogeneous equations:

$$\begin{aligned}
&\mu\Delta\mathbf{u}+(\lambda+\mu)\nabla\operatorname{div}\mathbf{u}-b_\alpha\nabla\varphi_\alpha+\beta'_\alpha\nabla p_\alpha+\varepsilon'_0\nabla\theta = \boldsymbol{\mathcal{F}}',\\
&(a_1\Delta-\alpha_1)\varphi_1+(a_3\Delta-\alpha_3)\varphi_2+b_1\operatorname{div}\mathbf{u}+m'_1p_1+m'_3p_2+\varepsilon'_1\nabla\theta = \mathcal{F}_4,\\
&(a_3\Delta-\alpha_3)\varphi_1+(a_2\Delta-\alpha_2)\varphi_2+b_2\operatorname{div}\mathbf{u}+m'_3p_1+m'_2p_2+\varepsilon'_2\nabla\theta = \mathcal{F}_5,\\
&(k_1\Delta+\gamma'_1)p_1+(k_3\Delta+\gamma'_3)p_2-\beta_1\operatorname{div}\mathbf{u}+m_1\varphi_1+m_3\varphi_2+\varepsilon'_3T_0\nabla\theta = \mathcal{F}_6,\\
&(k_3\Delta+\gamma'_3)p_1+(k_2\Delta+\gamma'_2)p_2-\beta_2\operatorname{div}\mathbf{u}+m_3\varphi_1+m_2\varphi_2+\varepsilon'_4T_0\nabla\theta = \mathcal{F}_7,\\
&(\kappa\Delta+a')\theta-\varepsilon'_0\operatorname{div}\mathbf{u}+\varepsilon_\alpha\varphi_\alpha+\varepsilon'_{\alpha+2}p_\alpha = \mathcal{F}_8,
\end{aligned} \tag{5.1}$$

where $\mathcal{F}_l$ $(l = 1, 2, \dots, 8)$ are smooth functions on $\mathbb{R}^3$, $\boldsymbol{\mathcal{F}}' = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$. Obviously, the system (5.1) may be written in the form

$$\mathbf{M}^\top(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \boldsymbol{\mathcal{F}}(\mathbf{x}), \tag{5.2}$$

where $\mathbf{M}^\top$ is the transpose of matrix $\mathbf{M}$, $\mathbf{U} = (\mathbf{u}, \varphi_1, \varphi_2, p_1, p_2, \theta)$, $\boldsymbol{\mathcal{F}} = (\boldsymbol{\mathcal{F}}', \mathcal{F}_4, \mathcal{F}_5, \mathcal{F}_6, \mathcal{F}_7, \mathcal{F}_8)$.

Applying the operator div to the first equation of (5.1) we obtain the following system:

$$\begin{aligned}
&\mu_0\Delta\operatorname{div}\mathbf{u}-b_\alpha\Delta\varphi_\alpha+\beta_\alpha\Delta p_\alpha+\varepsilon'_0\Delta\theta = \operatorname{div}\boldsymbol{\mathcal{F}}',\\
&(a_1\Delta-\alpha_1)\varphi_1+(a_3\Delta-\alpha_3)\varphi_2+b_1\operatorname{div}\mathbf{u}+m'_1p_1+m'_3p_2+\varepsilon'_1\nabla\theta = \mathcal{F}_4,\\
&(a_3\Delta-\alpha_3)\varphi_1+(a_2\Delta-\alpha_2)\varphi_2+b_2\operatorname{div}\mathbf{u}+m'_3p_1+m'_2p_2+\varepsilon'_2\nabla\theta = \mathcal{F}_5,\\
&(k_1\Delta+\gamma'_1)p_1+(k_3\Delta+\gamma'_3)p_2-\beta_1\operatorname{div}\mathbf{u}+m_1\varphi_1+m_3\varphi_2+\varepsilon'_3T_0\nabla\theta = \mathcal{F}_6,\\
&(k_3\Delta+\gamma'_3)p_1+(k_2\Delta+\gamma'_2)p_2-\beta_2\operatorname{div}\mathbf{u}+m_3\varphi_1+m_2\varphi_2+\varepsilon'_4T_0\nabla\theta = \mathcal{F}_7,\\
&(\kappa\Delta+a')\theta-\varepsilon'_0\operatorname{div}\mathbf{u}+\varepsilon_\alpha\varphi_\alpha+\varepsilon'_{\alpha+2}p_\alpha = \mathcal{F}_8,
\end{aligned} \tag{5.3}$$

where $\mu_0 = \lambda + 2\mu$. From (5.3) we have

$$\mathbf{A}(\Delta)\mathbf{V} = \boldsymbol{\Phi}, \tag{5.4}$$

where

$$\mathbf{V} = (\operatorname{div}\mathbf{u}, \varphi_1, \varphi_2, p_1, p_2, \theta) = (V_1, V_2, \dots, V_6),$$
$$\boldsymbol{\Phi} = (\operatorname{div}\boldsymbol{\mathcal{F}}', \mathcal{F}_4, \mathcal{F}_5, \mathcal{F}_6, \mathcal{F}_7, \mathcal{F}_8) = (\Phi_1, \Phi_2, \dots, \Phi_6)$$

and

$$\mathbf{A}(\Delta) = (A_{lj}(\Delta))_{6\times 6}$$
$$= \begin{pmatrix} \mu_0\Delta & -b_1\Delta & -b_2\Delta & \beta_1'\Delta & \beta_2'\Delta & \varepsilon_0'\Delta \\ b_1 & a_1\Delta-\alpha_1 & a_3\Delta-\alpha_3 & m_1' & m_3' & \varepsilon_1' \\ b_2 & a_3\Delta-\alpha_3 & a_2\Delta-\alpha_2 & m_3' & m_2' & \varepsilon_2' \\ -\beta_1 & m_1 & m_3 & k_1\Delta+\gamma_1' & k_3\Delta+\gamma_3' & \varepsilon_3'T_0 \\ -\beta_2 & m_3 & m_2 & k_3\Delta+\gamma_3' & k_2\Delta+\gamma_2' & \varepsilon_4'T_0 \\ -\varepsilon_0 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & \Delta+a \end{pmatrix}_{6\times 6}.$$

Let us introduce the notation

$$\Lambda_1(\Delta) = \frac{1}{a_0\kappa k_0\mu_0}\det\mathbf{A}(\Delta) = \Delta\prod_{j=1}^{5}(\Delta+\lambda_j^2), \tag{5.5}$$

where $a_0 = a_1a_2 - a_3^2$; $k_0 = k_1k_2 - k_3^2$; λ_1^2, λ_2^2, λ_3^2 and λ_4^2 are the roots of the following equation with respect to ξ

$$\det\begin{pmatrix} \mu_0 & -b_1 & -b_2 & \beta_1' & \beta_2' & \varepsilon_0' \\ b_1 & -a_1\xi-\alpha_1 & -a_3\xi-\alpha_3 & m_1' & m_3' & \varepsilon_1' \\ b_2 & -a_3\xi-\alpha_3 & -a_2\xi-\alpha_2 & m_3' & m_2' & \varepsilon_2' \\ -\beta_1 & m_1 & m_3 & -k_1\xi+\gamma_1' & -k_3\xi+\gamma_3' & \varepsilon_3'T_0 \\ -\beta_2 & m_3 & m_2 & -k_3\xi+\gamma_3' & -k_2\xi+\gamma_2' & \varepsilon_4'T_0 \\ -\varepsilon_0 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 & -\kappa\xi+a' \end{pmatrix}_{6\times 6} = 0.$$

We assume that $\operatorname{Im}\lambda_l > 0$, $\lambda_l \neq \lambda_j$ for $l, j = 1, 2, 3, 4$ and $l \neq j$.

From Eq. (5.4) we deduce that

$$\begin{aligned} &\Lambda_1(\Delta)\operatorname{div}\mathbf{u} = \Psi_1, && \Lambda_1(\Delta)\varphi_l = \Psi_{l+1}, \\ &\Lambda_1(\Delta)p_l = \Psi_{l+3}, && \Lambda_1(\Delta)\theta = \Psi_6, \quad l = 1, 2, \end{aligned} \tag{5.6}$$

where

$$\Psi_m = \frac{1}{a_0\kappa k_0\mu_0}\sum_{j=1}^{6}A_{jm}^*\Phi_j, \qquad m = 1, 2, \dots, 6 \tag{5.7}$$

and A_{jm}^* is the cofactor of the element A_{jm} of the matrix $\mathbf{A}$.

Now applying the operator $\Lambda_1(\Delta)$ to the first equation of system (5.1) and by virtue of (5.6) it follows that

$$\Lambda_2(\Delta)\mathbf{u} = \tilde{\mathbf{\Psi}}, \tag{5.8}$$

where $\Lambda_2(\Delta) = \Delta\Lambda_1(\Delta)$ and

$$\tilde{\mathbf{\Psi}} = \frac{1}{\mu}\Lambda_1(\Delta)\boldsymbol{\mathcal{F}}' - \frac{1}{\mu}\nabla[(\lambda+\mu)\Psi_1 - b_\alpha\,\Psi_{\alpha+1} + \beta'_\alpha\,\Psi_{\alpha+3} + \varepsilon'_0\Psi_6]. \tag{5.9}$$

In view of the relations (5.6) and (5.8), we can write

$$\mathbf{\Lambda}(\Delta)\mathbf{U} = \mathbf{\Psi}, \tag{5.10}$$

where $\mathbf{\Psi} = (\tilde{\mathbf{\Psi}}, \Psi_2, \Psi_3, \Psi_4, \Psi_5, \Psi_6)$ is a eight-component vector function and

$$\begin{aligned} &\mathbf{\Lambda} = (\Lambda_{lj})_{8\times 8}, \quad \Lambda_{11} = \Lambda_{22} = \Lambda_{33} = \Lambda_2, \quad \Lambda_{44} = \Lambda_{55} = \ldots = \Lambda_{88} = \Lambda_1, \\ &\Lambda_{lj} = 0, \quad l \neq j, \quad l, j = 1, 2, \ldots, 8. \end{aligned} \tag{5.11}$$

We introduce the notation

$$\begin{aligned} m_{l1}(\Delta) = &-\frac{1}{a_0\kappa k_0\mu\mu_0} \\ &\times [(\lambda+\mu)A^*_{l1}(\Delta) - b_\alpha A^*_{l;\alpha+1}(\Delta) + \beta'_\alpha A^*_{l;\alpha+3}(\Delta) + \varepsilon'_0 A^*_{l6}], \\ m_{lj}(\Delta) = &\frac{1}{a_0 k_0\kappa\mu_0}A^*_{lj}(\Delta), \quad l = 1, 2, \ldots, 6, \; j = 2, 3, 4, 5, 6. \end{aligned} \tag{5.12}$$

Taking into account (5.12), from (5.7) and (5.9) we obtain

$$\begin{aligned} \tilde{\mathbf{\Psi}} &= \frac{1}{\mu}\Lambda_1(\Delta)\boldsymbol{\mathcal{F}}' + m_{11}(\Delta)\nabla\operatorname{div}\boldsymbol{\mathcal{F}}' + \sum_{l=2}^{6} m_{l1}(\Delta)\nabla\mathcal{F}_{l+2}, \\ \Psi_j &= m_{1j}\operatorname{div}\boldsymbol{\mathcal{F}}' + \sum_{l=2}^{6} m_{lj}(\Delta)\mathcal{F}_{l+2}, \quad j = 2, 3, \ldots, 6. \end{aligned} \tag{5.13}$$

Then, we may derive from (5.13)

$$\mathbf{\Psi} = \mathbf{N}^\top(\mathbf{D_x})\boldsymbol{\mathcal{F}}, \tag{5.14}$$

where

$$\begin{aligned} &\mathbf{N}(\mathbf{D_x}) = (N_{lj}(\mathbf{D_x}))_{8\times 8}, \quad N_{lj}(\mathbf{D_x}) = \frac{1}{\mu}\Lambda_1\delta_{lj} + m_{11}\frac{\partial^2}{\partial x_l\partial x_j}, \\ &N_{l;r+2}(\mathbf{D_x}) = m_{1r}\frac{\partial}{\partial x_l}, \quad N_{r+2;j}(\mathbf{D_x}) = m_{r1}\frac{\partial}{\partial x_j}, \\ &N_{r+2;m+2}(\mathbf{D_x}) = m_{rm}(\Delta), \quad l, j = 1, 2, 3, \; r, m = 2, 3, \ldots, 6. \end{aligned} \tag{5.15}$$

Combining the relations (5.2) and (5.10) with (5.14) we may further conclude that $\mathbf{\Lambda U} = \mathbf{N}^\top \mathbf{M}^\top \mathbf{U}$. Obviously, from the last identity we get

$$\mathbf{M}(\mathbf{D}_\mathbf{x})\mathbf{N}(\mathbf{D}_\mathbf{x}) = \mathbf{\Lambda}(\Delta). \tag{5.16}$$

Let

$$\begin{aligned}
&\mathbf{\Upsilon}(\mathbf{x}) = (\Upsilon_{lj}(\mathbf{x}))_{8\times 8},\\
&\Upsilon_{11}(\mathbf{x}) = \Upsilon_{22}(\mathbf{x}) = \Upsilon_{33}(\mathbf{x}) = \sum_{r=0}^{5} \eta_{2r}\gamma^{(r)}(\mathbf{x}) + \eta_{10}\gamma_0'(\mathbf{x}),\\
&\Upsilon_{44}(\mathbf{x}) = \Upsilon_{55}(\mathbf{x}) = \ldots = \Upsilon_{88}(\mathbf{x}) = \sum_{r=0}^{5} \eta_{1r}\gamma^{(r)}(\mathbf{x}),\\
&\Upsilon_{lj}(\mathbf{x}) = 0, \quad l \neq j,\ l, j = 1, 2, \ldots, 8,
\end{aligned} \tag{5.17}$$

where we have used the notations

$$\gamma^{(0)}(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|}, \quad \gamma_0'(\mathbf{x}) = -\frac{|\mathbf{x}|}{8\pi}, \quad \gamma^{(j)}(x) = -\frac{e^{i\lambda_j|\mathbf{x}|}}{4\pi|\mathbf{x}|} \tag{5.18}$$

and

$$\begin{aligned}
&\eta_{10} = -\prod_{l=1}^{5} \lambda_l^{-2}, \quad \eta_{1j} = \lambda_j^{-2} \prod_{l=1;l\neq j}^{4} (\lambda_j^2 - \lambda_l^2)^{-1}, \quad \eta_{20}' = -\sum_{l=1}^{5} \lambda_l^{-2},\\
&\eta_{20} = \eta_{20}' \sum_{l=1}^{5} \lambda_l^{-2}, \quad \eta_{2j} = \lambda_j^{-5} \prod_{l=1;l\neq j}^{4} (\lambda_j^2 - \lambda_l^2)^{-1}, \quad j = 1, 2, \ldots, 5.
\end{aligned} \tag{5.19}$$

On the basis of (5.5), (5.11), (5.18) and (5.19) it is easy to prove

$$\mathbf{\Lambda}(\Delta)\mathbf{\Upsilon}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{J}, \tag{5.20}$$

i.e., $\mathbf{\Upsilon}(\mathbf{x})$ is the fundamental matrix of the operator $\mathbf{\Lambda}(\Delta)$.

Now we introduce the notation

$$\mathbf{G}(\mathbf{x}) = \mathbf{N}(\mathbf{D}_\mathbf{x})\mathbf{\Upsilon}(\mathbf{x}). \tag{5.21}$$

By virtue of (5.16), (5.20) and (5.21) we have

$$\mathbf{M}(\mathbf{D}_\mathbf{x})\mathbf{G}(\mathbf{x}) = \mathbf{M}(\mathbf{D}_\mathbf{x})\mathbf{N}(\mathbf{D}_\mathbf{x})\mathbf{\Upsilon}(\mathbf{x}) = \mathbf{\Lambda}(\Delta)\mathbf{\Upsilon}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{J}.$$

Consequently, $\mathbf{G}(\mathbf{x})$ is the fundamental matrix of the operator $\mathbf{M}(\mathbf{D}_\mathbf{x})$. We have thereby proved the following consequence.

THEOREM 6. *The matrix* $\mathbf{G}(\mathbf{x}) = (G_{lj}(\mathbf{x}))_{8\times 8}$ *which is defined by* (5.21) *is the fundamental solution of system* (2.11), *where* $\mathbf{N}(\mathbf{D_x})$ *and* $\mathbf{\Upsilon}(\mathbf{x})$ *are given by* (5.15) *and* (5.17), *respectively.*

Note that the matrix $\mathbf{G}(\mathbf{x})$ is constructed explicitly by means of seven elementary functions: $\gamma_0'(\mathbf{x})$, $\gamma^{(j)}(\mathbf{x})$ $(j = 0, 1, \dots, 5)$.

Theorem 6 leads to the following basic properties of the matrix $\mathbf{G}(\mathbf{x})$.

THEOREM 7. *Each column of the matrix* $\mathbf{G}(\mathbf{x})$ *is a solution of homogeneous equation*

$$\mathbf{M}(\mathbf{D_x})\mathbf{G}(\mathbf{x}) = \mathbf{0}$$

at every point $\mathbf{x} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$.

THEOREM 8. *The relations*

$$\begin{aligned}
&G_{lj}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \quad G_{rm}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \quad G_{r+2;m+2}(\mathbf{x}) = O(|x|^{-1}),\\
&G_{88}(\mathbf{x}) = O(|\mathbf{x}|^{-1}), \quad G_{ls}(\mathbf{x}) = O(1), \quad G_{sl}(\mathbf{x}) = O(1),\\
&G_{r;j+5}(\mathbf{x}) = O(1), \quad G_{j+5;r}(\mathbf{x}) = O(1), \quad G_{r+2;8}(\mathbf{x}) = O(1),\\
&G_{8;r+2}(\mathbf{x}) = O(1), \quad l, j = 1, 2, 3, \;\; r, m = 4, 5, \;\; s = 4, 5, \dots, 8
\end{aligned}$$

hold in the neighborhood of the origin of $\mathbb{R}^3$.

THEOREM 9. *The matrix* $\mathbf{G}^{(0)}(\mathbf{x}) = (G^{(0)}_{lj}(\mathbf{x}))_{8\times 8}$ *defined by*

$$\begin{aligned}
&G^{(0)}_{lj}(\mathbf{x}) = -\frac{\lambda + 3\mu}{8\pi\mu\mu_0}\frac{\delta_{lj}}{|\mathbf{x}|} - \frac{\lambda+\mu}{8\pi\mu\mu_0}\frac{x_l x_j}{|\mathbf{x}|^3}, \quad G^{(0)}_{44}(\mathbf{x}) = \frac{a_2}{a_0}\gamma^{(0)}(\mathbf{x}),\\
&G^{(0)}_{45}(\mathbf{x}) = G^{(0)}_{54}(\mathbf{x}) = -\frac{a_3}{a_0}\gamma^{(0)}(\mathbf{x}), \quad G^{(0)}_{55}(\mathbf{x}) = \frac{a_1}{a_0}\gamma^{(0)}(\mathbf{x}),\\
&G^{(0)}_{66}(\mathbf{x}) = \frac{k_2}{k_0}\gamma^{(0)}(\mathbf{x}), \quad G^{(0)}_{67}(\mathbf{x}) = G^{(0)}_{76}(\mathbf{x}) = -\frac{k_3}{k_0}\gamma^{(0)}(\mathbf{x}),\\
&G^{(0)}_{77}(\mathbf{x}) = \frac{k_1}{k_0}\gamma^{(0)}(\mathbf{x}), \quad G^{(0)}_{88}(\mathbf{x}) = \frac{1}{\kappa}\gamma^{(0)}(\mathbf{x}),\\
&G^{(0)}_{ls}(\mathbf{x}) = G^{(0)}_{sl}(\mathbf{x}) = G^{(0)}_{r;j+5}(\mathbf{x}) = G^{(0)}_{j+5;r}(\mathbf{x}) = G^{(0)}_{r+2;8}(\mathbf{x}) = G^{(0)}_{8;r+2}(\mathbf{x}) = 0,\\
&\qquad\qquad l, j = 1, 2, 3, \;\; r, m = 4, 5, \;\; s = 4, 5, \dots, 8
\end{aligned}$$

is the fundamental solution of the system

$$\begin{aligned}
&\mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\operatorname{div}\mathbf{u} = \mathbf{0}, \quad a_1\Delta\varphi_1 + a_3\Delta\varphi_2 = 0, \quad a_3\Delta\varphi_1 + a_2\Delta\varphi_2 = 0,\\
&k_1\Delta p_1 + k_3\Delta p_2 = 0, \quad k_3\Delta p_1 + k_2\Delta p_2 = 0, \quad \kappa\Delta\theta = 0.
\end{aligned}$$

Theorem 10. *The relations*

$$G_{lj}(\mathbf{x}) - G_{lj}^{(0)}(\mathbf{x}) = \text{const} + O(|\mathbf{x}|), \qquad l, j = 1, 2, \dots, 8$$

hold in the neighborhood of the origin of $\mathbb{R}^3$.

Thus, on the basis of Theorems 8 and 10 the matrix $\mathbf{G}^{(0)}(\mathbf{x})$ is the singular part of the fundamental solution $\mathbf{G}(\mathbf{x})$ in the neighborhood of the origin of $\mathbb{R}^3$.

6. Basic properties of potentials and singular integral operators

In this section, the surface (single-layer and double-layer) and volume potentials are defined, the useful singular integral operators are introduced, and the basic properties of these potentials and operators are established.

In the sequel we use the following matrix differential operator:

$$\begin{aligned}
&\tilde{\mathbf{R}}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = (\tilde{R}_{lj}(\mathbf{D}_\mathbf{x}, \mathbf{n}))_{8\times 8}, \qquad \tilde{R}_{lj}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = R_{lj}(\mathbf{D}_\mathbf{x}, \mathbf{n}),\\
&\tilde{R}_{l;r+5}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = -\beta'_r n_l, \qquad \tilde{R}_{l;8}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = -\varepsilon'_0 n_l,\\
&\tilde{R}_{ms}(\mathbf{D}_\mathbf{x}, \mathbf{n}) = R_{ms}(\mathbf{D}_\mathbf{x}, \mathbf{n}), \qquad l = 1, 2, 3, \ j = 1, 2, \dots, 5,\\
&\qquad\qquad\qquad\qquad\qquad\qquad r = 1, 2, \ m = 4, 5, \dots, 8, \ s = 1, 2, \dots, 8,
\end{aligned} \tag{6.1}$$

where $R_{lj}(\mathbf{D}_\mathbf{x}, \mathbf{n})$ is given by (3.2).

Let us now introduce the single-layer potential

$$\mathbf{P}^{(1)}(\mathbf{x}, \mathbf{g}) = \int\limits_S \mathbf{G}(\mathbf{x} - \mathbf{y})\mathbf{g}(\mathbf{y})\, d_\mathbf{y} S,$$

the double-layer potential

$$\mathbf{P}^{(2)}(\mathbf{x}, \mathbf{g}) = \int\limits_S \big[\tilde{\mathbf{R}}(\mathbf{D}_\mathbf{y}, \mathbf{n}(\mathbf{y}))\mathbf{G}^\top(\mathbf{x} - \mathbf{y})\big]^\top \mathbf{g}(\mathbf{y})\, d_\mathbf{y} S,$$

and the volume potential

$$\mathbf{P}^{(3)}(\mathbf{x}, \boldsymbol{\phi}, \Omega^\pm) = \int\limits_{\Omega^\pm} \mathbf{G}(\mathbf{x} - \mathbf{y})\boldsymbol{\phi}(\mathbf{y})\, d\mathbf{y},$$

where $\mathbf{G}$ is the fundamental matrix of the operator $\mathbf{M}(\mathbf{D}_\mathbf{x})$ and defined by (5.21), the operator $\tilde{\mathbf{R}}$ is given by (6.1), $\mathbf{g}$ and $\boldsymbol{\phi}$ are eight-component vector functions.

It is not very difficult to prove the basic properties of these potentials. Namely, we can obtain the following consequences.

THEOREM 11. *If* $S \in C^{r+1,\nu}$, $\mathbf{g} \in C^{r,\nu'}(S)$, $0 < \nu' < \nu \leq 1$, *and* r *is a nonnegative whole number, then:*

(i)
$$\mathbf{P}^{(1)}(\cdot, \mathbf{g}) \in C^{0,\nu'}(\mathbb{R}^3) \cap C^{r+1,\nu'}(\overline{\Omega^{\pm}}) \cap C^{\infty}(\Omega^{\pm}),$$

(ii)
$$\mathbf{M}(\mathbf{D_x})\, \mathbf{P}^{(1)}(\mathbf{x}, \mathbf{g}) = \mathbf{0}, \qquad \mathbf{x} \in \Omega^{\pm},$$

(iii) $\mathbf{R}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\, \mathbf{P}^{(1)}(\mathbf{z}, \mathbf{g})$ *is a singular integral for* $\mathbf{z} \in S$,

(iv)
$$\{\mathbf{R}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\, \mathbf{P}^{(1)}(\mathbf{z}, \mathbf{g})\}^{\pm} = \mp\frac{1}{2}\, \mathbf{g}(\mathbf{z}) + \mathbf{R}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\, \mathbf{P}^{(1)}(\mathbf{z}, \mathbf{g}), \tag{6.2}$$

for $\mathbf{z} \in S$,

(v)
$$\mathbf{P}^{(1)}(\mathbf{x}, \mathbf{g}) = O(|\mathbf{x}|^{-1}), \qquad \frac{\partial}{\partial x_l}\mathbf{P}^{(1)}(\mathbf{x}, \mathbf{g}) = O(|\mathbf{x}|^{-2})$$

for $|\mathbf{x}| \gg 1$ *and* $l = 1, 2, 3$.

THEOREM 12. *If* $S \in C^{r+1,\nu}$, $\mathbf{g} \in C^{r,\nu'}(S)$, $0 < \nu' < \nu \leq 1$, *then:*

(i)
$$\mathbf{P}^{(2)}(\cdot, \mathbf{g}) \in C^{r,\nu'}(\overline{\Omega^{\pm}}) \cap C^{\infty}(\Omega^{\pm}),$$

(ii)
$$\mathbf{M}(\mathbf{D_x})\, \mathbf{P}^{(2)}(\mathbf{x}, \mathbf{g}) = \mathbf{0}, \qquad \mathbf{x} \in \Omega^{\pm},$$

(iii) $\mathbf{P}^{(2)}(\mathbf{z}, \mathbf{g})$ *is a singular integral for* $\mathbf{z} \in S$,

(iv)
$$\{\mathbf{P}^{(2)}(\mathbf{z}, \mathbf{g})\}^{\pm} = \pm\frac{1}{2}\, \mathbf{g}(\mathbf{z}) + \mathbf{P}^{(2)}(\mathbf{z}, \mathbf{g}), \qquad \mathbf{z} \in S \tag{6.3}$$

for the non-negative integer r,

(v)
$$\mathbf{P}^{(2)}(\mathbf{x}, \mathbf{g}) = O(|\mathbf{x}|^{-2}), \qquad \frac{\partial}{\partial x_l}\mathbf{P}^{(2)}(\mathbf{x}, \mathbf{g}) = O(|\mathbf{x}|^{-3})$$

for $|\mathbf{x}| \gg 1$ *and* $l = 1, 2, 3$,

(vi)
$$\{\mathbf{R}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\, \mathbf{P}^{(2)}(\mathbf{z}, \mathbf{g})\}^{+} = \{\mathbf{R}(\mathbf{D_z}, \mathbf{n}(\mathbf{z}))\, \mathbf{P}^{(2)}(\mathbf{z}, \mathbf{g})\}^{-}$$

for the natural number m *and* $\mathbf{z} \in S$.

THEOREM 13. *If* $S \in C^{r+1,\nu}$, $\boldsymbol{\phi} \in C^{r,\nu'}(\Omega^{+})$, $0 < \nu' < \nu \leq 1$, *then:*

(i)
$$\mathbf{P}^{(3)}(\cdot, \boldsymbol{\phi}, \Omega^{+}) \in C^{1,\nu'}(\mathbb{R}^3) \cap C^{2}(\Omega^{+}) \cap C^{2,\nu'}(\overline{\Omega_0^{+}}),$$

(ii)
$$\mathbf{M}(\mathbf{D_x})\, \mathbf{P}^{(3)}(\mathbf{x}, \boldsymbol{\phi}, \Omega^{+}) = \boldsymbol{\phi}(\mathbf{x}), \qquad \mathbf{x} \in \Omega^{+},$$

where Ω_0^{+} *is a domain in* $\mathbb{R}^3$ *and* $\overline{\Omega_0^{+}} \subset \Omega^{+}$.

THEOREM 14. *If* $S\in C^{1,\nu}$, $\operatorname{supp}\boldsymbol{\phi}=\Omega\subset\Omega^-$, $\boldsymbol{\phi}\in C^{0,\nu'}(\Omega^-)$, $0<\nu'<\nu\le 1$, *then:*

(i) $\mathbf{P}^{(3)}(\cdot,\boldsymbol{\phi},\Omega^-)\in C^{1,\nu'}(\mathbb{R}^3)\cap C^2(\Omega^-)\cap C^{2,\nu'}(\bar{\Omega}_0^-)$,

(ii) $\mathbf{M}(\mathbf{D_x})\,\mathbf{P}^{(3)}(\mathbf{x},\boldsymbol{\phi},\Omega^-)=\boldsymbol{\phi}(\mathbf{x}), \quad \mathbf{x}\in\Omega^-$,

where Ω *is a finite domain in* $\mathbb{R}^3$ *and* $\overline{\Omega_0^-}\subset\Omega^-$.

Now we introduce the following integral operators

$$
\begin{aligned}
&\mathcal{H}^{(1)}\mathbf{g}(\mathbf{z})\equiv\frac{1}{2}\,\mathbf{g}(\mathbf{z})+\mathbf{P}^{(2)}(\mathbf{z},\mathbf{g}),\\
&\mathcal{H}^{(2)}\mathbf{g}(\mathbf{z})\equiv\frac{1}{2}\,\mathbf{g}(\mathbf{z})+\mathbf{R}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{P}^{(1)}(\mathbf{z},\mathbf{g}),\\
&\mathcal{H}^{(3)}\mathbf{g}(\mathbf{z})\equiv-\frac{1}{2}\,\mathbf{g}(\mathbf{z})+\mathbf{P}^{(2)}(\mathbf{z},\mathbf{g}),\\
&\mathcal{H}^{(4)}\mathbf{g}(\mathbf{z})\equiv-\frac{1}{2}\,\mathbf{g}(\mathbf{z})+\mathbf{R}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{P}^{(1)}(\mathbf{z},\mathbf{g}),\\
&\mathcal{H}_{\varsigma}\mathbf{g}(\mathbf{z})\equiv\frac{1}{2}\,\mathbf{g}(\mathbf{z})+\varsigma\,\mathbf{P}^{(2)}(\mathbf{z},\mathbf{g}),\qquad \mathbf{z}\in S,
\end{aligned}
\tag{6.4}
$$

where ς is an arbitrary complex number. On the basis of Theorems 11 and 12, we can prove that $\mathcal{H}^{(l)}$ $(l=1,2,3,4)$ and $\mathcal{H}_\varsigma$ are the singular integral operators.

On the other hand, if $\boldsymbol{\Gamma}^{(r)}=(\Gamma_{lj}^{(r)})_{8\times 8}$ is the symbol of the operator $\mathcal{H}^{(r)}$ $(r=1,2,3,4)$, then from (6.4) we have

$$
\begin{aligned}
\det\boldsymbol{\Gamma}^{(1)}&=\det\boldsymbol{\Gamma}^{(2)}=-\det\boldsymbol{\Gamma}^{(3)}=-\det\boldsymbol{\Gamma}^{(4)}\\
&=\Big(-\frac{1}{2}\Big)^8\Big(1-\frac{\mu^2}{(\lambda+2\mu)^2}\Big)=\frac{(\lambda+\mu)(\lambda+3\mu)}{256(\lambda+2\mu)^2}>0,
\end{aligned}
\tag{6.5}
$$

i.e., the operator $\mathcal{H}^{(r)}$ is of the normal type, where $r=1,2,3,4$.

Moreover, let $\boldsymbol{\Gamma}_\varsigma$ and $\operatorname{ind}\mathcal{H}_\varsigma$ be the symbol and the index of the operator $\mathcal{H}_\varsigma$, respectively. It may be easily shown that

$$
\det\boldsymbol{\Gamma}_\varsigma=\frac{(\lambda+2\mu)^2-\mu^2\varsigma^2}{256(\lambda+2\mu)^2}
$$

and $\det\boldsymbol{\Gamma}_\varsigma$ vanishes only at two points ς_1 and ς_2 of the complex plane. By virtue of (6.5) and $\det\boldsymbol{\Gamma}_1=\det\boldsymbol{\Gamma}^{(1)}$ we get $\varsigma_j\neq 1$ $(j=1,2)$ and

$$
\operatorname{ind}\mathcal{H}_1=\operatorname{ind}\mathcal{H}^{(1)}=\operatorname{ind}\mathcal{H}_0=0.
$$

Similarly we obtain

$$
\operatorname{ind}\mathcal{H}^{(2)}=-\operatorname{ind}\mathcal{H}^{(1)}=0,\qquad \operatorname{ind}\mathcal{H}^{(3)}=-\operatorname{ind}\mathcal{H}^{(4)}=0.
$$

Thus, the singular integral operator $\mathcal{H}^{(r)}$ $(r = 1, 2, 3, 4)$ is of the normal type with an index equal to zero and consequently, Noether's theorems are valid for $\mathcal{H}^{(r)}$.

For the definitions of a normal type singular integral operator, the symbol and the index of the 2D singular integral operators see, e.g., KUPRADZE *et al.* [34].

7. Existence theorems

In this section, applying the potential method and the theory of singular integral equations the existence of classical solutions of the internal and external basic BVPs $(K)^+_{\mathbf{F},\mathbf{f}}$ and $(K)^-_{\mathbf{F},\mathbf{f}}$ are proved, where $K = I, II$.

Taking into account Theorems 13 and 14 we deduce that the volume potential $\mathbf{P}^{(3)}(\mathbf{x}, \mathbf{F}, \Omega^\pm)$ is a particular solution of the nonhomogeneous equation (2.12), where $\mathbf{F} \in C^{0,\nu'}(\Omega^\pm)$, $0 < \nu' \leq 1$; $\operatorname{supp} \mathbf{F}$ is a finite domain in Ω^-. Because of this, we prove the existence theorems of a regular (classical) solution of the problems $(K)^+_{\mathbf{0},\mathbf{f}}$ and $(K)^-_{\mathbf{0},\mathbf{f}}$, where $K = I, II$.

Problem $(I)^+_{\mathbf{0},\mathbf{f}}$. We are looking for a regular solution to this problem in the form of the double-layer potential

$$\mathbf{U}(\mathbf{x}) = \mathbf{P}^{(2)}(\mathbf{x}, \mathbf{g}) \quad \text{for} \quad \mathbf{x} \in \Omega^+, \tag{7.1}$$

where $\mathbf{g}$ is the required eight-component vector function.

In view of Theorem 12 the vector function $\mathbf{U}$ is a solution of the following homogeneous equation

$$\mathbf{M}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{0} \tag{7.2}$$

for $\mathbf{x} \in \Omega^+$. By virtue of the boundary condition (3.3) and using (6.3) from (7.1) we obtain a singular integral equation

$$\mathcal{H}^{(1)}\,\mathbf{g}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \tag{7.3}$$

for determining the unknown vector function $\mathbf{g}$, where $\mathbf{z} \in S$. We prove that Eq. (7.3) is always solvable for an arbitrary vector $\mathbf{f}$.

Obviously, the homogeneous adjoint integral equation of (7.3) has the following form

$$\mathcal{H}^{(2)}\,\mathbf{h}(\mathbf{z}) = \mathbf{0} \quad \text{for } \mathbf{z} \in S, \tag{7.4}$$

where $\mathbf{h}$ is the required eight-component vector function. Now we prove that (7.4) has only the trivial solution.

Let $\mathbf{h}_0$ be a solution of the homogeneous equation (7.4). On the basis of Theorem 11 and Eq. (7.4) the vector function $\mathbf{V}(\mathbf{x}) = \mathbf{P}^{(1)}(\mathbf{x}, \mathbf{h}_0)$ is a regular solution of the external homogeneous BVP $(II)^-_{\mathbf{0},\mathbf{0}}$. By virtue of Theorem 5, the problem $(II)^-_{\mathbf{0},\mathbf{0}}$ has only the trivial solution, i.e.,

$$\mathbf{V}(\mathbf{x}) \equiv \mathbf{0} \quad \text{for } \mathbf{x} \in \Omega^-. \tag{7.5}$$

In addition, by Theorem 11 and (7.5) we get

$$\{\mathbf{V}(\mathbf{z})\}^+ = \{\mathbf{V}(\mathbf{z})\}^- = \mathbf{0} \quad \text{for } \mathbf{z} \in S.$$

Consequently, the vector $\mathbf{V}(\mathbf{x})$ is a regular solution of the internal homogegeous BVP $(I)^+_{\mathbf{0},\mathbf{0}}$ and using Theorem 3 it follows that

$$\mathbf{V}(\mathbf{x}) \equiv \mathbf{0} \quad \text{for } \mathbf{x} \in \Omega^+. \tag{7.6}$$

In view of relations (7.5), (7.6) and identity (6.2) we obtain

$$\mathbf{h}_0(\mathbf{z}) = \{\mathbf{R}(\mathbf{D}_\mathbf{z}, \mathbf{n})\mathbf{V}(\mathbf{z})\}^- - \{\mathbf{R}(\mathbf{D}_\mathbf{z}, \mathbf{n})\mathbf{V}(\mathbf{z})\}^+ = \mathbf{0} \quad \text{for } \mathbf{z} \in S.$$

Thus, the homogeneous equation (7.4) has only the trivial solution. On the basis of Noether's theorem the nonhomogeneous integral equation (7.3) is always solvable for an arbitrary vector $\mathbf{f}$. We have thereby proved the following result.

THEOREM 15. *If* $S \in C^{2,\nu}$, $\mathbf{f} \in C^{1,\nu'}(S)$, $0 < \nu' < \nu \le 1$, *then a regular solution of the internal BVP* $(I)^+_{\mathbf{0},\mathbf{f}}$ *exists, is unique and is represented by the double-layer potential* (7.1), *where* $\mathbf{g}$ *is a solution of the singular integral equation* (7.3) *which is always solvable for an arbitrary vector* $\mathbf{f}$.

Problem $(II)^-_{\mathbf{0},\mathbf{f}}$. Now we seek a regular solution to this problem in the form of the single-layer potential

$$\mathbf{U}(\mathbf{x}) = \mathbf{P}^{(1)}(\mathbf{x}, \mathbf{h}) \quad \text{for } \mathbf{x} \in \Omega^-, \tag{7.7}$$

where $\mathbf{h}$ is the required eight-component vector function. Clearly, by Theorem 11 the vector function $\mathbf{U}$ is a solution of (7.2) for $\mathbf{x} \in \Omega^-$. By virtue of the boundary condition (3.6) and using (6.2), from (7.7) we obtain the following singular integral equation for determining the unknown vector $\mathbf{h}$

$$\mathcal{H}^{(2)}\,\mathbf{h}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \quad \text{for} \quad \mathbf{z} \in S. \tag{7.8}$$

In Theorem 15, we proved that the corresponding homogeneous equation (7.4) has only the trivial solution. Hence, by Noether's theorem (7.8) is always solvable. We have the following consequence.

THEOREM 16. *If* $S \in C^{2,\nu}$, $\mathbf{f} \in C^{0,\nu'}(S)$, $0 < \nu' < \nu \le 1$, *then a regular solution of the external BVP* $(II)^-_{\mathbf{0},\mathbf{f}}$ *exists, is unique and is represented by single-layer potential* (7.7), *where* $\mathbf{h}$ *is a solution of the singular integral equation* (7.8) *which is always solvable for an arbitrary vector* $\mathbf{f}$.

Problem $(I)^{-}_{\mathbf{0},\mathbf{f}}$. We seek a regular solution to this problem in the sum of the single-layer and double-layer potentials

$$\mathbf{U}(\mathbf{x}) = \mathbf{P}^{(1)}(\mathbf{x},\mathbf{g}) + \mathbf{P}^{(2)}(\mathbf{x},\mathbf{g}) \quad \text{for } \mathbf{x} \in \Omega^{-}, \tag{7.9}$$

where $\mathbf{g}$ is the required eight-component vector function.

Obviously, by Theorems 11 and 12 the vector function $\mathbf{U}$ is a regular solution of (7.2) for $\mathbf{x} \in \Omega^{-}$. Keeping in mind the boundary condition (3.5) and using (6.3) from (7.9) we obtain, for determining the unknown vector $\mathbf{g}$, a singular integral equation

$$\mathcal{H}^{(5)}\,\mathbf{g}(\mathbf{z}) \equiv \mathcal{H}^{(3)}\,\mathbf{g}(\mathbf{z}) + \mathbf{P}^{(1)}(\mathbf{z},\mathbf{g}) = \mathbf{f}(\mathbf{z}) \quad \text{for } \mathbf{z} \in S. \tag{7.10}$$

We prove that Eq. (7.10) is always solvable for an arbitrary vector $\mathbf{f}$. We can easily verify that the singular integral operator $\mathcal{H}^{(5)}$ is of the normal type and $\operatorname{ind}\mathcal{H}^{(5)} = \operatorname{ind}\mathcal{H}^{(3)} = 0$.

Now we prove that the homogeneous equation

$$\mathcal{H}^{(5)}\,\mathbf{g}_0(\mathbf{z}) = \mathbf{0} \quad \text{for } \mathbf{z} \in S \tag{7.11}$$

has only a trivial solution. Let $\mathbf{g}_0$ be a solution of the homogeneous equation (7.11). Then the vector

$$\mathbf{V}(\mathbf{x}) \equiv \mathbf{P}^{(1)}(\mathbf{x},\mathbf{g}_0) + \mathbf{P}^{(2)}(\mathbf{x},\mathbf{g}_0) \quad \text{for } \mathbf{x} \in \Omega^{-} \tag{7.12}$$

is a regular solution of the external BVP $(I)^{-}_{\mathbf{0},\mathbf{0}}$. Using Theorem 5 we have (7.5).

Moreover, by identities (6.2) and (6.3) from (7.12) we get

$$\begin{aligned} &\{\mathbf{V}(\mathbf{z})\}^{+} - \{\mathbf{V}(\mathbf{z})\}^{-} = \mathbf{g}_0(\mathbf{z}), \\ &\{\mathbf{R}(\mathbf{D}_{\mathbf{z}},\mathbf{n})\mathbf{V}(\mathbf{z})\}^{+} - \{\mathbf{R}(\mathbf{D}_{\mathbf{z}},\mathbf{n})\mathbf{V}(\mathbf{z})\}^{-} = -\mathbf{g}_0(\mathbf{z}), \quad \text{for } \mathbf{z} \in S. \end{aligned} \tag{7.13}$$

In view of (7.5) from (7.13) it follows that

$$\{\mathbf{R}(\mathbf{D}_{\mathbf{z}},\mathbf{n})\mathbf{V}(\mathbf{z}) + \mathbf{V}(\mathbf{z})\}^{+} = \mathbf{0} \quad \text{for } \mathbf{z} \in S. \tag{7.14}$$

Obviously, the vector $\mathbf{V}$ is a solution of Eq. (7.2) in Ω^{+} satisfying the boundary condition (7.14). Now applying identity (4.5) for vector $\mathbf{V}$ we obtain

$$\{\mathbf{V}(\mathbf{z})\}^{+} = \mathbf{0} \quad \text{for } \mathbf{z} \in S. \tag{7.15}$$

Finally, by virtue of (7.5) and (7.15) from the first equation of (7.13) we get $\mathbf{g}_0(\mathbf{z}) \equiv \mathbf{0}$ for $\mathbf{z} \in S$.

Thus, the homogeneous equation (7.11) has only the trivial solution and therefore on the basis of Noether's theorem the integral equation (7.10) is always solvable for an arbitrary vector $\mathbf{f}$. We have thereby proved the following theorem.

Theorem 17. *If* $S \in C^{2,\nu}$, $\mathbf{f} \in C^{1,\nu'}(S)$, $0 < \nu' < \nu \le 1$, *then a regular solution of the external BVP* $(I)^-_{\mathbf{0},\mathbf{f}}$ *exists, is unique and is represented by the sum of double-layer and single-layer potentials* (7.9), *where* $\mathbf{g}$ *is a solution of the singular integral equation* (7.10) *which is always solvable for an arbitrary vector* $\mathbf{f}$.

Problem $(II)^+_{\mathbf{0},\mathbf{f}}$. Finally, we are looking for a regular solution to this problem in the form of a single-layer potential

$$\mathbf{U}(\mathbf{x}) = \mathbf{P}^{(1)}(\mathbf{x},\mathbf{g}) \quad \text{for } \mathbf{x} \in \Omega^+, \tag{7.16}$$

where $\mathbf{g}$ is the required eight-component vector function.

In view of Theorem 11 the vector function $\mathbf{U}$ is a solution of the homogeneous equation (7.2). Then, taking into account the identity (6.3) and the boundary condition (3.4) from (7.16) we obtain, for determining the vector $\mathbf{g}$, the following singular integral equation

$$\mathcal{H}^{(4)}\,\mathbf{g}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \quad \text{for } \mathbf{z} \in S. \tag{7.17}$$

To investigate the solvability of Eq. (7.17) we consider the homogeneous equation

$$\mathcal{H}^{(4)}\,\mathbf{g}(\mathbf{z}) = \mathbf{0} \quad \text{for } \mathbf{z} \in S. \tag{7.18}$$

Clearly, the adjoint homogeneous integral equation of (7.18) has the form

$$\mathcal{H}^{(3)}\,\mathbf{h}(\mathbf{z}) = \mathbf{0} \quad \text{for } \mathbf{z} \in S. \tag{7.19}$$

In our further analysis we need the following consequence.

Lemma 3. *The homogeneous equations* (7.18) *and* (7.19) *have six linearly independent solutions each and they constitute complete systems of solutions.*

Lemma 3 can be proved similarly to the corresponding result in the quasi-static theory of elasticity for single-porosity materials (see [32]).

Introduce now the eight-component vector functions $\boldsymbol{\vartheta}^{(j)}(\mathbf{x})$ $(j = 1,2,\ldots,6)$ by

$$\begin{aligned} &\boldsymbol{\vartheta}^{(1)}(\mathbf{x}) = (1,0,0,0,0,0,0,0), &&\boldsymbol{\vartheta}^{(2)}(\mathbf{x}) = (0,1,0,0,0,0,0,0),\\ &\boldsymbol{\vartheta}^{(3)}(\mathbf{x}) = (0,0,1,0,0,0,0,0), &&\boldsymbol{\vartheta}^{(4)}(\mathbf{x}) = (0,-x_3,x_2,0,0,0,0,0),\\ &\boldsymbol{\vartheta}^{(5)}(\mathbf{x}) = (x_3,0,-x_1,0,0,0,0,0), &&\boldsymbol{\vartheta}^{(6)}(\mathbf{x}) = (-x_2,x_1,0,0,0,0,0,0). \end{aligned} \tag{7.20}$$

Obviously, $\{\boldsymbol{\vartheta}^{(j)}(\mathbf{x})\}_{j=1}^{6}$ is the system of linearly independent vectors. Moreover, by Theorem 4 each vector $\boldsymbol{\vartheta}^{(j)}(\mathbf{x})$ is a regular solution of the internal homogeneous BVP $(II)^+_{\mathbf{0},\mathbf{0}}$ and the homogeneous singular integral equation (7.19), i.e., we have:

$$\mathbf{M}(\mathbf{D_x})\,\boldsymbol{\vartheta}^{(j)}(\mathbf{x}) = \mathbf{0} \quad \text{for } \mathbf{x} \in \Omega^+,$$
$$\{\mathbf{R}(\mathbf{D_z},\mathbf{n})\,\boldsymbol{\vartheta}^{(j)}(\mathbf{z})\}^+ = \mathbf{0}, \qquad \mathcal{H}^{(4)}\,\boldsymbol{\vartheta}^{(j)}(\mathbf{z}) = \mathbf{0} \quad \text{for } \mathbf{z} \in S$$

and $j = 1, 2, \ldots, 6$. Hence, $\{\boldsymbol{\vartheta}^{(j)}(\mathbf{x})\}_{j=1}^{6}$ is a complete system of linearly independent solutions of Eq. (7.19).

Applying Noether's theorem the necessary and sufficient condition for (7.17) to be solvable is of the form

$$\int_S \mathbf{f}(\mathbf{z}) \cdot \boldsymbol{\vartheta}^{(j)}(\mathbf{z})\, d_{\mathbf{z}}S = 0, \qquad j = 1, 2, \ldots, 6, \tag{7.21}$$

where $\boldsymbol{\vartheta}^{(j)}$ is determined by (7.20).

On the other hand, if $\mathbf{f} = (f_1, f_2, \ldots, f_8)$ and $\mathbf{f}^{(0)} = (f_1, f_2, f_3)$, then by virtue of (7.20) the condition (7.21) can be rewritten as:

$$\int_S \mathbf{f}^{(0)}(\mathbf{z})\, d_{\mathbf{z}}S = \mathbf{0}, \qquad \int_S \mathbf{z} \times \mathbf{f}^{(0)}(\mathbf{z})\, d_{\mathbf{z}}S = \mathbf{0}. \tag{7.22}$$

We have thereby proved the following result.

THEOREM 18. *If $S \in C^{1,\nu}$, $\mathbf{f} \in C^{0,\nu'}(S)$, $0 < \nu' < \nu \le 1$, then problem $(II)^+_{\mathbf{0},\mathbf{f}}$ is solvable only when conditions* (7.22) *are fulfilled. In this case, the solution of this problem is represented by the potential of single-layer* (7.16) *and is determined within an additive vector of $\tilde{\mathbf{U}} = (\tilde{\mathbf{u}}, \tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{p}_1, \tilde{p}_2, \tilde{\theta})$, where $\mathbf{g}$ is a solution of the singular integral equation* (7.17) *and*

$$\tilde{\mathbf{u}}(\mathbf{x}) = \tilde{\mathbf{a}} + \tilde{\mathbf{b}} \times \mathbf{x}, \qquad \tilde{\varphi}_l(\mathbf{x}) = \tilde{p}_l(\mathbf{x}) = \tilde{\theta}(\mathbf{x}) \equiv 0, \qquad l = 1, 2,$$

for $\mathbf{x} \in \Omega^+$, $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ are arbitrary three-component constant vectors.

8. Conclusion

In this paper, the basic internal and external BVPs of steady vibrations in the coupled linear quasi-static theory of thermoelasticity for materials with double porosity are investigated. The research yields the following key findings:

(i) Uniqueness theorems for classical solutions of the aforementioned BVPs are established through the application of Green's identity.

(ii) The fundamental solution of the system of steady vibration equations is explicitly constructed by using seven elementary functions.

(iii) Essential properties of the surface (single-layer and double-layer) and volume potentials are established.

(iv) The useful singular integral operators are studied for which Noether's theorems are valid.

(v) The existence theorems for classical solutions of the BVPs of steady vibrations are proved by using the potential method and the theory of singular integral equations.

The findings presented in this paper serve as a foundation for exploring the BVPs within the framework of coupled linear quasi-static theory of elasticity and thermoelasticity. These investigations pertain specifically to materials exhibiting triple porosity. The potential method emerges as a valuable tool in this endeavor.

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Received August 10, 2023.

Published online October 31, 2023.